АСТРОФИЗИКА

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r-MODE OSCILLATIONS AND ROCKET EFFECT IN ROTATING SUPERFLUID NEUTRON STARS. II. NUMERICAL RESULTS

G.COLUCCI[1], M.MANNARELLI[2], C.MANUEL[3]

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We estimate the timescale due to damping mechanism responsible of the *rocket effect*. We consider a two fluid model in which both components can oscillate and we consider two different kind of *r*-mode oscillations, one predominantly associated with comoving displacements of the two fluids and a second one associated with countermoving, out of phase, displacements. In the former case, we find that the dissipation mechanism associated with the rocket effect is not able to prevent the growth of the *r*-mode instability. In the latter case, the rocket effect prevents the growth of the *r*-mode instability for temperatures larger than about 10^{8} K.

Key words: *neutron stars:oscillations:rocket effect*

1. *Introduction*. The purpose of this paper is to estimate the characteristic damping timescale of *r*-mode oscillations of standard neutron stars due to the rocket effect, that is, in the presence of processes that change the particle densities of protons, neutrons and electrons. The hydrodynamical equations governing the evolution of the system have been derived in the accompanying paper [1] for two different kind of *r*-mode oscillations. We have considered the *standard r-mode oscillations*, which are predominantly toroidal comoving displacement of the superfluid and normal components, and the *superfluid r-mode oscillations*, which are associated to toroidal countermoving displacements of the two fluids. In principle, these two modes are coupled, however, in the limit of small rotation frequency and assuming that the star has a uniform mass density and is incompressible, they decouple.

The rocket effect is related to any mechanism that changes the relative abundance of the fluid components that are in relative motions. In neutron stars there are two mechanisms that can change the relative abundance of the various components of the outer core, Urca processes and crust-outer core (or inner core-outer core) *transfusions*. The Urca process can lead to a dissipative force if neutrons and protons are in relative motions and it has been shown by Gusakov [2], that the rocket effect can be described by considering in the Euler equations of the system the four bulk viscosity coefficients characteristic of a superfluid system [3]. In the transfusion processes the quantity of matter in the outer core of the neutron star increases (or decreases) due to fluxes of matter from (to) the crust or the inner core. One of the mechanism to produce such a flux of matter may be related to the neutron star accretion. The fall of matter on the top of the crust produces an increase of the pressure on the external surface of the crust and then the inner part of the crust melts and liberates neutrons, protons and electrons in the outer core. Or one may think that the transfusion is due to a compression of the crust due to a radial oscillation of the outer core. In this case the ionic constituents of the crust are squeezed from below and part of their nucleonic content is released and augments the fluid components of the outer core. The opposite mechanism, related to a reduction of the pressure on the crust, leads to the nucleonic capture by the ions of the lattice of the crust. A similar mechanism may produce the outer core-inner core transfusion, where a change of pressure at the interface between the outer core and the inner core may be accompanied to a flux of matter.

As already was shown in [1], all these processes lead to the appearance of dissipative terms in the hydrodynamical equations, and we have called the rocket effect the corresponding effect on the hydrodynamical evolution of the system.

In order to simplify the analysis we consider a simplified model of neutron star consisting of a fluid of neutrons, protons and electrons and no crust. Protons and electrons are locked together by the electromagnetic interaction and therefore the system effectively consists of two fluids. As a further simplification we assume that the star has a uniform mass distribution with density $\rho = 2.5\rho_0$ and a radius of 10 km. Since this simplified model of star does not comprise a crust we consider only number changing processes associated with Urca processes. We estimate the damping timescale of this mechanism for the "standard *r*-mode oscillations" and the "superfluid *r*-mode oscillations", whose hydrodynamical equations have been derived in [1]. We find that the dissipative mechanism associated with the rocket term does not efficiently damp standard *r*-mode oscillations. On the other hand, this mechanism becomes efficient in damping the superfluid *r*-mode instability at high temperatures, of the order of 10^9 K. Therefore the rocket effect due to Urca processes provides a generalization of the bulk viscosity dissipation. In agreement with [4] we find that the superfluid *r*-mode oscillations are damped as well by the mutual friction force for sufficiently large values of the entrainment parameter.

This paper is organized as follows. In Section 2 we compute the contribution of Urca processes to particle mass creation rates. In Section 3 we evaluate the timescale corresponding to the rocket effect, only in the case due to weak interactions, and compare the result with the timescale associated with other dissipative processes. We draw our conclusions in Section 4. In Appendix A we report the expressions of the decay widths of Urca processes in presence of various *nn* condensates.

2. *Urca processes and the rocket effect.* In order to evaluate the damping timescale associated with the rocket effect we need to evaluate the mass creation rate, Γ_n. There are two main processes that are responsible of the change of the densities of neutrons, protons and electrons in neutron stars. The capture/release of nucleons from the ions in the crust and the Urca processes. The first process regards the properties of ions at extremely high densities and it is difficult to take into account properly. One should consider all the microscopic processes as well as the macroscopic processes that can lead to a transfusion of material between the crust and the underlying fluid. We are not aware of any calculation of the mass creation rate associated with the crust-outer core or the inner core-outer core transfusion mechanisms. However, it is worth mentioning that such processes might become important for a self-consistent treatment of *r*-modes. *r*-mode oscillations have a radial component that can be several meters large at the surface of the star, see [5]. Such a radial displacement comes with a pressure oscillations that can lead to a change in composition of the crust and may lead to transfer of material between the crust and the underlying superfluid.

We postpone the treatment of the transfusion processes to future work. For the time being we restrict to Urca processes that take place among protons, neutrons and electrons. It was found in [6], that for certain realistic equations of state the direct Urca processes are allowed when the star density exceeds the nuclear saturation density $\rho_0 = 2.8\cdot 10^{14}$ g cm^{-3}, and the proton fraction exceeds the threshold value $x_p^c = 1/9$. We shall restrict to consider such processes which might be realized in the interior of massive neutron stars. Henceforth we shall take $x_p = x_p^c$, but our results do not strongly depend on the precise value of x_p as far as $x_p \geq x_p^c$. The direct Urca reactions are given by

$$n \to p + e^- + \bar{\nu}_e, \quad p + e^- \to n + \nu_e, \tag{1}$$

and the corresponding reaction rates are given by

$$\Gamma^d_{Urca} = \int \prod_{i=n,p,e,\nu} \left[\frac{d^3 p_i}{(2\pi)^3 2E_i}\right] f_n (1-f_p)(1-f_e) \sum_{\text{spins}} |M|^2 (2\pi)^4 \delta^{(4)}(P_i - P_f), \tag{2}$$

for the neutron decay process, and by

$$\Gamma^c_{Urca} = \int \prod_{i=n,p,e,\nu} \left[\frac{d^3 p_i}{(2\pi)^3 2E_i}\right] f_p f_e (1-f_n) \sum_{\text{spins}} |M|^2 (2\pi)^4 \delta^{(4)}(P_i - P_f), \tag{3}$$

for the electron capture process. In both reaction rates, $|M|^2$ is the squared of the scattering cross section for the weak interaction process (see [6-8] for its explicit value), and f_i is the Fermi-Dirac distribution function of the particle $i = n, p, e$.

In equilibrium, or quasi-equilibrium, the neutron decay and the electron capture processes balance, meaning that $\overline{\Gamma}^d_{Urca} = \overline{\Gamma}^c_{Urca}$ (hereafter the bar indicates the steady value of a quantity). In such a case, one has that $\overline{\Gamma}_n = \overline{\Gamma}_p = 0$ and the rocket terms in the perturbed hydrodynamical equations in Eqs. (17)-(18) in [1] vanish. At equilibrium the velocities of the proton and neutron fluids should be the same, and further, the chemical potentials should fulfill the relation

$$\overline{\mu}_n = \overline{\mu}_c = \overline{\mu}_p + \overline{\mu}_e \,. \qquad (4)$$

Therefore, a perturbation around the equilibrium distribution is related to a difference in the velocities of the two fluids and/or a deviation from chemical equilibrium. Out of equilibrium the neutron decay process and the electron capture process do not compensate, resulting in a net source (or sink) of neutrons, and thus, in a non-vanishing value of Γ_n, given by

$$\Gamma_n = \Gamma^d_{Urca} - \Gamma^c_{Urca} \,. \qquad (5)$$

In order. to evaluate this quantity we consider small perturbations of the equilibrium distribution functions, $f_i = \bar{f}_i + \delta f$; upon substituting the perturbed distribution function in Eqs.(2,3) and considering terms linear in the perturbation, we obtain the mass creation rate

$$\delta\Gamma_n = -\frac{1}{T}\left(\delta\mu_n - \delta\mu_c + \frac{m_n}{2}(1-\varepsilon_n-\varepsilon_p)(\delta \mathrm{w})^2\right)\overline{\Gamma}_{Urca} \,, \qquad (6)$$

in agreement with the results of [9]. Note that the last term in round parenthesis in Eq. (6) gives the correction to the chemical equilibrium due to the relative motion between the two fluids, $\delta \mathrm{w}$. Since it is of higher order in the fluctuations, it can be neglected.

The value of $\overline{\Gamma}_{Urca}$ depends on the particular neutron superfluid phase that is realized in the star. In [7] three phases have been considered, one corresponding to a $^1S_0$ condensate and two corresponding to the $^3P_2$ condensate with $m_j=0$ and $m_j=2$. In any superfluid phase the Urca reaction rate is suppressed with respect to the corresponding value in unpaired matter. The reason is that the phase space available for scattering is reduced in the superfluid phase. This leads to a suppression of the neutrino emission rate and of the bulk viscosity coefficient in the cores of superfluid neutron stars [7]. In our case this leads to a suppression of the rocket effect. However, it is worth remarking that while the suppression is exponentially large for the $^1S_0$ and for the $^3P_2$ condensate with $m_j=0$ (of order $\exp(-\Delta/T)$, with Δ the corresponding energy gap), it is much smaller for the $^3P_2$ condensate with $m_j=2$. This is due to the fact that in the latter case the quasiparticle fermionic dispersion law is gapless. We report the results of [7] regarding these three superfluid phases in Appendix A.

In the situation where the direct Urca processes are not allowed, one should consider modified Urca processes,

$$n+N \rightarrow p+N+e^{-}+\bar{\nu}_e, \quad p+N+e^{-} \rightarrow n+N+\nu_e, \tag{7}$$

where N is an additional nucleon. However, the reaction rates of the modified Urca processes are 3-5 orders of magnitude lower than those of direct Urca processes [10], and the effect of the rocket effect in the damping of r-modes will be weaker. For the time being, we will assume that the conditions for the direct Urca processes are allowed in the star.

In general, one can express the mass creation rate as

$$\delta\Gamma_n = \frac{\delta\beta^w}{k_B T}\Xi, \tag{8}$$

where k_B is the Boltzmann constant and we have defined

$$\delta\beta^w = m_n\,\delta\beta - \frac{m_n}{2}(1-\varepsilon_n-\varepsilon_c)(\delta \mathrm{w})^2, \tag{9}$$

where $\delta\beta$ has been defined in Eq. (19) of [1], and in the case when the Urca processes are allowed the value of $\Xi = \bar{\Gamma}_{Urca}$ depends on the superfluid phase considered.

3. *Stability analysis.* Given the mass creation rate, we can now evaluate the damping timescale associated with the rocket effect and compare the result with the timescale of different processes. In principle, one can determine the damping times associated with any mechanism from the imaginary part of the mode frequency. However, in the present analysis we shall limit ourselves to the energy integral estimates.

First we evaluate the energy associated with any oscillation of the star. In the frame corotating with the star, see [4] for more details, it can be written as

$$E = \frac{1}{2}\int\left(\rho|\delta \mathrm{v}|^2 + (1-\bar{\varepsilon})x_p(1-x_p)|\delta \mathrm{w}|^2 + \sum_{i,j=n,c}\mathcal{P}_{ij}\delta\rho_i\delta\rho_j^* - \frac{|\nabla\delta\Phi|^2}{4\pi G}dV\right), \tag{10}$$

where the integral is extended to the volume of the superfluid core of the star and we have defined $\mathcal{P}_{ij} = \partial\mu_i/\partial\rho_j$. The energy stored in the oscillations contains a kinetic energy contribution, E_k and a potential energy contribution, E_{pot}.

The variation of the kinetic energy and of the potential energy due to mutual friction and rocket effect can be obtained using the continuity and Euler equations for the oscillations, integrating by parts and discarding surface terms. The variation of the total energy due to the mutual friction turns out to be

$$\left(\frac{\partial E}{\partial t}\right)_{MF} = -\int dV|\delta \mathrm{w}\cdot \mathbf{f}_{MF}|. \tag{11}$$

Regarding the rocket effect, the variation of the kinetic energy can be written as

$$\begin{aligned}\left(\frac{\partial E_k}{\partial t}\right)_{RT} &= \frac{1}{2}\int dV\,\Gamma_n\left(|\delta\upsilon_n|^2 - |\delta\upsilon_c|^2\right) + \int dV\,\Gamma_n\delta\mathrm{v}_c\cdot\delta\mathrm{w} - \\ &- \int dV\rho_n\delta\mathrm{v}_n\cdot\nabla\delta\tilde{\mu}_n - \int dV\rho_c\delta\mathrm{v}_c\cdot\nabla\delta\tilde{\mu}_c,\end{aligned} \tag{12}$$

where the last two terms on the right hand side are related to the variation of the potential energy

$$\left(\frac{\partial E_p}{\partial t}\right)_{RT} = \int dV\,\Gamma_n(\delta\tilde{\mu}_c - \delta\tilde{\mu}_n) + \int dV\,\rho_n \delta\mathbf{v}_n \cdot \nabla\delta\tilde{\mu}_n + \int dV\,\rho_c \delta\mathbf{v}_c \cdot \nabla\delta\tilde{\mu}_c . \quad (13)$$

Upon substituting the expression above in the variation of the kinetic energy we have that

$$\left(\frac{\partial E}{\partial t}\right)_{RT} = -\int dV \left|\Gamma_n \delta\beta^{*}\right|, \quad (14)$$

and from Eq. (8), we have that at the leading order in the perturbation

$$\left(\frac{\partial E}{\partial t}\right)_{RT} = -m_n^2 \int dV |\delta\beta|^2 \frac{\Xi}{T}. \quad (15)$$

These expressions are in agreement with the general considerations for the entropy generation due to both mutual friction and the rocket effect of [9]. Further, the requirements of positive entropy production allow us to guarantee that these terms always represent energy losses and not gains.

The damping timescale associated to the mutual friction is given by

$$\frac{1}{\tau_{MF}} = -\frac{1}{2E}\left(\frac{\partial E}{\partial t}\right)_{MF} = \frac{1}{2E}\int dV \left|\delta\mathbf{w}\cdot\mathbf{f}_{MF}\right|, \quad (16)$$

while for the rocket term one has that

$$\frac{1}{\tau_{RT}} = -\frac{1}{2E}\left(\frac{\partial E}{\partial t}\right)_{RT} = \frac{m_n^2}{2E}\int dV |\delta\beta|^2 \frac{\Xi}{T}. \quad (17)$$

The timescale associated with gravitational-wave emission is instead given by

$$\frac{1}{\tau_{gw}} = -\frac{1}{2E}\left(\frac{\partial E}{\partial t}\right)_{gw} = \frac{1}{2E}\omega_r \sum_l N_l\, \omega_i^{2l+1}\left(\left|D_{lm}\right|^2 + \left|J_{lm}\right|^2\right), \quad (18)$$

where D_{lm} and J_{lm} are the mass and current multipoles respectively, ω_r is the r-mode frequency, $\omega_i = \omega_r - m\Omega$ is the frequency measured by an inertial observer and

$$N_l = \frac{4\pi G}{c^{2l+1}} \frac{(l+1)(l+2)}{l(l-1)[(2l+1)!!]^2}. \quad (19)$$

The three timescales τ_{MF}, τ_{RT} and τ_{gw} depend on the particular r-mode oscillation considered. In the following sections we shall separately analyze standard r-modes and superfluid r-modes. In order to make our results comparable with the analysis of [11] we define the entrainment parameter

$$\varepsilon = \frac{\varepsilon_c x_c}{1 - x_c - \varepsilon_c}, \quad (20)$$

and according with [11] we consider the range of values $\varepsilon \leq 0.06$.

3.1. *Timescales for standard r-modes.* In [1] we showed that in

Table 1

ORDER IN Ω OF THE COMOVING AND CONTERMOVING DISPLACEMENTS, AS OBTAINED IN [1] OF THE PRESSURE FLUCTUATION AND OF THE CHEMICAL POTENTIAL FLUCTUATION FOR THE STANDARD *r*-MODE OSCILLATION AND FOR THE SUPERFLUID *r*-MODE OSCILLATION

type of *r*-mode	K_{lm}	k_{lm}	ς_{lm}	τ_{lm}	S_{lm}, s_{lm}, Z_{lm}, z_{lm}
standard *r*-mode	$O(\Omega^0)$	$O(\Omega^2)$	$O(\Omega^2)$	$O(\Omega^4)$	$O(\Omega^2)$
superfluid *r*-mode	$O(\Omega^2)$	$O(\Omega^0)$	$O(\Omega^4)$	$O(\Omega^2)$	$O(\Omega^2)$

the slow rotation approximation the perturbations of the comoving and countermoving velocities can be written as

$$\delta \mathrm{v} = \partial_t \xi_+ \propto \Omega \xi_+ , \quad \delta \mathrm{w} = \partial_t \xi_- \propto \Omega \xi_- , \tag{21}$$

where

$$\xi_+ = r \sum_{l,m} \left(0, \frac{K_{lm}}{\sin\theta} \partial_\phi, -K_{lm}\, \partial_\theta \right) Y_{lm} + r \sum_{l,m} \left(S_{lm}, Z_{lm}\, \partial_\theta, \frac{Z_{lm}}{\sin\theta} \partial_\phi \right) Y_{lm} , \tag{22}$$

$$\xi_- = r \sum_{l,m} \left(0, \frac{k_{lm}}{\sin\theta} \partial_\phi, -k_{lm}\, \partial_\theta \right) Y_{lm} + r \sum_{l,m} \left(s_{lm}, z_{lm}\, \partial_\theta, \frac{z_{lm}}{\sin\theta} \partial_\phi \right) Y_{lm} \tag{23}$$

are respectively the comoving and countermoving displacement and Y_{lm} are the spherical harmonics. In the same approximation, one can write the fluctuations of the pressure and chemical potential as

$$\delta p = \rho g r \sum_{l,m} \zeta_{lm} Y_{lm} , \tag{24}$$

$$\delta\beta = g r \sum_{l,m} \tau_{lm} Y_{lm} , \tag{25}$$

where $g = \Omega_0^2 r$ (with $\Omega_0^2 = GM/R^3$). Therefore, according with Table I we have that $\delta \upsilon \propto \Omega$, while $\delta\omega \propto \Omega^3$ to leading order in the Ω expansion. Thus the leading term in the total energy, Eq. (10), is associated with the comoving oscillations and therefore $E \propto \Omega^2$.

We can now evaluate the power counting in Ω of the various timescales. For the mutual friction timescale, we have from [1] that $f_{MF} \propto \Omega^4$ and it follows from Eq.(16) that $\tau_{MF} \propto \Omega^{-5}$. As for the damping time of the rocket term in Eq.(17), we have that $\tau_{RT} \propto \Omega^{-6}$. This comes from the fact that $\delta\beta \propto \Omega^4$ at the leading order in Ω. Notice that in Eq. (14) one can consider only the first term on the right hand side, which is of order $O(\Omega^8)$ and one can neglect the second and the third term on the right hand side, because $\delta\beta(\delta \mathrm{w})^2 \propto \Omega^{10}$ and $(\delta \mathrm{w})^4 \propto \Omega^{12}$.

The timescale associated with the gravitational-wave emission for standard *r*-modes has been computed in [5,12-16]. For a star of constant density, the growth timescale of the $l=2$ current multipole is given by

$$\tau_{gw} \simeq 22\left(\frac{M}{1.4M_\odot}\right)^{-1}\left(\frac{R}{10\,\mathrm{km}}\right)^{-4}\left(\frac{P}{1\,\mathrm{ms}}\right)^{6} s \tag{26}$$

and therefore $\tau_{gw} \propto \Omega^{-6}$.

In r-mode analysis it is useful to introduce the frequency independent quantities τ^0_{MF}, τ^0_{RT}, τ^0_{gw} which are obtained from the corresponding timescales extracting the Ω dependence. For standard r-modes we have that

$$\frac{1}{\tau_{MF}}=\frac{1}{\tau^0_{MF}}\left(\frac{\Omega}{\sqrt{\pi G\rho}}\right)^5, \quad \frac{1}{\tau_{RT}}=\frac{1}{\tau^0_{RT}}\left(\frac{\Omega}{\sqrt{\pi G\rho}}\right)^6, \quad \frac{1}{\tau_{gw}}=\frac{1}{\tau^0_{gw}}\left(\frac{\Omega}{\sqrt{\pi G\rho}}\right)^6. \tag{27}$$

Since we have explicitly considered the mutual friction force in the Euler equations, the mutual friction timescale, τ^0_{MF} and the rocket effect timescale, τ^0_{RT}, depend on the drag parameter. In agreement with [17] we find that for $\mathcal{R}\sim 1$ the mutual friction is able to damp the r-mode oscillations, while for large or small values of $\mathcal{R}$ the damping time is too long.

Regarding the rocket effect damping timescale, we compute the numerical value of the coefficient Ξ in Eq. (8). In Fig.1 we report the plot of the rocket effect damping timescale versus entrainment. We consider the weak drag regime, $\mathcal{R} \ll 1$, and that nn pairing takes place in the $^3P_2(m_J=2)$ channel, which has the smallest reduction factor (reported in Appendix A, and corresponding to case C). However, results with different values of $\mathcal{R}$ and with pairing in the $^1S_0$ channel and in the $^3P_2(m_J=0)$ lead to very similar results. We assume throughout that the critical temperature of the superfluid phase is $T_c = 10^{10}$ K. We find that τ^0_{RT} does weakly depend on the entrainment parameter. The

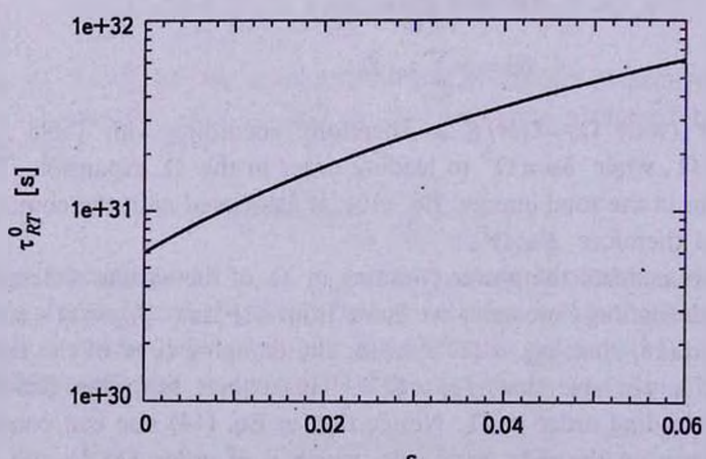

Fig.1. Damping time τ^0_{RT} evaluated for the standard r-modes in the weak drag regime. The damping time associated with the rocket effect decreases with increasing entrainment parameter ε (defined in Eq.(20)). We have taken $T=10^9$ K and a critical superfluid temperature $T_c=10^{10}$ K. The star has uniform density $\rho=2.5\rho_0$, with radius $R=10$ km and mass $M\simeq 1.47M_\odot$.

damping is stronger for small values of the entrainment, but in any case the damping time is extremely large.

More in detail, the rocket effect damps the r-mode instability for

$$\tau_{RT} \leq \tau_{gw} \tag{28}$$

and considering that the τ_{RT} has the same frequency dependence of τ_{gw}, it follows that the condition for damping is independent of the rotation frequency of the star and can be written as

$$\tau_{RT}^{0} \leq \tau_{gw}^{0}. \tag{29}$$

Thus, when this condition is realized, the star is rotationally stable for any value of Ω, as far as the Ω/Ω_K expansion makes sense, where Ω_K is its Kepler frequency.

However, τ_{gw}^{0} is of order of tens of second, and as is clear from Fig.1, the timescale of the rocket effect is way too big for damping the r-mode instability. More precisely, considering the temperature dependence of the rocket term, we find that for temperatures below 10^{15} K the damping timescale of the rocket effect is larger than the growth timescale of gravitational waves. Therefore the effect of the rocket term is completely negligible in the analysis of standard r-mode instabilities. The qualitative reason why the rocket term effect is negligible is related to the fact that it is associated with countermoving displacements, which are subleading in standard r-mode oscillations. The suppression in Ω can only be compensated by a very efficient particle conversion mechanism. In the present analysis we have assumed that particle conversion is due to the direct Urca process, which is the most efficient microscopic decay one can consider. Therefore, the rocket term can have a significant effect on standard r-mode oscillations only if a very efficient macroscopic crust-core transfusion mechanism of particle conversion is at work in the neutron star.

3.2. *Timescales for superfluid r-modes*. The superfluid r-mode oscillation for incompressible superfluids is dominated by toroidal countermoving displacements. According with Table 1, we have that $\delta\omega \propto \Omega$ and $\delta\upsilon \propto \Omega^3$, to leading order in the Ω expansion. The total energy in Eq. (10) is now dominated by the countermoving displacement and the power counting in Ω is the same one has for standard r-modes, i.e. $E \propto \Omega^2$.

The timescale associated to gravitational-wave emission for superfluid r-modes is much larger than the corresponding value for standard r-modes. The reason is that gravitational-wave emission is only associated with the comoving displacements, see [18,19], which are suppressed with respect to the countermoving modes. More in detail, one has that the mass and current multipoles in Eq. (18) are respectively given by (see e.g. [5])

$$D_{lm} = \int dVr^l \, \delta\rho Y^*_{lm} \quad \text{and} \quad J_{lm} = 2\sqrt{\frac{l}{l+1}} \int dVr^l (\rho\delta v + \delta\rho\Omega) \mathbf{Y}^{B*}_{lm} , \tag{30}$$

where $\mathbf{Y}^{B*}_{lm}$ are the magnetic vector harmonics, see [20]. Since for superfluid r-modes we have that $\delta\rho \propto \Omega^2$ and $\delta\upsilon \propto \Omega^3$, from Eq. (18) we obtain that $\tau_{gw} \propto \Omega^{-10}$. Thus, for superfluid r-modes we have that

$$\frac{1}{\tau_{gw}} = \frac{1}{\tau^0_{gw}} \left(\frac{\Omega}{\sqrt{\pi G \rho}} \right)^{10} \tag{31}$$

and performing the integral in Eq. (30) we find that $\tau^0_{gw} \simeq 0.91 s$. Notice that these results differ from the corresponding results reported in [4], where it is assumed that for superfluid r-modes $\tau_{gw} \propto \Omega^{-6}$ and it is found that $\tau^0_{gw} \sim 10^4$ s.

Regarding the mutual friction timescale, we have that $f_{MF} \propto \Omega^2$. Therefore in the case of superfluid r-modes the mutual friction force is much larger than in standard r-modes. This is clearly due to the fact that mutual friction force is proportional to the countermoving displacement, which are dominant for superfluid r-modes. From Eq.(16) we have that $\tau_{MF} \propto \Omega^{-1}$, which is four orders in Ω smaller than the corresponding expression for standard r-modes.

As for the damping time of the rocket term, we have that the three terms on the right hand side of Eq. (14) are of order $O(\Omega^4)$ and therefore from Eq. (17) we have that $\tau_{RT} \propto \Omega^{-2}$. Also in this case the timescale is much shorter than for standard r-modes, and for the same reason.

Summarizing, we have that for superfluid r-modes

$$\frac{1}{\tau_{MF}} = \frac{1}{\tau^0_{MF}} \left(\frac{\Omega}{\sqrt{\pi G \rho}} \right), \quad \frac{1}{\tau_{RT}} = \frac{1}{\tau^0_{RT}} \left(\frac{\Omega}{\sqrt{\pi G \rho}} \right)^2, \quad \frac{1}{\tau_{gw}} = \frac{1}{\tau^0_{gw}} \left(\frac{\Omega}{\sqrt{\pi G \rho}} \right)^{10} . \tag{32}$$

The timescale τ^0_{MF} strongly depends on the entrainment parameter. For very small values of the entrainment, this timescale can become very large, as an example for $\varepsilon = 0.0002$, we have that $\tau^0_{MF} \simeq 1.8 \times 10^3$ s. With increasing entrainment, the mutual friction timescale decreases, as an example for $\varepsilon = 0.02$ we find that $\tau^0_{MF} \simeq 1.6$s. The timescales of the rocket term does not strongly depend on mutual friction and we have that

$$\tau^0_{RT} \simeq 99.6 T_9^{-4} R_x^{-1} \, \text{s}, \tag{33}$$

where T_9 is the temperature in units of 10^9 K.

We shall now analyze in detail the stability of the superfluid r-modes. In addition to mutual friction and rocket term one has to consider shear and bulk viscosity as well. We have evaluated both of these quantities for superfluid r-modes and found that

$$\frac{1}{\tau_{sv}} = \frac{1}{\tau^0_{sv}}, \quad \frac{1}{\tau_{bv}} = \frac{1}{\tau^0_{bv}} \left(\frac{\Omega}{\sqrt{\pi G \rho}} \right)^2 \tag{34}$$

with

$$\tau_{sv}^0 = 7.5\times10^7 T_9^2 s \quad \text{and} \quad \tau_{bv}^0 = 2.68\times10^7 T_9^{-4} R_x^{-1} \text{s}. \tag{35}$$

Here the shear viscosity timescale is due to electron-electron scattering in a Fermi liquid and we have neglected Landau damping. The effect of Landau damping was considered in [21] and this leads to an even smaller value of the shear viscosity. In the evaluation of the rocket term and of the bulk viscosity timescales we have employed the results of [7], where the reduction factors R_x for direct Urca processes in the presence of the superfluid phase has been evaluated. The results that we shall present refer to the superfluid phase $^3P_2(m_J=2)$, see Appendix A. Notice that all the frequency independent timescales τ^0 have values very similar to the corresponding values found for standard r-modes. The reason is that for $\Omega=\Omega_K$ all the modes of oscillations are of the same order and therefore the difference between standard and superfluid r-modes is immaterial.

The study of the stability of r-mode oscillations now requires to take into account all the different timescales simultaneously. The relation for the stability is given by

$$-\frac{1}{\tau_{gw}}+\frac{1}{\tau_{sv}}+\frac{1}{\tau_{bv}}+\frac{1}{\tau_{MF}}+\frac{1}{\tau_{RT}}=0, \tag{36}$$

meaning that when this relation is satisfied one has a critical condition for stability. When the quantity on the left hand side is negative, then the mode is unstable.

In Fig.2 we report the result for the superfluid r-mode "instability window" for a star with uniform density, $\rho=2.5\rho_0$ and $R=10$ km. The dashed line represents the instability window in the absence of the rocket term and mutual friction. The region above the dashed line is unstable with respect to gravitational-wave emission when only shear and bulk viscosity damping mechanisms are considered. At low temperature, shear viscosity is the dominant dissipative mechanism. With increasing temperature shear viscosity is less efficient and it can damp r-mode oscillations for smaller and smaller values of the frequency. The behavior of bulk viscosity is the opposite and starts to damp r-mode oscillations for temperatures of the order of 10^{10} K.

As we expected, the shape of the instability window in this case is almost the same as for the standard r-mode. This is due to the fact that both the gravitational radiation and the viscosities are only related to the comoving degree of freedom and to the pressure. Therefore, the critical condition for the instability window remains the same.

The full line represents the effect of the rocket term. The instability window with the inclusion of the rocket term is much reduced. Notice that the rocket term has a behavior qualitatively similar to the bulk viscosity. The reason can be traced back to equations (32) and (34), where one can see that rocket term timescale and the bulk viscosity timescale have the same dependence on temperature and frequency, but the numerical coefficient of the rocket term timescale is much smaller than the numerical coefficient of the bulk viscosity timescale. Therefore the rocket term dominates the bulk viscosity at any frequency and temperature and it becomes effective at smaller temperatures.

In panel *a* of Fig.2 we have assumed zero entrainment and hence, vanishing mutual friction. In the panel *b* of Fig.2 we include the effect of mutual friction. In the simplified model of neutron star that we are considering, mutual friction is independent of the temperature. We consider only the case of small drag parameter, where $\mathcal{B}$ and $\mathcal{B}'$ are given in [22] and are functions of the entrainment. The lower horizontal line corresponds to $\varepsilon = 0.0002$, while the

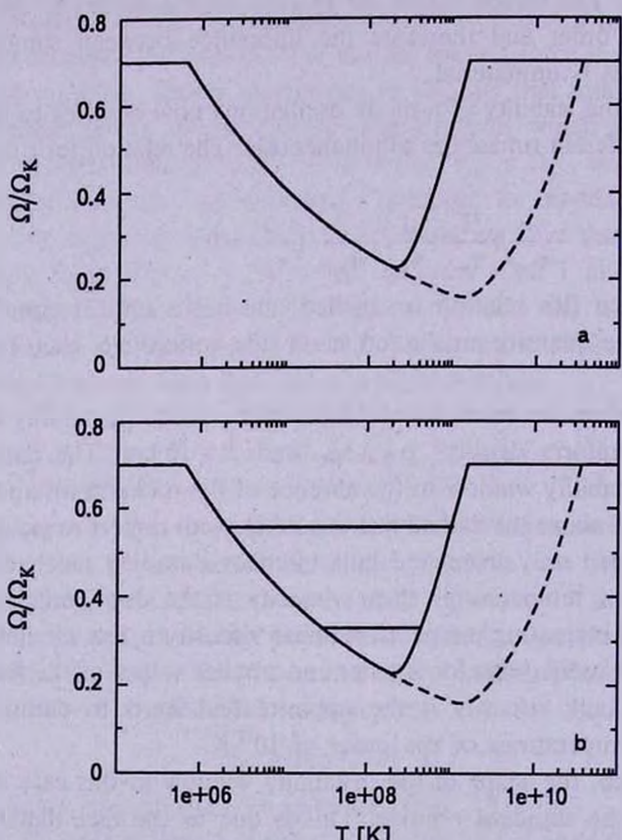

Fig.2. Instability window of the superfluid *r*-modes of a star with uniform density $\rho = 2.5\rho_0$, with radius $R = 10$ km and mass $M = 1.47 M_\odot$. In both panels, the dashed line represents the instability window in the absence of the rocket term. The full line represents the instability window with the inclusion of the rocket term. The panel *a* corresponds to the case of vanishing entrainment and therefore vanishing mutual friction. In the panel *b* we consider two values of entrainment. The horizontal full lines correspond to the effect of the mutual friction for $\varepsilon = 0.0002$, lower line and $\varepsilon = 0.002$ upper line. In our simplified model of star, the mutual friction is independent of the temperature.

upper line corresponds to $\varepsilon = 0.002$. In agreement with [4], we find that for values of $\varepsilon \sim 0.02$ the effect of the mutual friction is to damp superfluid *r*-modes for any value of the frequency.

4. *Conclusion*. We have evaluated the damping timescale associated to the rocket effect for a toy model neutron star comprising neutrons, protons and electrons, with uniform mass density and no crust. Since the charged components are locked together by the electromagnetic interaction, we have simplified the hydrodynamical equations considering a two-fluid system in which the mass densities of the two components are not separately conserved.

In this model one has to consider two different *r*-mode oscillations. The standard *r*-mode oscillations, which are predominantly toroidal comoving displacements of the two fluids, and the superfluid *r*-mode oscillations which are associated with toroidal countermoving displacements. In realistic neutron stars these two modes are coupled, however, in the limit of small rotation frequency and assuming that the star has a uniform density and is incompressible, they decouple.

We have evaluated the effect of the rocket term on the stability of these two *r*-mode oscillations by linearizing the corresponding Euler equations in the limit of slow rotation, i.e. at the leading order in Ω/Ω_K. In our analysis we have included the effect of mutual friction and of shear and bulk viscosities and compared the corresponding timescales with the timescale of gravitational-wave emission. We find that the dissipative mechanism associated with the rocket term is not efficient in damping standard *r*-mode oscillations. The reason is that standard *r*-modes are predominantly comoving modes, while the rocket term is associated with the countermoving flow, which for standard *r*-mode oscillations is subleading. On the other hand, this mechanism becomes efficient in damping the superfluid *r*-mode instability. The reason is that in this case the countermoving mode is the leading oscillation of the system and we find that for temperatures of the order of 10^9 K the rocket term has a typical timescale comparable with the timescale of gravitational-wave emission. Quite interestingly, for superfluid *r*-modes the rocket term becomes an efficient damping mechanism before bulk viscosity sets in.

In agreement with [4] we find that the superfluid *r*-mode oscillations can be damped as well by the mutual friction force for sufficiently large values of the entrainment parameter. However, unlike the mutual friction force, the rocket term is not strongly dependent on the entrainment parameter and it reduces the instability window even for vanishing entrainment. Since the entrainment parameter is a poorly known coefficient, the rocket term represents an interestingly mechanism for superfluid *r*-mode damping.

It would be interesting to extend the present analysis to more realistic equation of states, in order to see whether the effect persists. It would be also quite interesting to study the effect of transfusion of neutrons and protons from the crust to the outer core. For standard r-modes we find that the damping timescale of the rocket term is quite large, however this timescale would be reduced if the transfusion of nucleons from the crust to the core results in a mass rate larger than the one due to weak processes. In this case the rocket term might be able to reduce the instability window for standard r-modes as well.

We have limited our analysis to the region of small chemical potential fluctuations, $\delta\beta < T$. As in [23], it would be interesting to extend the present analysis to the supra-thermal region, $\delta\beta > T$, in order to study the spin-down time of unstable stars. The rocket term might also be relevant for hybrid stars with a quark matter core. In that case one should consider weak processes between quarks of different flavors and transfusion processes between nuclear matter and quark matter.

Appendix A: Reaction rates for the Urca processes in different superfluid phases

The reaction rates for the direct Urca processes that may occur in superfluid neutron stars depend on the particular pp and nn superfluid condensates that can be realized. Various superfluid phases have been considered in [7], and the corresponding reaction rates have been evaluated; here we review some of their results.

If there is no superfluidity, the reaction rate for the direct Urca process can be expressed as

$$\overline{\Gamma}_{Urca} = (\Delta I) 1.667\times 10^{32}(1-\varepsilon_c)\left(1-\varepsilon_c \frac{x_c}{1-x_c}\right)\left(\frac{\rho_c}{\rho_0}\right)^{1/3} T_9^5\, \Theta_{npe}\ \mathrm{cm}^{-3}\,\mathrm{s}^{-1}, \qquad (A1)$$

where T_9 is the temperature in units of 10^9 K. The step function Θ_{npe} is 1 if the direct Urca process is allowed (see [6]), it is 0 otherwise. The factor (ΔI) is a statistical factor that depends on the phase of nuclear matter under consideration. When all particles are in the normal phase, one finds that

$$\Delta I = \Delta I_0 = \frac{17\pi^4}{60}. \qquad (A2)$$

When neutrons or protons are superfluid this quantity is multiplied by a reduction factor, R_X and one has that

$$\Delta I = \Delta I_0 R_X . \qquad (A3)$$

Here we consider only nn pairing and the index $X=A$, B, C, depending on whether neutrons pair in the ${}^1S_0$ channel (which corresponds to $X=A$), in the ${}^3P_2(m_J=0)$ channel (which corresponds to $X=B$), or in the ${}^3P_2(m_J=2)$ channel (which corresponds to $X=C$). In [7] it is found that for temperatures smaller than the critical temperature of the corresponding superfluid phase, the reduction factors are well approximated by

$$R_A = \left(0.2787 + \sqrt{(0.7213)^2 + (0.1564\,\upsilon_A)^2}\right)^{3.5} \exp\left(2.9965 - \sqrt{(2.9965)^2 + \upsilon_A^2}\right), \quad (A4)$$

$$R_B = \left(0.2854 + \sqrt{(0.7146)^2 + (0.1418\,\upsilon_B)^2}\right)^{3} \exp\left(2.0350 - \sqrt{(2.0350)^2 + \upsilon_B^2}\right), \quad (A5)$$

$$R_C = \frac{0.5 + (0.1086\,\upsilon_C)^2}{1 + (0.2347\,\upsilon_C)^2 + (0.2023\,\upsilon_C)^4} + 0.5\exp\left(1 - \sqrt{1 + (0.5\,\upsilon_C)^2}\right), \quad (A6)$$

where $\upsilon_X = \Delta_X(T)/k_B T$ and $\Delta_X(T)$ is the temperature dependent superfluid gap of the phase under consideration. For temperatures larger than the critical temperature the reduction factors are equal to one. It is worth noticing that while in the phase A and B the reaction rates are suppressed exponentially at low T, the reduction factor for the phase C varies as T^2. The case when both neutrons and protons are in a superfluid phase is also considered in [7], where the reduction factors are also numerically computed.

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[1] Università di Bari and I.N.F.N. Sezione di Bari, Italy, e-mail: colucci@th.physik.uni-frankfurt.de

[2] Departament d'Estructura i Constituents de la Matèria and Institut de Ciències del Cosmos (ICCUB), Universitat de Barcelona, Spain I.N.F.N., Laboratori Nazionali del Gran Sasso, Assergi (AQ), Italy, e-mail: massimo@lngs.infn.it

[3] Instituto de Ciencias del Espacio (IEEC/CSIC), Campus Universitat Autònoma de Barcelona, Facultat de Cències, Spain, e-mail: cmanuel@ieec.uab.es

r-МОДА ОСЦИЛЛЯЦИЙ И РЕАКТИВНЫЙ ЭФФЕКТ ВО ВРАЩАЮЩИХСЯ НЕЙТРОННЫХ ЗВЕЗДАХ. II. ЧИСЛЕННЫЕ РЕЗУЛЬТАТЫ

Д.КОЛУЧЧИ[1], М.МАННАРЕЛЛИ[2], К.МАНУЕЛ[3]

Получены оценки для характерного времени механизма затухания из-за реактивного эффекта. Рассмотрена двухжидкостная модель, в которой обе жидкости могут осциллировать в двух типах r-моды осцилляций. Одна из них в основном связана с совместным смешением двух жидкостей, а вторая - с противоположным безфазовым смешением. В первом случае найдено, что диссипативный механизм, связянный с реактивным эффектом, не в состоянии предотвратить рост нестабильности r-моды. Во втором случае реактивный эффект предотвращает рост нестабильности r-моды при температурах выше 10^9 К.

Ключевые слова: *нейтронные звезды:осцилляция:реактивный еффект*

REFERENCES

1. *G.Colucci*, *M.Mannarelli*, *C.Manuel*, Astrophysics, 56, 101, 2013.
2. *M.E.Gusakov*, Phys. Rev. D, 76(8), 083001, 2007.
3. *I.M.Khalatnikov*, Introduction to the theory of superfluidity, 1965.
4. *U.Lee*, *S.Yoshida*, Astrophys. J., 586, 403, 2003.
5. *N.Andersson*, *K.D.Kokkotas*, International Journal of Modern Physics D, 10, 381, 2001.
6. *J.M.Lattimer*, *M.Prakash*, *C.J.Pethick*, *P.Haensel*, Phys. Rev. Lett., 66, 2701, 1991.
7. *K.P.Levenfish*, *D.G.Yakovlev*, Astron. Lett., 20, 43, 1994.
8. *P.Haensel*, *K.P.Levenfish*, *D.G.Yakovlev*, Astron. Astrophys., 357, 1157, 2000.
9. *R.Prix*, Phys. Rev. D, 69(4),043001, 2004.
10. *P.Haensel*, *K.P.Levenfish*, *D.G.Yakovlev*, Astron. Astrophys., 372, 130, 2001.
11. *L.Lindblom*, *G.Mendell*, Phys. Rev. D, 61(10), 104003, 2000.
12. *L.Lindblom*, ArXiv Astrophysics e-prints, January 2001.
13. *N.Andersson*, *K.Kokkotas*, *B.F.Schutz*, Astrophys. J., 510, 846, 1999.
14. *K.D.Kokkotas*, *N.Stergioulas*, Astron. Astrophys., 341, 110, 1999.
15. *L.Lindblom*, *G.Mendell*, *B.J.Owen*, Phys. Rev. D, 60(6), 064006, 1999.
16. *L.Lindblom*, *B.J.Owen*, *S.M.Morsink*, Phys. Rev. Lett., 80, 4843, 1998.
17. *B.Haskell*, *N.Andersson*, *A.Passamonti*, Mon. Notic. Roy. Asrtron. Soc., 397, 1464, 2009.
18. *A.Sedrakian*, *I.Wasserman*, Phys. Rev. D, 63(2), 024016, 2001.
19. *N.Andersson*, *K.Glampedakis*, *B.Haskell*, Phys. Rev. D, 79(10), 103009, 2009.
20. *K.S.Thorne*, Reviews of Modern Physics, 52, 299, 1980.
21. *H.Heiselberg*, *C.J.Pethick*, Phys. Rev. D, 48, 2916, 1993.
22. *M.A.Alpar*, *S.A.Langer*, *J.A.Sauls*, Astrophys. J., 282, 533, 1984.
23. *M.G.Alford*, *S.Mahmoodifar*, *K.Schwenzer*, Journal of Physics G Nuclear Physics, 37(12), 125202, 2010.