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r-MODE OSCILLATIONS AND ROCKET EFFECT IN ROTATING SUPERFLUID NEUTRON STARS. I. FORMALISM

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We derive the hydrodynamical equations of *r*-mode oscillations in neutron stars in presence of a novel damping mechanism related to particle number changing processes. The change in the number densities of the various species leads to new dissipative terms in the equations which are responsible of the *rocket effect*. We employ a two-fluid model, with one fluid consisting of the charged components, while the second fluid consists of superfluid neutrons. We consider two different kind of *r*-mode oscillations, one associated with comoving displacements, and the second one associated with countermoving, out of phase, displacements.

Key words: neutron stars:oscillations:rocket effect

1. Introduction. Rapidly rotating neutron stars have been the subject of intensive investigation in the last years. Of particular interest are neutron star oscillations, which might be useful to shed light on the internal structure of these stars [1,2]. Stars have various modes of oscillations, among them r-mode oscillations are probably the most interesting ones, because they provide a severe limitation on the star's rotation frequency through coupling to gravitational radiation emission [3,4]. The oscillations of compact stars can be damped by various dissipative process [5,6], which take place in the interior of the star. However, if dissipative phenomena are not strong enough, the r-mode oscillations will grow exponentially fast in time until the star slows down, by emission of gravitational waves, to a rotation frequency where some dissipative mechanism efficiently damps these oscillations. Since neutron stars are observed to rotate at very high frequencies, any model of neutron star must provide an efficient mechanism of dissipation of r-mode oscillations. In this way, the study of r-mode damping is useful in constraining the stellar structure and can be used to rule out some exotic phases of matter [7.9].

Standard neutron stars are stellar objects with a mass of about $1.4 M_{\odot}$ and a radius of about 10 km. They are believed to have a crust of about 1 km, with an outer part made of a lattice of ions embedded in a liquid of electrons and an inner part made of nuclei embedded in a liquid of ${}^{1}S_{0}$ superfluid neutrons. In the interior of the star nuclei are melted and both neutrons and protons

are expected to condense into BCS-like superfluids. However, neutron interaction in the ${}^{1}S_{0}$ state at supernuclear matter density is repulsive, but it is still possible to form Cooper pairs in the ${}^{3}P_{2}$ channel [10]. The proton density is much smaller than the neutron density and therefore the formation of pp Cooper pairs in the isotropic ${}^{1}S_{0}$ channel is allowed. Pairing between protons and neutrons does not take place because of the large mismatch between their Fermi energies. In the core of neutron stars muons might be present (when $\mu_{e} > m_{\mu}$), or deconfined quarks in a color superconducting phase, moreover pion or kaon condensates might be realized. In the present paper we shall not consider any of these possibilities and assume the core of the neutron star comprises only neutrons, protons and electrons.

r-mode oscillations have been studied extensively in the literature [1], and it is known that shear and bulk viscosities are able to suppress the instability in two different ranges of temperatures [5,6]. For temperatures smaller than about 105 K the fluid damping due to shear viscosity suffices to suppress the r-mode instability, but with increasing temperature the effect of shear viscosity is gradually suppressed. On the other hand, for temperatures larger than about 10¹⁰ - 10¹¹ K, bulk viscosity becomes an efficient mechanism for damping r-mode oscillations. Bulk viscosity does not lead to sufficient damping at lower temperatures because neutron matter is likely to be in the superfluid phase where bulk viscosity is suppressed by Pauli blocking. However, at temperatures above 1010-1011 K nuclear matter is believed to be in the normal phase with a large bulk viscosity coefficient. Therefore, one has an "instability window" for standard neutron stars corresponding to a range of temperature approximately given by 10⁶ - 10¹⁰ K. The exact values depend on the details of the model considered and on the rotation frequency of the star. The instability window is in part reduced by the "surface rubbing" between the core and the crust of the star [11,12]. This mechanism results in a viscous boundary layer between the core and the crust of the star which damps r-mode oscillations for temperatures less than about 10¹⁰ K and for sufficiently small frequencies.

In [13-16] it is studied the effect of mutual friction in reducing the instability window. It is shown that the typical timescale of mutual friction is of the order of 10^4 s and is therefore too long for damping the *r*-mode instability. Indeed, the timescale associated with gravitational-wave emission is of the order of few seconds (for a millisecond pulsar). However, mutual friction can reduce the instability for certain values of the entrainment parameter [13] or for large values of the drag parameter [16].

In the present paper we determine the hydrodynamical equations for rmode oscillations in presence of a novel dissipative mechanism associated with the change in the number of protons, neutrons and electrons which we shall refer to as the rocket effect. In real neutron stars this mechanism can take place in the outer core and in the inner crust of the star and is related to beta decays and interactions between the neutron fluid and the crust. The rocket effect is dissipative because when two or more fluids move with different velocities a change of one component into the other results in a momentum transfer between the fluids. This change in momentum is not reversible, because it is always the faster moving fluid that will lose momentum. The resulting dissipative force is proportional to the mass rate change and to the relative velocity between the fluids. The name "rocket effect" reminds that the same phenomenon takes place in the dynamical evolution of a rocket whose mass is changing in time as it consumes its fuel.

In order to simplify the analysis we consider a simplified model of neutron star consisting of a fluid of neutrons, protons and electrons and no crust. Protons and electrons are locked together by the electromagnetic interaction and therefore we consider that the system consists of two fluids. As a further simplification we assume that the star has a uniform mass distribution with density $\rho = 2.5\rho_0$ and a radius of 10 km. Since this simplified model of star does not comprise a crust we consider only number changing processes associated with weak interactions.

In our analysis we consider two different *r*-mode oscillations. One is associated predominantly with toroidal comoving displacements and the second one is dominated by toroidal countermoving displacements. We shall refer to these oscillations as "standard *r*-mode" and as "superfluid *r*-mode", respectively. These two modes decouple for a star made by uniform and incompressible matter, and we shall restrict to treat such a case. By performing an expansion in the parameter Ω/Ω_K , where Ω is the frequency of the star and Ω_K is its Kepler frequency, we find that the linearized equations for both the standard and superfluid *r*-modes present additional dissipative terms due to the rocket effect, but for each oscillation they appear in a different order in our expansion parameter. The numerical evaluation of the timescale related to this mechanism is performed in the accompanying paper [17].

This paper is organized as follows. In Section II we review the hydrodynamic equations in the two fluid approximation. In the hydrodynamical equations we neglect shear and bulk viscosities but consider the mutual friction force and the rocket effect. In Section III we present the differential equations governing the deviations from equilibrium of a rotating two fluid system. We study the linearized problem, neglecting deformations of the star due to rotation and assuming uniform mass density. Although not very realistic, these assumptions allow us to simplify the study of the *r*-modes, obtaining an analytical expression for the fluid displacements. We draw our preliminary conclusions in Section IV. In Appendix A we report some details about the evolution equations for the comoving and countermoving displacements.

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2. Hydrodynamical equations for the multifluid neutron star model. The equations describing the dissipative processes of neutrons, protons and electrons in the outer core of a standard neutron star have been studied in detail in [15]. In general, the entropy production rate depends on 19 independent coefficients which are related to the various dissipative processes. However, for low temperatures, well below the critical temperature for superfluidity, and neglecting viscosities one obtains the expression given in [18] which depends only on two different coefficients. One is related to mutual friction and the second one with the so-called rocket term. Here we review the basic hydrodynamical equations in presence of these two different dissipative mechanisms, neglecting the presence of shear and bulk viscosities.

For a system consisting of neutrons, protons and electrons, the mass conservation law is given by, see e.g. [15,18,19],

$$\partial_t \rho_x + \nabla_t (\rho_x v_x') = \Gamma_x , \qquad (1)$$

where Γ_x is the particle mass creation rate per unit volume and the index x=n, p, e refer to the particle species, that is, neutrons, protons and electrons. In these equations we have considered that some process can convert neutrons in protons and electrons and vice versa. Therefore, we are assuming that the various components are not separately conserved. One possible mechanism leading to a change in the particle number densities is given by the weak processes

$$n \to p + e^- + \overline{\nu}_e, \quad p + e^- \to n + \nu_e.$$
 (2)

These reactions lead to a change in the chemical potentials of the various species and therefore are associated with number density changes.

A different process can lead to a non-vanishing mass creation rate, which we shall call *crust-core transfusion*. In this process when a compression takes place, the ionic constituent of the crust are squeezed and part of their nucleonic content is released and augments the fluid components. The opposite mechanism, related to a reduction of the pressure leads to the nucleonic capture by the ions of the crust. In the present section we consider that a generic mechanism is at work to produce a change in the number densities. In the accompanying paper [17] we shall evaluate the particle mass creation rate corresponding to the beta decay processes. Regarding the crust-core transfusion processes the corresponding creation rates are difficult to evaluate and we postpone their calculations to future work.

In any case, the three particle creation rates are not independent quantities, because charge conservation implies that $\Gamma_e/m_e = \Gamma_p/m_p$, whereas baryon number conservation leads to $\Gamma_p/m_p = -\Gamma_n/m_n$, meaning that only one creation rate is independent.

It is possible to simplify the treatment of the system considering that our

analysis regards processes that happen at a time scale much larger than the electromagnetic time scale. Therefore, we can consider that electrons and protons are locked together to move with the same velocity [20], (see however [21]). Moreover, charge neutrality implies that the number densities of electrons and protons are equal, i.e. $n_e = n_p$, meaning that electrons and protons can be treated as a single charge-neutral fluid and henceforth we shall refer to this fluid as the "charged" component, employing the subscript c to label it. As a matter of fact, the system can be viewed as consisting of two fluids, with mass densities $\rho_n = m_n n_n$ and $\rho_c = m_n n_c$, where $m_n = m_p + m_e$ and $n_e = n_p = n_c$. For non vanishing mass creation rates, the Euler equations have an extra term, see [18] and are given by

$$\left(\partial_{i} + \upsilon_{n}^{j} \nabla_{j}\right) \left(\upsilon_{i}^{n} + \varepsilon_{n} w_{i}\right) + \nabla_{i} \left(\widetilde{\mu}_{n} + \Phi\right) + \varepsilon_{n} w^{j} \nabla_{i} \upsilon_{j}^{n} = \frac{f_{i}^{MF}}{\rho_{n}}, \qquad (3)$$

$$\left(\partial_{i} + \upsilon_{c}^{j} \nabla_{j}\right) \left(\upsilon_{i}^{c} - \varepsilon_{c} w_{i}\right) + \nabla_{i} \left(\widetilde{\mu}_{c} + \Phi\right) - \varepsilon_{c} w^{j} \nabla_{i} \upsilon_{j}^{c} = -\frac{f_{i}^{MP}}{\rho_{c}} + \left(1 - \varepsilon_{n} - \varepsilon_{c}\right) \frac{\Gamma_{n}}{\rho_{c}} w_{i}, \quad (4)$$

where *i*, *j* label the space components, we have defined a chemical potential by mass $\tilde{\mu}_x = \mu_x/m_n$ and $\mathbf{w} = \mathbf{v}_c - \mathbf{v}_n$ represents the relative velocity between the two fluids. The quantities ε_n and ε_c are the entrainment parameters, that are related to the fact that momenta and velocities of quasiparticles may not be aligned [22], and the the gravitational potential, Φ , obeys the Poisson equation

$$\nabla^2 \Phi = 4\pi G(\rho_\mu + \rho_c). \tag{5}$$

The force term f_i^{MF} entering into both Euler equations corresponds to the mutual friction force between the superfluid and normal component. This force appears when a suerfluid is put in rotation [23-25] and at the microscopic level it is due to the scattering of the normal component off the superfluid vortices [20]. In the present case it is due to the scattering of electrons off the neutron superfluid vortices. Indeed, as a consequence of the entrainment between neutrons and protons the superfluid vortices are accompanied by a magnetic field. The expression of the mutual friction force valid for small values of w has been determined in [23] and is given by

$$f_i^{MF} = 2\rho_n B' \epsilon_{ijk} \Omega_j w^k + 2\rho_n B \epsilon_{ijk} \Omega^j \epsilon^{klm} \Omega_l w_m, \qquad (6)$$

where the coefficients B, B' can be written as

$$B = \frac{\mathcal{R}}{1 + \mathcal{R}^2}$$
 and $B' = \frac{\mathcal{R}^2}{1 + \mathcal{R}^2}$, (7)

where \mathcal{R} is the dimensionless "drag" parameter [15]. The actual strength of the drag is not precisely known, see e.g. [19,20,26] and one can consider three different regimes: the weak drag regime, $\mathcal{R} \ll 1$, the strong drag regime $\mathcal{R} \gg 1$

and the intermediate drag regime, $\mathcal{R} \sim 1$. For small values of the drag parameter one can express the coefficients B, B' as a function of the entrainment parameter. Considering scattering of electrons off vortices, according with [20], one has that

$$B = 4 \times 10^{-4} \frac{\varepsilon_c^2}{\sqrt{1 - \varepsilon_c}} \left(\frac{x_c}{0.05}\right)^{1/6} \left(\frac{\rho}{10^{14} \text{ g/cm}^3}\right)^{1/6} \text{ and } B' = B^2.$$
(8)

The last term on the right hand side of Eq. (4) is the so-called rocket term. This force is due to the fact that when two fluids move with different velocities a change of one component into the other results in a variation of the momentum of each fluid component. This change in momentum can be viewed as a force proportional to the mass rate change, Γ_n and to the relative velocity between the two fluids, w. Actually, in Eqs. (3,4), one can see that the rocket term acts only on the charged component. The reason is that in the presence of the rocket term, the mutual friction is not uniquely determined because part of the rocket term contribution can be included in the definition of the mutual friction force. In the present analysis we have employed the same definition given in [18]. One can show, see [18] for more details, that the mutual friction force is given by the expression in Eq. (6).

In summary, in the presence of the rocket effect one has to consider a nonvanishing mass creation rate in Eq. (1) and the rocket term force in Eq. (4). As we shall show in accompanying paper [17], the rocket effect leads to energy dissipation, and we shall estimate the corresponding damping timescale for *r*-mode oscillations. In previous analysis of the possible dissipative mechanisms of star oscillations the rocket term has been neglected. Indeed, it was assumed that the neutron, proton and electron numbers are separately conserved quantities, that is $\Gamma_n = \Gamma_e = \Gamma_n = 0$.

3. Perturbed hydrodynamical equations. A non-vanishing mass creation rate influences the evolution of the various hydrodynamical quantities. Indeed, the continuity equation (1) as well as the Euler equations (4) depend on Γ_n . Therefore, in the analysis of the various modes of oscillations of a neutron star one has to take into account effects related to this term. In the present paper we only discuss its effect on the evolution of the *r*-modes of a superfluid neutron star, although it would be equally interesting to study its effect on other pulsation modes. In the following analysis of the hydrodynamical equations we also include the mutual friction force and we follow very closely the recent analysis of the *r*-mode oscillations developed in [27] for normal fluid stars and extended to superfluid stars in [16,28].

As in [16], we study the linearized hydrodynamical equations for the perturbations around an equilibrium configuration of a neutron star rotating with constant angular velocity Ω and we assume that the background configuration

is such that the two fluids move with the same velocity, thus at equilibrium w = 0.

It is useful to write the Euler equations for the perturbed quantities using as degrees of freedom (dof) the center of mass displacement and the relative displacements between the neutron fluid and the charged fluid. We define the comoving velocity as

$$\delta \mathbf{v} = \frac{\rho_n}{\rho} \delta \mathbf{v}_n + \frac{\rho_c}{\rho} \delta \mathbf{v}_c , \qquad (9)$$

and the countermoving, or relative, velocity as

$$\delta \mathbf{w} = \delta \mathbf{v}_c - \delta \mathbf{v}_n \,. \tag{10}$$

The continuity equation for the comoving degree of freedom is not affected by the rocket effect and is given by

$$\partial_{i}\delta\rho + \nabla_{j}(\rho\delta\upsilon^{j}) = 0, \qquad (11)$$

on the other hand the continuity equation for the countermoving dof depends on it. We shall assume that the mass creation rate is given by

$$\Gamma_n = \overline{\Gamma}_n + \delta \Gamma_n \,, \tag{12}$$

where $\overline{\Gamma}_n$ is a steady mass creation rate and $\delta\Gamma_n$ is a small fluctuation on the top of it. Then, employing as a second continuity equation the one for the charged fraction, $x_c = \rho_c / \rho$, we have that

$$\partial_t \delta x_c = -\frac{1}{\rho} \nabla \cdot [x_c (1 - x_c) \rho \delta \mathbf{w}] - \delta \mathbf{v} \cdot \nabla x_c - \frac{\delta \Gamma_n}{\rho}.$$
 (13)

The linearized Euler equations for both the comoving and countermoving velocities are given

$$\partial_{i}\delta v_{i} + 2\epsilon_{ijk}\Omega^{j}\delta v^{k} + \frac{1}{\rho}\nabla_{i}\delta p - \frac{\delta\rho}{\rho^{2}}\nabla_{i}p + \nabla_{i}\delta\Phi = (1-\overline{\epsilon})\frac{\Gamma_{n}}{\rho}\delta w_{i}, \qquad (14)$$

$$\partial_{i} (1 - \overline{\varepsilon}) \delta w_{i} + \nabla_{i} (\delta \beta) + 2 \overline{B}' \epsilon_{ijk} \Omega_{j} \delta w^{k} - 2 \overline{B} \epsilon_{ijk} \overline{\Omega}^{j} \epsilon^{klm} \Omega_{i} \delta w_{m} = (1 - \overline{\varepsilon}) \frac{\overline{\Gamma}_{n}}{\rho_{c}} \delta w_{i} , \quad (15)$$

where here we have defined $\overline{\varepsilon} = \varepsilon_c + \varepsilon_n = \varepsilon_n (1 + \rho_n / \rho_c) = \varepsilon_n / x_c$ and where $\delta \beta = \delta \widetilde{\mu}_c - \delta \widetilde{\mu}_n$ (16)

and $\overline{B} = B/x_c$, $\overline{B'} = 1 - B'/x_c$. The hydrodynamical equations can be studied employing a perturbative expansion of the various hydrodynamical variables in Ω , the star rotation frequency. Actually, the expansion is in the parameter Ω/Ω_K , where Ω_K is the Kepler frequency of the star. For superfluid systems, this expansion is particularly convenient, as one can show that the complicated system of equations for the comoving and countermoving degrees of freedom decouple as these variables depend on different powers of Ω .

In the study of the evolution equations we shall restrict to the case where $\overline{\Gamma}_n = 0$ and therefore the rocket terms in Eqs. (14) and (15) will be neglected.

The only contribution to dissipation will arise from the mass creation rate in Eq. (13), and we shall evaluate the corresponding damping timescale employing the energy integral approximation, see e.g. [16].

For our study we consider some simplifying, admittedly unrealistic, assumptions. We neglect the deformation of the star due to rotation, which affects the hydrodynamical variables at order Ω^2 . We use the Cowling approximation, that is, we neglect perturbations of the gravitational potential associated with the oscillations of the star. As a further simplification, we also consider a model where the mass density of the star is uniform. As emphasized in the *Introduction*, our goal is to study the impact of the rocket effect in the evolution of the star.

Oscillations of a fluid element of a stars can be described by the Lagrangian displacement vector ξ , which can be decomposed into a sum of toroidal and spheroidal components. Since neutron stars can be described employing the two fluid model, one defines comoving and countermoving displacements, respectively ξ_+ and ξ_- , by means of the equations

$$\delta \mathbf{v} = \partial_1 \xi_+ \propto \Omega \xi_+, \quad \delta \mathbf{w} = \partial_1 \xi_- \propto \Omega \xi_-. \tag{17}$$

These two displacements describe the center of mass oscillation and the out of phase oscillation of the two fluids, respectively. We then expand these quantities in terms of toroidal and spheroidal components

$$\xi_{+} = r \sum_{l,m} \left(0, \frac{K_{lm}}{\sin \theta} \partial_{\phi}, -K_{lm} \partial_{\theta} \right) Y_{lm} + r \sum_{l,m} \left(S_{lm}, Z_{lm} \partial_{\theta}, \frac{Z_{lm}}{\sin \theta} \partial_{\phi} \right) Y_{lm} , \qquad (18)$$

$$\xi_{-} = r \sum_{l,m} \left(0, \frac{k_{lm}}{\sin\theta} \partial_{\phi}, -k_{lm} \partial_{\theta} \right) Y_{lm} + r \sum_{l,m} \left(s_{lm}, z_{lm} \partial_{\theta}, \frac{z_{lm}}{\sin\theta} \partial_{\phi} \right) Y_{lm} , \qquad (19)$$

where Y_{lm} are the spherical harmonics. The fluctuations of the pressure and of the chemical potential difference can be written respectively as

$$\delta p = \rho gr \sum_{l,m} \varsigma_{lm} Y_{lm} , \qquad (20)$$

$$\delta\beta = gr \sum_{I,m} \tau_{Im} Y_{Im} , \qquad (21)$$

where $g = \Omega_0^2 r$ (with $\Omega_0^2 = GM/R^3$) fixes the scale of pressure and chemical potential fluctuations. Notice that with these definitions, τ_{lm} and ς_{lm} are dimensionless.

Since in a superfluid star one has two different kind of displacements, in principle one can have two different kind of *r*-mode oscillations, one associated with the comoving dof and one associated with the countermoving dof. Actually, the hydrodynamical variables defined above obey a complicated set of coupled differential equations, see [16], with couplings between comoving and countermoving displacements. However, as shown in [16], at the leading order in Ω , one finds that the equation for the comoving displacement decouples and

one can determine an analytic expression for the standard r-mode oscillation. Regarding the mode associated with the countermoving dof, it turns out to be a general inertial mode. That is, it is not a mode dominated by the toroidal components. However, for incompressible stars with uniform density one has that this inertial mode turns into an r-mode. We shall restrict to this case and analyze this r-mode oscillation in Sec. 3.2. We explicitly consider the effect of the mutual friction in the equations of motion, the reason is that in this way we can analyze the regime where the mutual friction coefficients, Band B', are large. Therefore our results will explicitly depend on the values of these parameters.

3.1. Standard r-mode oscillations. For the standard r-mode oscillations one assumes that the comoving toroidal displacement, K_{low} , is of order unity, while the spheroidal comoving displacements are of order Ω^2 . All the countermoving displacements turn out to be of order Ω^2 as well. Since the standard r-mode oscillation is dominated by K_{low} , it is very similar to the r-mode oscillation in normal fluids [1], and can be easily determined after imposing proper boundary conditions [16]. To first order in the rotation frequency of the star, one has that the typical frequency of the oscillations (measured in the corotating frame) is

$$\rho_r = \frac{2\,m\,\Omega}{l(l+1)}.\tag{22}$$

In our analysis we restrict to analyze the case l = m = 2, which corresponds to the most unstable *r*-mode.

Regarding the pressure perturbations, they are of order Ω^2 , whereas $\delta\beta \propto \Omega^4$ [13,16]. The order in Ω of the toroidal oscillations and of the pressure and chemical potential fluctuations are reported in the first line of Table 1.

For the purpose of estimating the damping time scales associated to both mutual friction and the rocket effect, carried out in the accompanying paper [17], we have to determine the solutions for the countermoving dof. The equations governing the evolution of the various dynamical variables are

Table 1

ORDER IN Ω OF THE COMOVING AND CONTERMOVING DISPLACEMENTS, OF THE PRESSURE FLUCTUATION AND OF THE CHEMICAL POTENTIAL FLUCTUATION FOR THE STANDARD *r*-MODE OSCILLATION AND FOR THE SUPERFLUID *r*-MODE OSCILLATION

type of r-mode	K _{im}	k _{im}	Sim	₹/m	$S_{lm}, s_{lm}, Z_{lm}, z_{lm}$
standard <i>r</i> -mode	$O(\Omega^0)$	$O(\Omega^2)$	$O(\Omega^2)$	$O(\Omega^4)$	$O(\Omega^2)$
superfluid <i>r</i> -mode	$O(\Omega^2)$	$O(\Omega^0)$	$O(\Omega^4)$	$O(\Omega^2)$	$O(\Omega^2)$

reported in the Appendix A. Assuming constant mass density and hydrostatic equilibrium we find that τ_{l+1} obeys the following differential equation

$$r^{2}\tau_{l+1}^{*} = (A_{l} + B_{l} - 1)r\tau_{l+1}^{*} + (A_{2}B_{2} - A_{1}B_{1})\tau_{l+1} - A_{2}B_{4}\frac{r^{*}}{R^{2} - r^{2}},$$
 (23)

where the prime indicates differentiation with respect to r and the coefficients A_p , B_j are reported in Appendix A. As shown there, the last term on the right hand side of this equation arises because we have assumed that matter is in hydrostatic equilibrium. The differential equation has solution given by

$$c_{l+1}(r) = f(r) + C_1 r^{n_1} + C_2 r^{n_2} , \qquad (24)$$

where f(r) is the particular solution of the differential equation and where C_1 and C_2 are the coefficients of the homogeneous solution, to be fixed by the boundary conditions. The exponents of the homogeneous solution are given by

$$n_{1,2} = \frac{A_1 + B_1 \pm \sqrt{(A_1 + B_1)^2 + 4(A_2 B_2 - A_1 B_1)}}{2},$$
 (25)

and it turns out that n_2 is negative, meaning that in order to avoid divergences at r=0, it must be $C_2=0$. It is interesting to note that for vanishing mutual friction one has that $n_1=l-1$ and $n_2=-(l+4)$. As a second boundary condition we assume that the chemical potential difference vanishes at the surface of the star, that is $\delta\beta(R)=0$.

For completeness we report the equation obeyed by the radial component of the countermoving spheroidal displacement, which is given by

$$\xi'_{-} = \frac{r^2 \tau'_{l+1}}{A_2} - \frac{A_1 r \tau_{l+1}}{A_2}.$$
 (26)

3.2. Superfluid r-mode oscillations. Assuming that k_{i} is of order unity one finds that the spheroidal countermoving displacements are of order Ω^2 . The driving force on the countermoving displacement is the chemical potential difference which turns out to be of order Ω^2 . The order of the toroidal comoving displacement depends on the compressibility of the fluid. For a compressible fluid it is of order Ω^0 , while for an incompressible fluid it is of order Ω^2 . The reason can be traced back to the fact that comoving oscillations are driven by pressure oscillations and it turns out that $K_{lm} \propto \Omega^{-2} \zeta$. For compressible fluids the pressure oscillations are proportional to chemical potential oscillations and therefore $\zeta \propto \Omega^2$ and thus K_{lm} must be of order unity. Moreover, for this kind of mode, comoving spheroidal displacements turn out to be of the same order in Ω of comoving toroidal displacements, meaning that for a compressible fluid this oscillation is a generic inertial mode and not an r-mode. Since for a compressible fluid various components of the displacements are of the same order in Ω , one has to solve a system of coupled differential equations.

The situation is much easily tractable for incompressible fluids. In this case one can assume that spheroidal oscillations are of order $O(\Omega^2)$ and then toroidal oscillations turn out to be of the same order. The order in Ω of the various displacements and of the pressure and chemical potential fluctuations for incompressible matter are reported in Table 1. We shall restrict the analysis to the case of incompressible fluids, where the comoving and countermoving dof decouple, with the superfluid *r*-mode oscillation dominated by the toroidal displacement k_{low} . To first order in the rotation frequency of the star and to first order in the entrainment parameter, the typical frequency of the superfluid *r*-mode oscillation (measured in the corotating frame) is

$$\omega_r = \frac{2m\Omega}{l(l+1)} (1+\overline{\varepsilon}). \tag{27}$$

As for the standard *r*-mode, we restrict to analyze the case l=m=2, which corresponds to the most unstable *r*-mode. Moreover we consider only small values of the entrainment.

The analysis of the superfluid *r*-mode oscillation is very similar to the one we have performed for the standard *r*-mode oscillations, with the roles of K_{lm} and of the pressure oscillations interchanged with k_{lm} and the chemical potential oscillations. We find that for superfluid *r*-modes, k_{lm} obeys the same equation that K_{lm} obeys for standard *r*-modes, and the chemical potential fluctuation obeys the same equation that pressure fluctuation obey for standard *r*-modes. Regarding the pressure oscillation, ζ_{lm} , one has to solve an equation analogous to Eq. (23), but without the last term on the right hand side, because we are now considering an incompressible fluid. We find that

$$\zeta_{l+1} = C_1 r^{s_1} + C_2 r^{s_2} , \qquad (28)$$

where $s_{1,2}$ depend on the parameters of the model. One of the two coefficients is always negative, and therefore in order to avoid the divergence at the origin, we have that

$$\zeta_{j+1} = Cr^s \,. \tag{29}$$

We fix C by demanding that the comoving toroidal displacement, K_{lm} , is properly normalized, as in [14].

4. Conclusion. Superfluid neutron stars are characterized by various oscillation modes. Of particular interest are r-mode oscillations, because in the absence of efficient dissipative mechanisms they lead to a rapid spin-down of the compact star. The reason is that r-mode oscillations couple to gravitational waves, and the emission of gravitational waves (which spins down the star) makes these oscillations larger. This unstable mechanism can however be damped by dissipative forces, which tend to reduce the amplitude of r-mode oscillations. Indeed, if the characteristic timescale of the dissipative force is

comparable with the timescale associated to the gravitational wave emission, the *r*-mode oscillation becomes stable, meaning that the compact star does not quickly spin-down by gravitational wave emission.

We have derived the perturbed hydrodynamical equations for two different r-mode oscillations in presence of the rocket effect, that is, in the presence of processes that change the number of protons, neutrons and electrons. The two different r-mode oscillations considered are the "standard r-mode oscillation". which is a predominantly toroidal comoving displacement of the two superfluid and normal components, and the "superfluid r-mode oscillation", which is associated to toroidal countermoving displacements of the two fluids. In realistic neutron stars these two modes are coupled, however, in the limit of small rotation frequency and assuming that the star has a uniform mass density and is incompressible, they decouple. For both kind of oscillations we have determined the linearized Euler equations and found that for both the standard r-mode oscillation and the superfluid r-mode oscillation the rocket effect leads to the appearance of additional dissipative terms in the perturbed hydrodynamical equations. These terms might give a relevant contribution to the energy dissipation of the oscillations, with a damping timescale comparable to the one associated to gravitational wave emission. The numerical evaluation of the corresponding damping timescale and the comparison with those deriving from other dissipative mechanisms is performed in the accompanying paper [17].

APPENDIX A: Evolution equations

1. Standard r-mode. We derive the evolution equations for the standard r-modes, assuming uniform mass density of the star. For a star with uniform mass density one can impose hydrostatic equilibrium obtaining

$$P(r) = G \frac{2\pi}{3} (R^2 - r^2) \rho^2, \qquad (A1)$$

where we have assumed that the pressure vanishes at the surface of the star. Here R = 10 km is the radius of the star and we shall consider a mass density $\rho = 2.5\rho_0$, where ρ_0 is the saturation density of nuclear matter. With these values we obtain that the mass of the star is $M \approx 1.47 M_{\odot}$, where M_{\odot} is the mass of the sun. In this case we have that the pressure and the various components of the countermoving mode obey the following set of equations [16]

$$k_{lm} = as_{l+1} + bz_{l+1} \tag{A2}$$

$$\tau_{l+1} = ck_{lm} + dz_{l+1} + es_{l+1} \tag{A3}$$

$$r\frac{d\tau_{l+1}}{dr} = -2\tau_{l+1} - fk_{lm} - gz_{l+1} + hs_{l+1}$$
(A4)

$$r\frac{ds_{l+1}}{dr} = -3s_{l+1} - \frac{V}{\Gamma}\frac{1}{1 - x_{p}}\zeta_{l+1} + pz_{l+1}, \qquad (A5)$$

where the various coefficients have been derived in [16] and for l=m are given by

$$\begin{aligned} a &= \frac{B - iB'}{(1 - \overline{\varepsilon}) - \overline{B}' - i\overline{B}} \frac{1}{\sqrt{2l + 3}} \qquad b = (l + 2)a \\ c &= l \,\omega^2 \left(\overline{B} - i\overline{B}'\right) \frac{1}{\sqrt{2l + 3}} \qquad d = \frac{\omega^2}{l + 2} \left((l + 2)(1 - \overline{\varepsilon}) - l\overline{B}' - i\overline{B} \frac{2 + 11l + 8l + 2l^2}{5 + 2l} \right) \\ e &= -\frac{\omega^2}{l + 2} \left(l\overline{B}' - iB \frac{2 - l}{5 + 2l} \right) \qquad f = -(l + 1)c \\ g &= -pe \qquad h = \omega^2 \left((1 - \overline{\varepsilon}) - 2i\overline{B} \frac{(l + 1)^2}{5 + 2l} \right) \\ p &= (l + 1)(l + 2), \end{aligned}$$

where $\omega = \sigma/\Omega_0$, with $\Omega_0 = \sqrt{GM/R^3}$. In Eq. (A5) we have that

$$V = \frac{gr\rho}{P}$$
 and $\Gamma = \frac{d\log P}{d\log \rho}$, (A6)

which depend on the equation of state. Since we assume hydrostatic equilibrium we have from Eq. (A1) that

$$V = \frac{2r^2}{R^2 - r^2}$$
 and $\Gamma = 2$. (A7)

Notice that we shall assume that both the background and the perturbations will obey the same equation of state, therefore the coefficient Γ determined above, characterizes both the background and the perturbation.

Finally, according with [16], we have that the pressure fluctuations are given by

$$\zeta_{l+1} = 2 \frac{\omega \widetilde{\omega}}{\sqrt{2l+3}} \frac{l}{l+1} \frac{r^{l-1}}{R^{l-1}},$$
 (A8)

where $\tilde{\omega} = \Omega/\Omega_0$. Upon substituting the expressions above in the equations (A2), (A3), (A4) and (A5) and expressing k_{lm} and z_{l-1} in terms of s_{l-1} and τ_{l+1} we have two coupled differential equations for s_{l-1} and τ_{l+1} . These equations can be written as a second order differential equation

$$r^{2}\tau_{l+1}'' = (A_{l}+B_{l}-1)r\tau_{l+1}' + (A_{2}B_{2}-A_{1}B_{1})\tau_{l+1} - A_{2}B_{4}\frac{r^{l+1}}{R^{2}-r^{2}},$$
 (A9)

where

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 $\begin{aligned} A_1 &= -2 - \frac{fb+g}{cb+d}, \qquad A_2 &= -f\frac{ad-be}{cb+d} + g\frac{ac+e}{cb+d} + h, \\ B_1 &= -3 - p\frac{ac+e}{cb+d}, \qquad B_2 &= \frac{p}{cb+d}, \\ B_4 &= \frac{x}{1-x_p}. \end{aligned}$

The analysis of the superfluid *r*-modes is analogous to the one we have done for the standard *r*-modes. However, in order to have an *r*-mode oscillation and not a generic inertial mode, one has to assume that the fluid is incompressible [16]. In this case $\Gamma \rightarrow \infty$ and the differential equations one has to solve are simpler.

2. Superfluid r-mode. In the case of the superfluid r-mode we assume that the fluid is incompressible, that means $\Gamma \rightarrow \infty$. Therefore, in this case we are left with the following equations for the countermoving degree of freedom:

$$r\frac{d\tau_{l+1}}{dr} = (l-1)\tau_{l+1}$$
(A10)

$$k_{lm} = -\frac{l+1}{2i\omega\widetilde{\omega}Q_{l+1}(l\overline{B}'+im\overline{B})}\tau_{l+1}$$
(A11)

while for the comoving degree of freedom

$$r\frac{dS_{l+1}}{dr} = -3S_{l+1} + aZ_{l+1} \tag{A12}$$

$$r\frac{d\zeta_{l+1}}{dr} = -2\zeta_{l+1} + bS_{l+1} + cZ_{l+1} + dK_{lm}$$
(A13)

$$\zeta_{l+1} = eK_{lm} + fZ_{l+1} + gS_{l+1} \tag{A14}$$

$$K_{lm} = hS_{l+1} + jZ_{l+1}, \qquad (A15)$$

where we derived similar coefficients as in [16] and, for l = m, are given by

$$a = (l+1)(l+2) \qquad b = \omega^{2}$$

$$c = -2l\,\omega\widetilde{\omega} \qquad d = -2i\,\omega\widetilde{\omega}lQ_{l+1}$$

$$e = \frac{d}{l+1} \qquad f = \omega \left[\omega - \frac{2\widetilde{\omega}}{(l+1)(l+2)}\right]$$

$$g = -\frac{2}{l+1} \qquad h = -\frac{i\,\omega_{0}}{\omega - \omega_{0}}Q_{l+1}$$

$$i = (l+2)h.$$

Now, by expressing K_{in} and Z_{i-1} in terms of S_{i-1} and ζ_{i+1} we are left with two differential equations for the latter variables, that can also be rearranged as a second order differential equation for S_{i-1}

$$r^{2}S_{l+1}'' = (A_{l} + B_{l} - 1)rS_{l+1}' + (A_{2}B_{2} - A_{l}B_{l})S_{l+1}, \qquad (A16)$$

where

$$\begin{aligned} A_1 &= -3 - a \frac{eh + g}{ej + f}, \qquad A_2 &= -c \frac{eh + g}{ej + f} - d \frac{gh - fh}{ej + f} + b, \\ B_1 &= -2 + \frac{c}{ej + f} + \frac{dj}{ej + f}, \qquad B_2 &= \frac{a}{ej + f}. \end{aligned}$$

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г-МОДА ОСЦИЛЛЯЦИЙ И РЕАКТИВНЫЙ ЭФФЕКТ ВО ВРАЩАЮЩИХСЯ НЕЙТРОННЫХ ЗВЕЗДАХ. I. ФОРМАЛИЗМ

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Получены гидродинамические уравнения *г*-моды осцилляций в нейтронных звездах при наличии нового механизма затухания, связанного с процессами изменения числа частиц. Изменения плотности числа различных сортов частиц приводят к возникновению в этих уравнениях новых диссипативных членов, которые ответственны за реактивный эффект. Использована двухжидкостная модель, причем одна из жидкостей представляет собой заряженную компоненту звезды, а вторая жидкость состоит из сверхтекучих нейтронов. Рассмотрены два различных типа *г*-моды осцилляций, первая - связанная с совместным смещением, а вторая - с противоположным и безфазовым смещением этих жидкостей.

Ключевые слова: нейтронные звезды:осцилляция:реактивный эффект

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