АСТРОФИЗИКА

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RADIATION TRANSFER IN AN INHOMOGENEOUS MEDIUM. OPTICAL DEPTH DEPENDING ABSORPTION COEFFICIENT

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The Invariance principle is applied to obtain the equations for finding the radiation field intensity in an inhomogeneous atmosphere. Though the behaviour of the inhomogeneity is not specified definitely but the absorption coefficient is assumed to depend on the optical depth. Such kind of depth dependence is needed because this case encounters when the elemental diffusion is considered in the atmospheres of Ap stars. The corresponding equations are obtained to solve by numerical methods.

Key words: *radiation transfer:inhomogeneous medium*

1. *Introduction.* This work has been started to tackle a very specific problem concerning mainly Ap-Bp stars atmospheres, but it may have more general applications for studies, which need radiation transfer calculations in optically thin and inhomogeneous media.

Ap-Bp stars are main sequence stars with effective temperature between about 8000 K and 16000 K, and their main characteristic is that they present strong abundance anomalies: in a given star, some metals may be overabundant by a factor up to 10^5 compared to the solar abundances, while some other elements may be underabundant (see for instance the review by Smith [1]). Presently, these anomalies are explained by atomic diffusion [2-3] which is very efficient in Ap-Bp stars because these stars are supposed to be very quiet regarding to superficial mixing processes such as turbulence, convection, etc. (see the seminal work of Michaud [4] and numerous following papers). Therefore, atomic diffusion cannot be neglected for such stars: elements are pushed upward when photons are absorbed through atomic transitions (radiative acceleration). According to the diffusion model, when for a given element the radiative acceleration is strong enough to counterbalance gravity, the element moves upward, otherwise it sinks. Elements accumulate at some places in the star according to their atomic properties and according to the way the particle flux varies.

The time-dependent stratification of elements due to the diffusion processes has been thoroughly studied in optically thick media by mean of heavy numerical calculations [5], but only recently and very approximately in optically thin media such as Ap-Bp stars' atmosphere [6]. The reason of this situation is the following: the study of the stratification process needs to solve numerically the time-dependent continuity equation (for concentrations). This is heavy and very computer-time consuming. In optically thick case, radiative accelerations can be determined through local radiation flux. This makes such calculations possible, even with thousands of atomic transitions. Notice that, in these calculations for stellar interiors, atomic transitions are not considered individually: numerical codes use large opacity tables and computations are done with the technique of opacity sampling. This method is not possible to use in optically thin medium, because radiative acceleration is very sensitive to the line profile and line profiles are narrow in atmospheres. So, the opacity sampling would require too high frequency resolution in the atmosphere. On another hand, the continuity equation is coupled to the transfer one, and both of them depend on elements concentration in the whole medium (the problem being non-local in the optically thin case).

Alecian [7] speculated about a scenario with an unstable behavior of elements stratification in Ap-Bp star's atmosphere. Detailed numerical computations such as those made in optically thick case cannot be used to confirm this scenario, because of the difficulties we have just mentioned (see also the theoretical study by Alecian & Grappin [8]). We think that a simple model involving only a fictitious element, as done in numerical calculations by Alecian et. al (with ions having only few energy levels), can be a helpful approach, and could give interesting insight on the relevant physical mechanisms (namely to check the instability hypothesis) [6]. To prepare the ground for a better theoretical studies about the behavior of the abundance stratifications build-up, and because atomic diffusion is strongly dependent on the radiative acceleration, we need to study the problem of the radiation transfer in the case of strongly stratified abundances.

In this paper, we consider the equations of radiation transfer in the framework of the formalism of Ambartsumian's Invariance Principle, generalized for the case of inhomogeneous media [9] (see, also [10] for details). It is noteworthy that up to novadays mostly the medium inhomogeneity was described by the dependence on optical thickness of the photon's survival parameter or, so called, the single scattering albedo (see, [9-12] and references therein).

2. *Green function formalism*. It is known (see, [13]) that the probabilistic approach is rather fruitful for many problems in radiation transfer theory. We will use this approach to investigate the radiation transfer problem in a semi-infinite inhomogeneous atmosphere. Atmosphere is considered to be one-dimensional; however the obtained results can be easily generalized for a three-dimensional plane-parallel medium.

In the present paper, at first sight a little more generalized problem is considered than could be of interest for our main purpose. We assume that the atmosphere consists in an undefined mixture with known absorbing/ scattering properties, plus a trace element (hereafter "A" type particles) with known physical characteristics (two-level ion).

To operate with the formalism adopted in the radiative transfer theory, we use the following variables: the optical depth τ and a dimensionless frequency x which is defined here as the distance from the center of the considered spectral line divided by the Doppler width.

The interaction between radiation and matter has been divided into two processes. The first one involves "A" type particles only and the second one the remaining matter. Interactions with "A" are described by an absorption coefficient $\sigma(\tau, x) \equiv q(\tau)\alpha(x)$ where $q(\tau)$ represents the spatial distribution of "A" and $\alpha(x)$ the absorption profile (Voigt function for instance). The probability that a photon survives after interaction with "A" and returns to the radiation field is denoted by $\lambda(\tau)$. Interactions with remaining matter are considered to represent true absorptions only described by the absorption coefficient $\beta(\tau, x)$.

Details of methods based on the *Invariance Principle* can be found in many monographs [13-15]. We would like, however, to emphasize that a generalization for inhomogeneous atmosphere was proposed by Sobolev (see [9-11] and references therein). The main idea is to consider, instead of a given atmosphere, a family of atmospheres for which an upper layer with optical thickness h has been removed (truncated atmosphere). To go further, it is useful to define a *Green function* for such a truncated source free atmosphere.

Let us denote by $G^-(h, \tau', x', \tau, x)d\tau\, dx$ [$G^+(h, \tau', x', \tau, x)d\tau\, dx$] the probability that a photon at the optical depth τ' moving in any direction and having initially the frequency x', will be (after a series of scattering's) in the domain ($\tau, \tau + d\tau$) moving upwards [inwards], in the frequency interval ($x, x+dx$). For farther consideration it is more appropriate to separate this *Green function* into two functions according to the direction of the initial photon (X for outward, Z for inward) and express it as a sum of these components:

$$G = X + Z\,. \tag{1}$$

Invariance Principle allows deriving the principal integro-differential equations for *Green functions'* components. If one adds a very thin layer $\Delta\tau$ to the boundary of truncated atmosphere and takes into account all processes at first order of $\Delta\tau$, the following equation is obtained:

$$\frac{\partial G^{\mp}}{\partial \tau'} + \frac{\partial G^{\mp}}{\partial \tau} - \frac{\partial G^{\mp}}{\partial h} = l(0) \int_{-\infty}^{+\infty} G^-(h, \tau', x', 0, x'')dx'' \int_{-\infty}^{+\infty} r(x', x'')X^{\mp}(h, 0, x''', \tau, x)dx''' \,, \tag{2}$$

where

$$l(\tau) = \frac{\lambda(\tau + h)q(\tau + h)}{2}\,, \tag{3}$$

$r(x', x)$ is the so called *frequency redistribution function* (see [16-19] and references therein) which depends on the physical process of elementary scattering and describes the probability of photons reemission in a frequency domain $(x, x+dx)$.

On the other hand, applying a common procedure of Invariance Principle approach, one can express the value of the *Green function* defined at a depth $\tau+\Delta\tau$ $[\tau'+\Delta\tau']$ with respect to its value at τ $[\tau']$. Then two more pairs of integro-differential equations can be obtained. One of them is given as follows:

$$\mp\frac{\partial G^{\mp}}{\partial\tau} = -\upsilon(x,\tau)G^{\mp}(h,\tau',x',\tau,x) + l(\tau)\int_{-\infty}^{\infty} G^{\mp}(h,\tau',x',\tau,x'')dx'', \quad (4)$$

where $\nu(x,\tau) = \alpha(x)q(\tau+h) + \beta(\tau+h,x)$ is the total absorption coefficient and:

$$\mp\frac{\partial G^{\mp}}{\partial\tau'} = -\upsilon(x',\tau')\left[Z^{\mp}(h,\tau',x',\tau,x) - X^{\mp}(h,\tau',x',\tau,x)\right], \quad (5)$$

which can be written in a combined form as well:

$$\mp\frac{\partial G}{\partial\tau'} = -\upsilon(x',\tau')\left[Z(h,\tau',x',\tau,x) - X(h,\tau',x',\tau,x)\right]. \quad (6)$$

In equation (6) the quantities without superscripts are the sums of corresponding "half-functions". It is worth mentioning that functions X and Z in their turn are the solutions of the following equations:

$$\begin{aligned} -\frac{\partial X}{\partial\tau'} &= -\upsilon(x',\tau')X(h,\tau',x',\tau,x) + l(\tau')\int_{-\infty}^{\infty} r(x',x'')G(h,\tau',x'',\tau,x)dx'' \\ \frac{\partial Z}{\partial\tau'} &= -\upsilon(x',\tau')Z(h,\tau',x',\tau,x) + l(\tau')\int_{-\infty}^{\infty} r(x',x'')G(h,\tau',x'',\tau,x)dx''. \end{aligned} \quad (7)$$

It should be kept in mind that the $G^{\mp}$ functions from one side and functions X and Z from other side describe physical processes which are actually *reciprocal in relation to each other*. Mathematically such a reciprocity principle can be represented for these quantities in the following form:

$$\begin{aligned} G^{-}(h,\tau',x',\tau,x) &= X(h,\tau,x,\tau',x') \\ G^{+}(h,\tau',x',\tau,x) &= Z(h,\tau,x,\tau',x'). \end{aligned} \quad (8)$$

To conclude these developments, it is necessary to write the following combined equations

$$\begin{aligned} \frac{\partial G^{\mp}}{\partial\tau'} - \frac{\partial G^{\mp}}{\partial h} &= \mp\upsilon(x,\tau)G^{\mp}(h,\tau',x',\tau,x) \pm l(\tau)\int_{-\infty}^{\infty} G^{\mp}(h,\tau',x',\tau,x'')dx'' + \\ &+ l(0)\int_{-\infty}^{\infty} G^{\mp}(h,\tau',x',0,x'')dx'' \int_{-\infty}^{\infty} r(x'',x''')X^{\mp}(h,0,x''',\tau,x)dx'''. \end{aligned} \quad (9)$$

The analogous equation (9) derived for a surface *Green function* was obtained by Harutyunian [12].

3. *Reflection from semi-infinite atmosphere.* Let us consider the particular case $\tau' = 0$ and the probabilities related to a photon falling on the boundary of a semi-infinite atmosphere, the so called reflection problem. Here, reflection problem is not to be understood in a sense that we deal only with the quantities describing the emerging from the boundary of the medium intensities. For the further analysis the probabilities of appearing the initial photon at any depth must be of an importance as well.

In that instance, only X functions are important. Thus, using the equations (2), (4) and (7), one can write down the equation:

$$[\upsilon(x',0)\pm\nu(x,\tau)]X^{\mp}(h,0,x',\tau,x)-\frac{\partial X^{\mp}}{\partial h}=l(0)\int_{-\infty}^{\infty}r(x',x'')G^{\mp}(h,0,x'',\tau,x)dx''\pm$$

$$\pm l(\tau)\int_{-\infty}^{\infty}X^{\mp}(h,0,x',\tau,x'')r(x'',x')dx''+ \tag{10}$$

$$+l(0)\int_{-\infty}^{\infty}X^{\mp}(h,0,x',0,x'')dx''\int_{-\infty}^{\infty}r(x'',x''')X^{\mp}(h,0,x''',\tau,x)dx'''.$$

It is easy to notice that the system of two equations (10) comes to the much simpler equation for finding the reflection probability on the boundary of the medium when the final optical thickness is equal to zero - $\tau = 0$. Only X^- has a non-trivial physical meaning in this case and instead of two equations, one obtains:

$$[\upsilon(x',0)+\nu(x,0)]\rho(h,x',x)-\frac{\partial\rho}{\partial h}=l(0)r(x',x)+l(0)\int_{-\infty}^{\infty}r(x',x'')\rho(h,x'',x)dx''+$$

$$+l(0)\int_{-\infty}^{\infty}\rho(h,x',x'')r(x'',x)dx''+ \tag{11}$$

$$+l(0)\int_{-\infty}^{\infty}\rho(h,x',x'')dx''\int_{-\infty}^{\infty}r(x'',x''')\rho(h,x''',x)dx''',$$

where $\rho(h,x',x)\equiv X^-(h,0,x',0,x)$. The equation (11) was derived taking into account the obvious identity:

$$Z^-(h,0,x',0,x)=\delta(x-x'). \tag{12}$$

The equation (11) describes the simplest problem which is connected only with probabilities of photons reflection calculated on the surface of atmosphere. However, this concerns a rather wide series of problems.

One can write down the formal solution of (11) as follows:

$$\rho(t,x',x)=\int_t^{\infty}K(t,x',x)e^{-T(t,t')}dt', \tag{13}$$

where

$$T(t,t')=\int_t^{t'}[\nu(x',t'')+\upsilon(x,t'')]dt'' \tag{14}$$

and $K(t, x', x)$ involves the right hand side expression in (11). So having all the necessary quantities given for the all relevant optical depths one can solve the equation (13) for chosen redistribution function. It can be done by a simple iteration method, for instance.

The second principal quantity very important for practical uses is the integral of X function:

$$R^{\mp}(h,\tau,x)=\int_{-\infty}^{\infty}X^{\mp}(h,0,x',\tau,x)dx'. \tag{15}$$

Integrating the equation over the all frequencies x', one finds:

$$\begin{gathered}\pm\upsilon(x,\tau)R^{\mp}(h,\tau,x)-\frac{\partial R^{\mp}}{\partial h}=\int_{-\infty}^{\infty}\upsilon(x',0)X^{\mp}(h,0,x',\tau,x)dx'+\\ +l(0)\int_{-\infty}^{\infty}\alpha(x')G^{\mp}(h,0,x',\tau,x)dx'\pm l(\tau)\int_{-\infty}^{\infty}R^{\mp}(h,\tau,x)r(x',x)dx'+\\ +l(0)\int_{-\infty}^{\infty}R^{-}(h,0,x')dx'\int_{-\infty}^{\infty}r(x',x)X^{\mp}(h,0,x'',\tau,x)dx'',\end{gathered} \tag{16}$$

where the normalization condition of the redistribution function is taken into account as well:

$$\alpha(x)=\int_{-\infty}^{\infty}r(x',x)dx'. \tag{17}$$

The physical meaning of the quantities $R^{\mp}(h,\tau,x)$ is rather evident. It is easy to see that $R^{\mp}(h,\tau,x)d\tau dx$ is the probability that a photon will appear at the depth domain $\tau,\tau+\Delta\tau$, having a frequency in x, $x+dx$ moving upwards or downwards if the semi-infinite atmosphere is illuminated by a radiation of intensity equals to 1 in all frequencies. At $\tau=0$ one can find the value of the mentioned quantity on the boundary surface noting also that $R^{+}(h,0,x)=0$.

On the other hand the quantity $R^{\mp}(h,0,x)$ could be interpreted in other way if the reciprocity principle for the ρ function is taken into account. Then, one may interpret the quantity $R^{\mp}(h,\tau,x)d\tau dx$ as a probability that a photon of frequency x incident on the boundary of semi-infinite atmosphere will be at the depth τ moving upwards (inwards). It is worth mentioning that the same principle of reciprocity for Green functions allows giving various physical explanations for quantities $R^{\mp}(h,\tau,x)d\tau dx$.

4. *Intensities of the radiation field*. Up to now, all our investigation concerned the probabilities of certain physical processes in a source-free medium. The quantities introduced had clearly probabilistic meaning and described only the properties of the medium. However, having all the necessary probabilities, one can describe the radiation field in an atmosphere for any distribution of primary energy sources.

Let us consider now the existence of energy sources in the atmosphere which are distributed according to some function $\varepsilon(h,\tau,x)[\equiv\varepsilon(h+\tau,x)]$ with a time-independent behavior. Using the probabilistic meaning of Green function, it might be seen that the expression for intensities of the radiation field at any depth τ in such an atmosphere will have the following form:

$$I^{\mp}(h,\tau,x)=\int_0^{\infty}d\tau'\int_{-\infty}^{\infty}\varepsilon(h+\tau',x')G^{\mp}(h,\tau',x',\tau,x)dx', \tag{18}$$

where the superscript "-" and "+" again corresponds to the radiation fluxes which have outward and inward directions of motion.

Thus to obtain the equations for intensities one needs to multiply any of equations written for Green functions by $\varepsilon(h+\tau',x')d\tau'dx'$ and integrate those over all the depths and frequencies. So, various equations can be written down some combinations of which are most efficient for our purposes in further investigations. We will use here the equation (9) for obtaining the equations for intensities. As it is described above, for this purpose we are to multiply the both sides of the mentioned equation by $\varepsilon(h+\tau',x')d\tau'dx'$. Let us notice first before overall integration that the left hand side allows preliminary transformations while taking the integral over the optical depth:

$$\int_0^{\infty}\varepsilon(h+\tau',x')\left[\frac{\partial G^{\mp}}{\partial\tau'}-\frac{\partial G^{\mp}}{\partial h}\right]d\tau=-\varepsilon(h,x')G^{\mp}(h,0,x',\tau,x)-$$
$$-\frac{\partial}{\partial\tau'}\int_0^{\infty}\varepsilon(h+\tau',x')G^{\mp}(h,\tau',x',\tau,x)d\tau'. \tag{19}$$

Here the first term in the left hand side is the integrated by parts and the obvious identity:

$$\frac{\partial\varepsilon(h+\tau',x')}{\partial h}\equiv\frac{\partial\varepsilon(h+\tau',x')}{\partial\tau'} \tag{20}$$

is taken into account. The overall integration of the equation (9) with the mentioned energy sources gives the following equations for the intensities:

$$\frac{\partial I^{\mp}(h,\tau,x)}{\partial h}=\pm\upsilon(x,\tau)I^{\mp}(h,\tau,x)-\int_{-\infty}^{\infty}\varepsilon(h,x')G^{\mp}(h,0,x',\tau,x)dx'\mp$$
$$\mp l(\tau)\int_{-\infty}^{\infty}I^{\mp}(h,\tau,x')r(x,x')dx'- \tag{21}$$
$$-l(0)\int_{-\infty}^{\infty}I^{-}(h,0,x')dx'\int_{-\infty}^{\infty}r(x',x'')X^{\mp}(h,0,x'',\tau,x)dx'',$$

which has a construction very similar to equation (16) written for the quantity $R^{\mp}(h,\tau,x)$. This means that the solutions of these two pairs of equations are closely related and can be expressed each by other. Moreover, in one particular case when energy sources are distributed according the law:

$$\varepsilon(\tau,x)=[1-\lambda(\tau)]q(\tau)\alpha(x)-\beta(\tau,x), \tag{22}$$

it can be shown that the quantities $R^{\mp}$ and $I^{\mp}$ are connected by a linear relation. We will not study this question in detail here and will give only the result for the surface values of the mentioned quantities:

$$I^{-}(h,0,x)=1-R^{-}(h,0,x). \tag{23}$$

Generally speaking, all such properties show the same physical consequences for a photon born somewhere in the medium. In any case such a photon has only two possible ways: it will escape from its boundary or it will be "truly absorbed" transferring its energy to the medium.

Returning back to our problem, it must be mentioned that the purposes of the problem under discussion is the finding the net flux:

$$H(0,\tau,x) = I^{-}(0,\tau,x) - I^{+}(0,\tau,x). \tag{24}$$

So one can obtain instead of (21) a pair of equations in relation to the difference and sum of the quantities I^- and I^+ to write, for example, the formal solutions for them and find any method for numerical realization of it. The same could be done for equations (21) without any transformation.

5. *Conclusion*. Equations and relations obtained here generalize ones derived earlier for the homogeneous atmosphere and for comparatively simpler cases of inhomogeneous media. This research shows that the *Invariance principle* allows describing the multiple scattering of light in the media which show different types of inhomogeneity. The problem can be solved for both the general lows of the redistribution function and changing the form of inhomogeneity. A separate paper will be devoted to the numerical methods applicable for solution of this problem.

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ПЕРЕНОС ИЗЛУЧЕНИЯ В НЕОДНОРОДНОЙ СРЕДЕ. КОЭФФИЦИЕНТ ПОГЛОЩЕНИЯ, ЗАВИСЯЩИЙ ОТ ОПТИЧЕСКОЙ ГЛУБИНЫ

Ж.АЛЕСИАН, Г.А.АРУТЮНЯН

Принцип инвариантности применен, чтобы получить уравнения для определения интенсивности поля излучения в неоднородной атмосфере. Несмотря на то, что поведение неоднородности не конкретизировано, но предполагается, что коэффициент поглощения зависит от оптической глубины. Такого типа зависимость нужна, поскольку она встречается в задаче, когда рассматривается диффузия химических элементов в атмосферах Ар-звезд. Получены соответствующие уравнения для численного решения.

Ключевые слова: *перенос излучения:неоднородная среда*

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