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A CHARACTERIZATION OF WEIGHTED CENTRAL CAMPANATO SPACES

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Abstract. In this paper, we introduce the weighted central Campanato spaces $\dot{C}^{p,\lambda}(\omega)$ and characterize $\dot{C}^{p,\lambda}(\omega)$ by the boundedness of the commutators $[b,H]$ and $[b,H^*]$ from weighted central Morrey spaces to weighted central Morrey spaces for $\omega \in A_1$, where the commutators are generated by n-dimensional Hardy operators and symbol b. In particular, the Weighted Lipschitz estimates for the Commutators of Hardy operators are obtained if $0<\lambda<1/n$.

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1. Introduction

In 1963, Campanato space $C^{p,\lambda}(\mathbb{R}^n)$ was first introduced by Campanato [1] in order to study elliptic regularity in the context of the heat equation. Let $-1/p<\lambda<1/n$ and $1\le p<\infty$, a locally integrable function f is said to belong to the Campanato space $C^{p,\lambda}(\mathbb{R}^n)$, if

$$\|f\|_{C^{p,\lambda}(\mathbb{R}^n)}=\sup_B\left(\frac{1}{|B|^{1+\lambda p}}\int_B|f(x)-f_B|^p dx\right)^{\frac{1}{p}}<\infty,$$

where the supremum is taken over all balls $B\subset\mathbb{R}^n$, $f_B=\frac{1}{|B|}\int_B f(y)dy$, where $|B|$ is the Lebesgue measure of B. If the supremum is taken over all balls $B(0,r)$, it is the central Campanato space $\dot{C}^{p,\lambda}(\mathbb{R}^n)$. The excellent structures of Campanato spaces render them useful in the studies of the regularity theory of PDEs, which allows us to give an integral characterization of the spaces of Hölder continuous functions. This leads to a generalization of the classical Sobolev embedding theorem [2, 3, 4, 5, 6]. It is well known that $C^{1,\frac{1}{n}(\frac{1}{p}-1)}$ is the dual space of the Hardy space H^p when $0<p<1$ [7]. Especially, $C^{1,0}=BMO(\mathbb{R}^n)$.

Many authors have focused on the researches of commutators for which the symbol functions belong to BMO spaces and Lipschitz spaces which are the special

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cases of Campanato spaces. More precisely,

$$C^{p,\lambda}(\mathbb{R}^n) = \begin{cases} BMO(\mathbb{R}^n) & \lambda = 0, \\ Lip_\beta(\mathbb{R}^n) & 0 < \lambda < 1/n. \end{cases}$$

Recently, there are lots of studies concerning Campanato spaces and central Campanato spaces. In 2013, Shi and Lu [8, 9] characterized the space $C^{p,\lambda}$ via the boundedness of fractional integral and Calderón-Zygmund singular integral operator on Morrey spaces. In [10], Zhao and Lu gave some creative characterizations of central Campanato spaces via the boundedness of commutators associated with the Hardy operators for $\lambda > 0$. In 2015, Shi got another characterization via the boundedness of commutators associated with the Hardy operators for $-1/p < \lambda < 0$ [11].

As is well known, Lipschitz spaces and Campanato spaces have equivalent norms if $1 \leq p < \infty$. In 2018, Wang and Zhou [12] proved that they are still equivalent to $0 < p < 1$. In the weighted setting, J. García-Cuerva [13] proved the equivalence of weighted Lipschitz spaces and weighted Campanato spaces, which is stated as follows:

$$\begin{aligned} \|f\|_{Lip_{\beta,\omega}} &\approx \sup_Q \frac{1}{\omega(Q)^{\frac{\beta}{n}}} \left(\frac{1}{\omega(Q)} \int_Q |f(x) - f_Q|^p \omega(x)^{1-p} dx \right)^{\frac{1}{p}} \\ &\approx \sup_Q \frac{1}{\omega(Q)^{1+\frac{\beta}{n}}} \int_Q |f(x) - f_Q| dx \end{aligned}$$

if $1 \leq p \leq \infty$, $0 < \beta < 1$ and $\omega \in A_1$. Moreover, Hu and Zhou [14] extended its equivalence to $0 < p < 1$.

Inspired by the above works, in this paper, we introduce the weighted central Campanato spaces and characterize the weighted central Campanato spaces via the boundedness of commutators associated with the Hardy operators. In particular, we obtain characterizations of weighted central BMO spaces if $\lambda = 0$, the weighted Lipschitz estimates for the commutators of Hardy operators are derived according to the equivalence of weighted Lipschitz spaces and weighted Campanato spaces if $0 < \lambda < 1/n$.

2. Some preliminaries and notations

Most the notations we use are standard. $B(x, r)$ denotes the ball centered at x with radius r. For any $a > 0$, $aB(x, r) = B(x, ar)$. For a locally integrable function f, $f_B = \frac{1}{|B|} \int_B f(x)dx$, the Lebesgue measure of B by $|B|$. Also, ω is a nonnegative locally integrable function i.e. $\omega(E) = \int_E \omega(x)dx$, p' is the conjugate of p satisfying $1/p + 1/p' = 1$. C always stands for a constant independent of the main parameters and not necessarily the same at each occurrence.

In 1995, Christ and Grafakos [15] gave the definitions of the n-dimensional Hardy operator and its adjoint operator,

$$Hf(x) = \frac{1}{|x|^n}\int_{|y|<|x|} f(y)dy, \quad H^*f(x) = \int_{|y|\geq|x|} \frac{f(y)}{|y|^n}dy, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

H and H^* satisfy

$$\int_{R^n} g(x)Hf(x)dx = \int_{R^n} f(x)H^*g(x)dx.$$

Let b be a measurable locally integrable function and T be a linear operator. Then the commutator $[b, T]$ is defined by

$$[b, T]f = bTf - T(bf).$$

In [16], R. Coifman, R. Rochberg and G. Weiss proved that the commutator $[b, T]$ is bounded on $L^p(\mathbb{R}^n)$ if $b \in BMO(\mathbb{R}^n)$ and $1 \leq p < \infty$, where T is a Calderón-Zygmund singular integral operator.

In this article, the commutators of H and H^* are defined by

$$H_b f = [b, H]f = bHf - H(bf), \quad H_b^* f = [b, H^*]f = bH^*f - H^*(bf).$$

This article will prove that $H_b f$ and $H_b^* f$ are bounded from weighted central Morrey spaces to weighted central Morrey spaces if and only if b belongs to the weighted central Campanato spaces.

In the following, we give the definitions of weighted central Campanato spaces and weighted Morrey spaces.

Definition 2.1 Let ω be a nonnegative locally integrable function, a function $f \in L^p_{loc}(\mathbb{R}^n)$ is said to belong to the weighted central Campanato space $\dot{C}^{p,\lambda}(\omega)(\mathbb{R}^n)$ for $-1/p < \lambda < 1/n$ and $1 \leq p < \infty$, if

$$\|f\|_{\dot{C}^{p,\lambda}(\omega)} = \sup_{r>0}\left(\frac{1}{\omega(B(0,r))^{1+\lambda p}}\int_{B(0,r)} |f(x) - f_{B(0,r)}|^p \omega(x)^{1-p}dx\right)^{\frac{1}{p}} < \infty.$$

If $\omega = 1$, $\dot{C}^{p,\lambda}(\omega)(\mathbb{R}^n) = \dot{C}^{p,\lambda}(\mathbb{R}^n)$. If the supremum is taken over all balls $B \subset \mathbb{R}^n$ and $\omega = 1$, $\dot{C}^{p,\lambda}(\omega)(\mathbb{R}^n) = C^{p,\lambda}(\mathbb{R}^n)$, if $\lambda = 0$, it is the weighted central BMO space $CMO^p(\omega)$.

Definition 2.2 Let $1 \leq p < \infty$, ω is a nonnegative locally integrable function, a function $f \in L^p_{loc}(\mathbb{R}^n)$ is said to belong to the weighted central BMO space $CMO^p(\omega)$ if

$$\|f\|_{CMO^p(\omega)} = \sup_{r>0}\left(\frac{1}{\omega(B(0,r))}\int_{B(0,r)} |f(x) - f_{B(0,r)}|^p \omega(x)^{1-p}dx\right)^{\frac{1}{p}} < \infty.$$

Obviously, $CMO^p(\omega) \subseteq CMO^q(\omega)$ if $1 \leq q < p < \infty$. When $\omega = 1$, $CMO^p(\omega) = CMO^p(\mathbb{R}^n)$. In particular, $BMO(\mathbb{R}^n) \subset CMO^p(\mathbb{R}^n)$ if $1 \leq p < \infty$, $CMO^p(\mathbb{R}^n) \subseteq CMO^q(\mathbb{R}^n)(1 \leq q < p < \infty)$. There is no analysis of the famous John-Nirenberg inequality of $BMO(\mathbb{R}^n)$ for $CMO^p(\mathbb{R}^n)$, so $CMO^p(\mathbb{R}^n)$ and $CMO(\mathbb{R}^n)$ are not equivalent.

Definition 2.3 [17] Let $1 \le p < \infty$, $-1/p < \lambda < 0$, ω_1, ω_2 are nonnegative locally integrable functions, a function $f \in L^p_{loc,\omega_2}(\mathbb{R}^n)$ is said to belong to the weighted Morrey space $M^{p,\lambda}(\omega_1, \omega_2)$ if

$$\|f\|_{M^{p,\lambda}(\omega_1,\omega_2)} = \sup_B \left(\frac{1}{\omega_1(B)^{1+\lambda p}} \int_B |f(x)|^p \omega_2(x) dx \right)^{1/p} < \infty.$$

If $\omega_1 = \omega_2 = 1$, $M^{p,\lambda}(\omega_1, \omega_2)(\mathbb{R}^n)$ is the classical Morrey space $M^{p,\lambda}(\mathbb{R}^n)$. In particular, Sakamoto and Yabuta [18] pointed out that $C^{p,\lambda}(\mathbb{R}^n)$ is equivalent to $M^{p,\lambda}(\mathbb{R}^n)$ when $1 \le p < \infty$ and $-1/p < \lambda < 0$. But Lin [19] gave a counterexample to verify that $M^{p,\lambda}(\mathbb{R}^n) \subseteq C^{p,\lambda}(\mathbb{R}^n)$ when $1 \le p < \infty$ and $-1/p < \lambda < 0$.

In order to characterize the weighted central Campanato spaces, we give the following definition of the weighted central Morrey space $\dot{M}^{p,\lambda}(\omega_1, \omega_2)$.

Definition 2.4 Let $1 \le p < \infty$, $-1/p < \lambda < 0$, ω_1, ω_2 are nonnegative locally integrable functions, a function $f \in L^p_{loc,\omega_2}(\mathbb{R}^n)$ is said to belong to the weighted central Morrey space $\dot{M}^{p,\lambda}(\omega_1, \omega_2)$ if

$$\|f\|_{\dot{M}^{p,\lambda}(\omega_1,\omega_2)} = \sup_{B(0,r)} \left(\frac{1}{\omega_1(B(0,r))^{1+\lambda p}} \int_{B(0,r)} |f(x)|^p \omega_2(x) dx \right)^{1/p} < \infty.$$

If $\omega_1 = \omega_2 = 1$, it is the central Morrey space $\dot{M}^{p,\lambda}(\mathbb{R}^n)$.

Definition 2.5 Let $1 < p < \infty$, we say $\omega \in A_p$ if

$$\sup_B \left(\frac{1}{|B|} \int_B \omega(x) dx \right) \left(\frac{1}{|B|} \int_B \omega(x)^{-\frac{1}{p-1}} dx \right)^{p-1} < \infty.$$

For the case $p = 1$, we say $\omega \in A_1$ if

$$\frac{1}{|B|} \int_B \omega(x) dx \le C ess \inf_{x \in B} \omega(x)$$

for every ball $B \subset \mathbb{R}^n$. A weight function $\omega \in A_\infty$ if it satisfies the A_p condition for some $1 \le p < \infty$.

Lemma 2.6 [20]. Let $\omega \in A_1$, then there are constants C_1, C_2 and $0 < \delta < 1$ for any measurable subset $E \subset B$,

$$C_1 \frac{|E|}{|B|} \le \frac{\omega(E)}{\omega(B)} \le C_2 \left(\frac{|E|}{|B|} \right)^\delta. \tag{2.1}$$

Lemma 2.7 [21]. Let $\omega \in A_1$, then for $1 < p < \infty$,

$$\int_B \omega(x)^{1-p'} dx \le C |B|^{p'} \omega(B)^{1-p'}, \tag{2.2}$$

where $1/p + 1/p' = 1$.

Proof: Since $A_1 \subset A_p$, ω satisfies the condition of the weight A_p. The above lemma can be obtained by simple calculation.

Lemma 2.8 [22]. The function class is called the reverse Hölder class if a function f satisfies the following condition

$$\sup_{x\in B}|f(x)-f_B| \leq \frac{C}{|B|}\int_B |f(x)-f_B|dx. \tag{2.3}$$

Reverse Hölder class contains many kinds of functions, such as polynomial functions [23]. For more theories about reverse Hölder class, see [24].

3. A characterization of weighted central Campanato spaces

The main theorems are as follows.

Theorem 3.1. Let $\omega \in A_1(\mathbb{R}^n)$, $1<p<\infty$, $-1/p<\lambda<0$, $-1/p_i<\lambda_i<0 (i=1,2)$, $\lambda=\lambda_1+\lambda_2$, $1/p=1/p_1+1/p_2$, b satisfies (2.3), then the following statements are equivalent:

(i) $b \in \dot{C}^{p_1,\lambda_1}(\omega)$;

(ii)$[b,H]$ and $[b,H^*]$ are bounded from $\dot{M}^{p_2,\lambda_2}(\omega,\omega)$ to $\dot{M}^{p,\lambda}(\omega,\omega^{1-p})$.

Theorem 3.2. Let $\omega \in A_1(\mathbb{R}^n)$, $1<p<\infty$, $1/p+1/p'=1$, $-\min\{1/(2p),1/(2p')\}<\lambda<0$, then the following statements are equivalent:

(i) $b \in \dot{C}^{\max(p,p'),\lambda}(\omega)$;

(ii) $[b,H]$ and $[b,H^*]$ are bounded from $\dot{M}^{p,\lambda}(\omega,\omega)$ to $\dot{M}^{p,2\lambda}(\omega,\omega^{1-p})$. In addition, $[b,H]$ and $[b,H^*]$ are bounded from $\dot{M}^{p',\lambda}(\omega,\omega)$ to $\dot{M}^{p',2\lambda}(\omega,\omega^{1-p'})$.

Theorem 3.3. Let $\omega \in A_1(\mathbb{R}^n)$, $1<p<\infty$, $1/p+1/p'=1$, $-\min\{1/(p),1/(p')\}<\lambda<0$, then the following statements are equivalent:

(i) $b \in CMO^{\max(p,p')}(\omega)$;

(ii) $[b,H]$ and $[b,H^*]$ are bounded from $\dot{M}^{p,\lambda}(\omega,\omega)$ to $\dot{M}^{p,2\lambda}(\omega,\omega^{1-p})$. In addition, $[b,H]$ and $[b,H^*]$ are bounded from $\dot{M}^{p',\lambda}(\omega,\omega)$ to $\dot{M}^{p',2\lambda}(\omega,\omega^{1-p'})$.

Proof of Theorem 3.1:

$(i)\Rightarrow(ii)$, for simplicity, we write $B_k=\{x\in\mathbb{R}^n : |x|\leq 2^k\}$, $C_k=B_k\backslash B_{k-1}$ for $k\in\mathbb{Z}$. For a fixed ball $B=B(0,r)\subset\mathbb{R}^n$, let $B(0,r)=B_{k_0}$ with $k_0\in\mathbb{Z}$, we just need to prove that

$$\left(\frac{1}{\omega(B_{k_0})^{1+\lambda p}}\int_{B_{k_0}}|H_bf(x)|^p\omega(x)^{1-p}dx\right)^{\frac{1}{p}} \leq C\|f\|_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)} \tag{3.1}$$

and

$$\left(\frac{1}{\omega(B_{k_0})^{1+\lambda p}}\int_{B_{k_0}}|H_b^*f(x)|^p\omega(x)^{1-p}dx\right)^{\frac{1}{p}} \leq C\|f\|_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)}. \tag{3.2}$$

On the one hand,

$$\int_{B_{k_0}}|H_bf(x)|^p\omega(x)^{1-p}dx=\int_{B_{k_0}}\left|\frac{1}{|x|^n}\int_{|y|<|x|}\big(b(x)-b(y)\big)f(y)dy\right|^p\omega(x)^{1-p}dx$$

$$\leq C\sum_{k=-\infty}^{k_0}\int_{C_k}\left|\frac{1}{|x|^n}\int_{B_k}|b(x)-b_{B_k}||f(y)|dy\right|^p\omega(x)^{1-p}dx$$

$$+C\sum_{k=-\infty}^{k_0}\int_{C_k}\left|\frac{1}{|x|^n}\int_{B_k}|b(y)-b_{B_k}||f(y)|dy\right|^p\omega(x)^{1-p}dx=I+II.$$

Applying Hölder's inequality, (2.1) and (2.2), we have the following estimates,

$$I=C\sum_{k=-\infty}^{k_0}\int_{C_k}\left|\frac{1}{|x|^n}\sum_{i=-\infty}^{k}\int_{C_i}|b(x)-b_{B_k}||f(y)|dy\right|^p\omega(x)^{1-p}dx$$

$$\leq C\sum_{k=-\infty}^{k_0}2^{-kpn}\int_{C_k}|b(x)-b_{B_k}|^p\omega(x)^{1-p}dx\left|\sum_{i=-\infty}^{k}\int_{C_i}|f(y)|dy\right|^p$$

$$=C\sum_{k=-\infty}^{k_0}2^{-kpn}\int_{C_k}|b(x)-b_{B_k}|^p\omega(x)^{\frac{(1-p_1)p}{p_1}}\omega(x)^{\frac{p_1-p}{p_1}}dx\left|\sum_{i=-\infty}^{k}\int_{C_i}|f(y)|\omega(y)^{\frac{1}{p_2}}\omega(y)^{-\frac{1}{p_2}}dy\right|^p$$

$$\leq C\sum_{k=-\infty}^{k_0}2^{-kpn}\left(\int_{B_k}|b(x)-b_{B_k}|^{p_1}\omega(x)^{1-p_1}dx\right)^{\frac{p}{p_1}}\omega(B_k)^{1-\frac{p}{p_1}}$$

$$\times\left|\sum_{i=-\infty}^{k}\left(\int_{C_i}|f(y)|^{p_2}\omega(y)dy\right)^{\frac{1}{p_2}}\left(\int_{C_i}\omega(y)^{1-p_2'}dy\right)^{\frac{1}{p_2'}}\right|^p$$

$$\leq C\|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)}\|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)}\sum_{k=-\infty}^{k_0}2^{-kpn}\omega(B_k)^{1+\lambda_1p}\left|\sum_{i=-\infty}^{k}|B_i|\omega(B_i)^{\lambda_2}\right|^p$$

$$\leq C\|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)}\|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)}\sum_{k=-\infty}^{k_0}\omega(B_k)^{1+\lambda p}\left|\sum_{i=-\infty}^{k}2^{(i-k)n(1+\lambda_2)}\right|^p$$

$$\leq C\|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)}\|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)}\omega(B_{k_0})^{1+\lambda p}\sum_{k=-\infty}^{k_0}2^{(k-k_0)n\delta(1+\lambda p)}$$

$$\leq C\|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)}\|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)}\omega(B_{k_0})^{1+\lambda p}.$$

Due to $1/p=1/p_1+1/p_2$, using Hölder's inequality, (2.1) and (2.2), we can get

$$II=C\sum_{k=-\infty}^{k_0}\int_{C_k}\left|\frac{1}{|x|^n}\sum_{i=-\infty}^{k}\int_{C_i}|b(y)-b_{B_k}||f(y)|dy\right|^p\omega(x)^{1-p}dx\leq C\sum_{k=-\infty}^{k_0}2^{-kpn}$$

$$\times\int_{C_k}\left|\sum_{i=-\infty}^{k}\left(\int_{B_i}\big(|b(y)-b_{B_k}||f(y)|\big)^p\omega(y)^{1-p}dy\right)^{\frac{1}{p}}\omega(B_i)^{\frac{1}{p'}}\right|^p\omega(x)^{1-p}dx$$

$$\leq C\sum_{k=-\infty}^{k_0}2^{-kpn}\int_{C_k}\bigg|\sum_{i=-\infty}^{k}\bigg(\int_{B_i}|b(y)-b_{B_k}|^{p_1}\omega(y)^{1-p_1}dy\bigg)^{\frac{1}{p_1}}$$
$$\times\bigg(\int_{B_i}|f(y)|^{p_2}\omega(y)dy\bigg)^{\frac{1}{p_2}}\omega(B_i)^{\frac{1}{p'}}\bigg|^p\omega(x)^{1-p}dx$$
$$\leq C\|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)}\|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)}\sum_{k=-\infty}^{k_0}2^{-knp}\omega(B_k)^{p\lambda_1+\frac{p}{p_1}}$$
$$\times\int_{B_k}\omega(x)^{1-p}dx\bigg|\sum_{i=-\infty}^{k}\omega(B_i)^{\frac{1}{p_2}+\lambda_2+\frac{1}{p'}}\bigg|^p$$
$$\leq C\|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)}\|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)}\sum_{k=-\infty}^{k_0}\omega(B_k)^{1+\lambda p}$$
$$\leq C\|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)}\|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)}\omega(B_{k_0})^{1+\lambda p}\sum_{k=-\infty}^{k_0}2^{(k-k_0)n\delta(1+\lambda p)}$$
$$\leq C\|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)}\|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)}\omega(B_{k_0})^{1+\lambda p}.$$

Based on I and II, we obtain (3.1). For (3.2),

$$\int_{B_{k_0}}|H_b^*|^p\omega(x)^{1-p}dx=\int_{B_{k_0}}\bigg|\int_{|y|>|x|}\frac{\big(b(x)-b(y)\big)}{|y|^n}f(y)dy\bigg|^p\omega(x)^{1-p}dx$$
$$\leq\int_{B_{k_0}}\bigg|\int_{|x|<|y|<2^{k_0a}}\frac{\big(b(x)-b(y)\big)}{|y|^n}f(y)dy\bigg|^p\omega(x)^{1-p}dx$$
$$+\int_{B_{k_0}}\bigg|\int_{2^{k_0a}<|y|}\frac{\big(b(x)-b(y)\big)}{|y|^n}f(y)dy\bigg|^p\omega(x)^{1-p}dx$$
$$=I'+II'.$$

For I', using the same discussion as (3.1), we omit the details. The analysis of II' is different.

$$I'\leq\int_{B_{k_0}}\bigg|\frac{1}{|x|^n}\int_{|y|<2^{k_0a}}|b(x)-b(y)||f(y)|dy\bigg|^p\omega(x)^{1-p}dx$$
$$\leq C\sum_{k=-\infty}^{k_0}2^{-knp}\int_{B_k}\bigg|\sum_{i=-\infty}^{k}\int_{B_i}|b(x)-b(y)||f(y)|dy\bigg|^p\omega(x)^{1-p}dx$$
$$\leq C\|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)}\|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)}\omega(B_{k_0})^{1+\lambda p}.$$

For the term II', we proceed to show that

$$II'\leq\int_{B_{k_0}}\bigg|\sum_{k=k_0}^{\infty}\int_{C_k}\frac{|b(x)-b_{B_{k_0}}|}{|y|^n}|f(y)|dy\bigg|^p\omega(x)^{1-p}dx$$
$$+\int_{B_{k_0}}\bigg|\sum_{k=k_0}^{\infty}\int_{C_k}\frac{|b(y)-b_{B_{k_0}}|}{|y|^n}|f(y)|dy\bigg|^p\omega(x)^{1-p}dx=II_1'+II_2'.$$

Employing Hölder's inequality, (2.1) and (2.2),

$$\begin{aligned}
II_1' &\leq \int_{B_{k_0}} |b(x)-b_{B_{k_0}}|^p \omega(x)^{1-p} dx \bigg| \sum_{k=k_0}^{\infty} \int_{C_k} \frac{|f(y)|}{|y|^n} dy \bigg|^p \\
&\leq \left(\int_{B_{k_0}} |b(x)-b_{B_{k_0}}|^{p_1} \omega(x)^{1-p_1} dx \right)^{\frac{p}{p_1}} \omega(B_{k_0})^{1-\frac{p}{p_1}} \\
&\quad \times \bigg| \sum_{k=k_0}^{\infty} 2^{-kpn} \left(\int_{C_k} |f(y)|^{p_2} \omega(y) dy \right)^{\frac{1}{p_2}} \left(\int_{C_k} \omega(y)^{1-p_2'} dy \right)^{\frac{1}{p_2'}} \bigg|^p \\
&\leq C \|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)} \|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)} \omega(B_{k_0})^{1+\lambda_1 p} \bigg| \sum_{k=k_0}^{\infty} \omega(B_k)^{\lambda_2} \bigg|^p \\
&\leq C \|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)} \|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)} \omega(B_{k_0})^{1+\lambda p} \sum_{k=k_0}^{\infty} 2^{(k-k_0)n\delta\lambda_2} \\
&\leq C \|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)} \|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)} \omega(B_{k_0})^{1+\lambda p}.
\end{aligned}$$

To complete the proof, we divide II_2' into two parts:

$$\begin{aligned}
II_2' &\leq \int_{B_{k_0}} \bigg| \sum_{k=k_0}^{\infty} \int_{C_k} \frac{|b(y)-b_{B_k}|}{|y|^n} |f(y)| dy \bigg|^p \omega(x)^{1-p} dx \\
&\quad + \int_{B_{k_0}} \bigg| \sum_{k=k_0}^{\infty} \int_{C_k} \frac{|b_{B_k}-b_{B_{k_0}}|}{|y|^n} |f(y)| dy \bigg|^p \omega(x)^{1-p} dx = II_{21}' + II_{22}'.
\end{aligned}$$

For II_{21}', we have

$$\begin{aligned}
II_{21}' &\leq \int_{B_{k_0}} \bigg| \sum_{k=k_0}^{\infty} \left(\int_{C_k} \left(\frac{|b(y)-b_{B_{k_0}}|}{|y|^n} |f(y)| \right)^p \omega(y)^{1-p} dy \right)^{\frac{1}{p}} \omega(B_k)^{\frac{1}{p'}} \bigg|^p \omega(x)^{1-p} dx \\
&\leq \int_{B_{k_0}} \bigg| \sum_{k=k_0}^{\infty} \left(\int_{C_k} |b(y)-b_{B_{k_0}}|^{p_1} \omega(y)^{1-p_1} dy \right)^{\frac{1}{p_1}} \left(\int_{C_k} \bigg| \frac{|f(y)|}{|y|^n} \bigg|^{p_2} \omega(y) dy \right)^{\frac{1}{p_2}} \bigg|^p \omega(x)^{1-p} dx \\
&\leq C \|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)} \|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)} \omega(B_{k_0})^{1-p} \bigg| \sum_{k=k_0}^{\infty} 2^{(k_0-k)n} \omega(B_k)^{1+\lambda} \bigg|^p \\
&\leq C \|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)} \|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)} \omega(B_{k_0})^{1+\lambda p} \bigg| \sum_{k=k_0}^{\infty} 2^{(k-k_0)n\lambda} \bigg|^p \\
&\leq C \|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)} \|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)} \omega(B_{k_0})^{1+\lambda p}.
\end{aligned}$$

Applying Hölder's inequality, (2.1), (2.2) and $|b_{B_k}-b_{B_{k_0}}| \leq C(k-k_0)\|b\|_{\dot{C}^{p_1,\lambda_1}(\omega)} \frac{\omega(B_{k_0})^{1+\lambda_1}}{|B_{k_0}|}$, we can get estimate of II_{22}'. Indeed,

$$|b_{B_k}-b_{B_{k_0}}| \leq |b_{B_{k_0}}-b_{B_{k_0+1}}| + \cdots + |b_{B_{k-1}}-b_{B_k}|,$$

$$\begin{aligned}|b_{B_{k_0}}-b_{B_{k_0+1}}| &\leq \left|\frac{1}{|B_{k_0}|}\int_{B_{k_0+1}} b(x)-b_{B_{k_0+1}}dx\right| \\ &\leq \frac{1}{|B_{k_0}|}\left(\int_{B_{k_0+1}}|b(x)-b_{B_{k_0+1}}|^{p_1}\omega(x)^{1-p_1}dx\right)^{\frac{1}{p_1}}\omega(B_{k_0+1})^{1-\frac{1}{p_1}} \\ &\leq \|b\|_{\dot{C}^{p_1,\lambda_1}(\omega)}\frac{\omega(B_{k_0+1})^{1+\lambda_1}}{|B_{k_0}|}.\end{aligned}$$

So, we get the following inequalities,

$$\begin{aligned}|b_{B_k}-b_{B_{k_0}}| &\leq \|b\|_{\dot{C}^{p_1,\lambda_1}(\omega)}\sum_{j=k_0}^{k-1}\frac{\omega(B_{j+1})^{1+\lambda_1}}{|B_j|} \\ &\leq \|b\|_{\dot{C}^{p_1,\lambda_1}(\omega)}\omega(B_{k_0+1})^{\lambda_1}\sum_{j=k_0}^{k-1}\frac{\omega(B_{j+1})}{|B_j|} \\ &\leq (k-k_0)\|b\|_{\dot{C}^{p_1,\lambda_1}(\omega)}\frac{\omega(B_{k_0+1})^{1+\lambda_1}}{|B_{k_0}|} \\ &\leq C(k-k_0)\|b\|_{\dot{C}^{p_1,\lambda_1}(\omega)}\frac{\omega(B_{k_0})^{1+\lambda_1}}{|B_{k_0}|}.\end{aligned}$$

In the next step,

$$\begin{aligned}II_{22}' &\leq C\|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)}\left|\frac{\omega(B_{k_0})^{1+\lambda_1}}{|B_{k_0}|}\right|^p\int_{B_{k_0}}\omega(x)^{1-p}dx \\ &\times\left|\sum_{k=k_0}^{\infty}2^{-kn}(k-k_0)\left(\int_{C_k}|f(y)|^{p_2}\omega(y)dy\right)^{\frac{1}{p_2}}\left(\int_{C_k}\omega(y)^{1-p_2'}dy\right)^{\frac{1}{p_2'}}\right|^p \\ &\leq C\|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)}\|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)}\omega(B_{k_0})^{1+\lambda_1 p}\left|\sum_{k=k_0}^{\infty}\omega(B_k)^{\lambda_2}(k-k_0)\right|^p \\ &\leq C\|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)}\|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)}\omega(B_{k_0})^{1+\lambda p}\left|\sum_{k=k_0}^{\infty}2^{(k-k_0)n\delta\lambda_2}(k-k_0)\right|^p \\ &\leq C\|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)}\|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)}\omega(B_{k_0})^{1+\lambda p}.\end{aligned}$$

Summarizing, one has

$$II^{'} \leq C\|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)}\|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)}\omega(B_{k_0})^{1+\lambda p}.$$

Combining I' with II', we proved (3.2).

Next we prove $(ii)\Rightarrow(i)$. For a fixed ball $B=B(0,r)$, we assume b satisfies reverse Hölder condion (2.3), we just need to prove

$$\frac{1}{\omega(B)^{1+\lambda_1 p_1}}\int_B|b(x)-b_B|^{p_1}\omega(x)^{1-p_1}dx \leq C. \tag{3.3}$$

Indeed,

$$\begin{aligned}\frac{1}{\omega(B)^{1+\lambda_1 p_1}}&\int_B |b(x)-b_B|^{p_1}\omega(x)^{1-p_1}dx\\ &\le \omega(B)^{-1-\lambda_1 p_1}\int_B \omega(x)^{1-p_1}dx\big(\sup_{x\in B}|b(x)-b_B|\big)^{p_1}\\ &\le C\omega(B)^{-p_1-\lambda_1 p_1}|B|^{p_1}\left(\frac{1}{|B|}\int_B|b(x)-b_B|dx\right)^{p_1}\\ &\le C\omega(B)^{-\lambda_1 p_1-\frac{p_1}{p}}\left(\int_B|b(x)-b_B|^p\omega(x)^{1-p}dx\right)^{\frac{p_1}{p}}.\end{aligned}$$

Next we estimate $\int_B |b(x)-b_B|^p\omega(x)^{1-p}dx$,

$$\begin{aligned}\int_B |b(x)-b_B|^p\omega(x)^{1-p}dx &\le \frac{1}{|B|^p}\int_B\left|\int_B(b(x)-b(y))dy\right|^p\omega(x)^{1-p}dx\\ &\le \frac{1}{|B|^p}\int_B|x|^{np}\left|\frac{1}{|x|^n}\int_{|y|<|x|}(b(x)-b(y))\chi_B(y)dy\right|^p\omega(x)^{1-p}dx\\ &+\frac{1}{|B|^p}\int_B\left|\int_{|y|>|x|}|y|^n\frac{(b(x)-b(y))\chi_B(y)}{|y|^n}dy\right|^p\omega(x)^{1-p}dx\\ &=I+II.\end{aligned}$$

Considering I and II, respectively

$$\begin{aligned}I&\le\int_B|H_b\chi_B(x)|^p\omega(x)^{1-p}dx=\omega(B)^{1+\lambda p}\|H_b\chi_B\|^p_{\dot{M}^{p,\lambda}(\omega,\omega^{1-p})}\\ &\le C\omega(B)^{1+\lambda p}\|\chi_B\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)}\le C\omega(B)^{1+\lambda_1 p}.\end{aligned}$$

Similarly,

$$\begin{aligned}II&\le\int_B|H_b^*\chi_B(x)|^p\omega(x)^{1-p}dx=\omega(B)^{1+\lambda p}\|H_b^*\chi_B\|^p_{\dot{M}^{p,\lambda}(\omega,\omega^{1-p})}\\ &\le C\omega(B)^{1+\lambda p}\|\chi_B\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)}\le C\omega(B)^{1+\lambda_1 p}.\end{aligned}$$

Hence,

$$\frac{1}{\omega(B)^{1+\lambda_1 p_1}}\int_B|b(x)-b_B|^{p_1}\omega(x)^{1-p_1}dx\le C\omega(B)^{-\lambda_1 p_1-\frac{p_1}{p}}\left(C\omega(B)^{1+\lambda_1 p}\right)^{\frac{p_1}{p}}\le C.$$

Combining (3.1), (3.2) and (3.3), the proof of Theorem 3.1 is completed.

Proof of Theorem 3.2:

$(i)\Rightarrow(ii)$. For a fixed ball $B(0,r)=B_{k_0}$ with $k_0\in\mathbb{Z}$, we just need to prove that

$$\left(\frac{1}{\omega(B_{k_0})^{1+2\lambda p}}\int_{B_{k_0}}|H_bf(x)|^p\omega(x)^{1-p}dx\right)^{\frac{1}{p}}\le C\|f\|_{\dot{M}^{p,\lambda}(\omega,\omega)}\tag{3.4}$$

and

$$\left(\frac{1}{\omega(B_{k_0})^{1+2\lambda p}}\int_{B_{k_0}}|H_b^*f(x)|^p\omega(x)^{1-p}dx\right)^{\frac{1}{p}}\le C\|f\|_{\dot{M}^{p,\lambda}(\omega,\omega)}.\tag{3.5}$$

The Hölder's inequality and $1/p + 1/p' = 1$ show that

$$\int_{B_{k_0}} |H_b f(x)|^p \omega(x)^{1-p} dx = \int_{B_{k_0}} \left| \frac{1}{|x|^n} \int_{|y|<|x|} \big(b(x) - b(y)\big) f(y) dy \right|^p \omega(x)^{1-p} dx$$

$$\leq C \sum_{k=-\infty}^{k_0} \int_{C_k} \left| \frac{1}{|x|^n} \int_{B_k} |b(x) - b_{B_k}||f(y)| dy \right|^p \omega(x)^{1-p} dx$$

$$+ C \sum_{k=-\infty}^{k_0} \int_{C_k} \left| \frac{1}{|x|^n} \int_{B_k} |b(y) - b_{B_k}||f(y)| dy \right|^p \omega(x)^{1-p} dx = I + II.$$

For I, we show that

$$I \leq C \sum_{k=-\infty}^{k_0} 2^{-knp} \int_{B_k} |b(x) - b_{B_k}|^p \omega(x)^{1-p} dx \left| \sum_{i=-\infty}^{k} \int_{B_i} |f(y)| dy \right|^p$$

$$\leq C \|b\|^p_{\dot{C}^{p,\lambda}(\omega)} \sum_{k=-\infty}^{k_0} 2^{-knp} \omega(B_k)^{1+\lambda p} \left| \sum_{i=-\infty}^{k} \left(\int_{B_i} |f(y)|^p \omega(y) dy \right)^{\frac{1}{p}} \left(\int_{B_i} \omega(y)^{1-p'} dy \right)^{\frac{1}{p'}} \right|^p$$

$$\leq C \|b\|^p_{\dot{C}^{p,\lambda}(\omega)} \|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)} \sum_{k=-\infty}^{k_0} \omega(B_k)^{1+2\lambda p} \left| \sum_{i=-\infty}^{k} 2^{(i-k)n(1+\lambda)} \right|^p$$

$$\leq C \|b\|^p_{\dot{C}^{p,\lambda}(\omega)} \|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)} \omega(B_{k_0})^{1+2\lambda p}.$$

To get the boundedness for the term II, we require the following decomposition

$$II \leq C \sum_{k=-\infty}^{k_0} 2^{-knp} \int_{B_k} \left| \sum_{i=-\infty}^{k} \int_{B_i} |b(y) - b_{B_k}||f(y)| dy \right|^p \omega(x)^{1-p} dx$$

$$\leq C \sum_{k=-\infty}^{k_0} 2^{-knp} \int_{B_k} \left| \sum_{i=-\infty}^{k} \int_{B_i} |b(y) - b_{B_i}||f(y)| dy \right|^p \omega(x)^{1-p} dx$$

$$+ C \sum_{k=-\infty}^{k_0} 2^{-knp} \int_{B_k} \left| \sum_{i=-\infty}^{k} \int_{B_i} |b_{B_k} - b_{B_i}||f(y)| dy \right|^p \omega(x)^{1-p} dx$$

$$= II' + II''.$$

Discussing II' and II'', respectively, if $p > p'$, then

$$II' \leq C \sum_{k=-\infty}^{k_0} 2^{-knp} \int_{B_k} \left| \sum_{i=-\infty}^{k} \left(\int_{B_i} |b(y) - b_{B_i}|^{p'} \omega(y)^{1-p'} dy \right)^{\frac{1}{p'}} \right.$$

$$\left. \times \left(\int_{B_i} |f(y)|^p \omega(y) dy \right)^{\frac{1}{p}} \right|^p \omega(x)^{1-p} dx$$

$$\leq C \|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)} \sum_{k=-\infty}^{k_0} 2^{-knp} \int_{B_k} \left| \sum_{i=-\infty}^{k} \left(\int_{B_i} |b(y) - b_{B_i}|^p \omega(y)^{1-p} dy \right)^{\frac{1}{p}} \right.$$

$$\left. \times\, \omega(B_i)^{\frac{1}{p'}+\lambda} \right|^p \omega(x)^{1-p} dx$$

$$\leq C\|b\|^p_{\dot{C}^{p,\lambda}(\omega)}\|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)}\sum_{k=-\infty}^{k_0}\omega(B_k)^{1+2\lambda p}\left|\sum_{i=-\infty}^{k}2^{(i-k)n\delta(1+2\lambda)}\right|^p$$
$$\leq C\|b\|^p_{\dot{C}^{p,\lambda}(\omega)}\|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)}\omega(B_{k_0})^{1+2\lambda p}.$$

On the other hand, applying (2.1), (2.2) and $|b_{B_k}-b_{B_i}|\leq C(k-i)\|b\|_{\dot{C}^{p,\lambda}(\omega)}\frac{\omega(B_i)^{1+\lambda}}{|B_i|}$, we show

$$II''=C\sum_{k=-\infty}^{k_0}2^{-knp}\int_{B_k}\left|\sum_{i=-\infty}^{k}\int_{B_i}|b_{B_k}-b_{B_i}||f(y)|dy\right|^p\omega(x)^{1-p}dx$$
$$\leq C\|b\|^p_{\dot{C}^{p,\lambda}(\omega)}\sum_{k=-\infty}^{k_0}2^{-knp}\int_{B_k}\left|\sum_{i=-\infty}^{k}(k-i)\frac{\omega(B_i)^{1+\lambda}}{|B_i|}\left(\int_{B_i}|f(y)|^p\omega(y)dy\right)^{\frac{1}{p}}\right.$$
$$\left.\times\left(\int_{B_i}\omega(y)^{1-p'}dy\right)^{\frac{1}{p'}}\right|^p\omega(x)^{1-p}dx$$
$$\leq C\|b\|^p_{\dot{C}^{p,\lambda}(\omega)}\|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)}\sum_{k=-\infty}^{k_0}\omega(B_k)^{1-p}\left|\sum_{i=-\infty}^{k}(k-i)\omega(B_i)^{1+2\lambda}\right|^p$$
$$\leq C\|b\|^p_{\dot{C}^{p,\lambda}(\omega)}\|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)}\sum_{k=-\infty}^{k_0}\omega(B_k)^{1+2\lambda p}\leq C\|b\|^p_{\dot{C}^{p,\lambda}(\omega)}\|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)}\omega(B_{k_0})^{1+2\lambda p}.$$

If $p'>p$, we can obtain that $[b,H]$ are bounded from $\dot{M}^{p',\lambda}(\omega,\omega)$ to $\dot{M}^{p',2\lambda}(\omega,\omega^{1-p'})$.

By slightly modifying Theorem 3.1, we can obtain the proof of $[H^*,b]$. Here, we omit its proof for the similarity.

$(ii)\Rightarrow(i)$, case 1: $p>p'$, we want to get

$$\frac{1}{\omega(B)^{1+\lambda p}}\int_B|b(x)-b_B|^p\omega(x)^{1-p}dx\leq C. \tag{3.6}$$

Indeed,

$$\frac{1}{\omega(B)^{1+\lambda p}}\int_B|b(x)-b_B|^p\omega(x)^{1-p}dx$$
$$\leq\frac{1}{\omega(B)^{1+\lambda p}}\frac{1}{|B|^p}\int_B|x|^{np}\left|\frac{1}{|x|^n}\int_{|y|<|x|}(b(x)-b(y))\chi_B(y)dy\right|^p\omega(x)^{1-p}dx$$
$$+\frac{1}{\omega(B)^{1+\lambda p}}\frac{1}{|B|^p}\int_B\left|\int_{|y|>|x|}|y|^n\frac{(b(x)-b(y))}{|y|^n}\chi_B(y)dy\right|^p\omega(x)^{1-p}dx=K+L.$$

Considering K and L, respectively

$$K\leq\frac{\omega(B)^{1+2\lambda p}}{\omega(B)^{1+\lambda p}}\|H_b\chi_B\|^p_{\dot{M}^{p,2\lambda}(\omega,\omega^{1-p})}\leq C\omega(B)^{\lambda p}\|\chi_B\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)}\leq C.$$

Also,

$$L\leq\frac{\omega(B)^{1+2\lambda p}}{\omega(B)^{1+\lambda p}}\|H^*_b\chi_B\|^p_{\dot{M}^{p,2\lambda}(\omega,\omega^{1-p})}\leq C\omega(B)^{\lambda p}\|\chi_B\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)}\leq C.$$

Case 2: $p'>p$, with the $\left(\dot{M}^{p',\lambda}(\omega,\omega),\dot{M}^{p',2\lambda}(\omega,\omega^{1-p'})\right)$ boundedness of H_b and H^*_b, the similar arguments of case 1 can be applied to this and show that

$$\frac{1}{\omega(B)^{1+\lambda p'}}\int_B |b(x)-b_{B_{\gamma_0}}|^{p'}\omega(x)^{1-p'}dx \le C. \tag{3.7}$$

So, combining(3.4), (3.5), (3.6) and (3.7), The proof of Theorem 3.2 is completed.

The Proof of Theorem 3.3 is similar to that of Theorem 3.2.

4. Weighted Lipschitz estimates

Definition 4.1 [13]. Let $1 \le p \le \infty$, $0 < \beta < 1$, and $\omega \in A_\infty$, a locally integrable function f is said to belong to the weighted Lipschitz space $Lip^p_{\beta,\omega}$ if

$$\|f\|_{Lip^p_{\beta,\omega}} = \sup_B \frac{1}{\omega(B)^{\frac{\beta}{n}}}\left(\frac{1}{\omega(B)}\int_B |f(x)-f_B|^p\omega(x)^{1-p}dx\right)^{\frac{1}{p}} < \infty.$$

Modulo constants, the Banach space of such functions is denoted by $Lip^p_{\beta,\omega}$. Put $Lip_{\beta,\omega} = Lip^1_{\beta,\omega}$, obviously, if $\omega = 1$, then the $Lip_{\beta,\omega}$ is the classical Lipschitz space Lip_β, if $\omega \in A_1$, J. García-Cuerva [13] proved that the spaces $Lip^p_{\beta,\omega}$ coincide, and the norm of $Lip^p_{\beta,\omega}$ are equivalent with respect to different values of provided that $1 \le p \le \infty$. That is $Lip^p_{\beta,\omega} \sim Lip_{\beta,\omega}$ where $1 \le p \le \infty$.

Theorem 4.1. Let $\omega \in A_1(\mathbb{R}^n)$, $1 < p < \infty$, $-\frac{1}{p} < \lambda < \lambda_1 < 0$, $0 < \beta < 1$, and $\lambda_1 = \lambda + \beta/n$, $b \in Lip_{\beta,\omega}$, then commutators $[b,H]$ and $[b,H^*]$ are bounded from $\dot{M}^{p,\lambda}(\omega,\omega)$ to $\dot{M}^{p,\lambda_1}(\omega,\omega^{1-p})$.

Proof of Theorem 4.1. For a fixed ball $B(0,r) = B_{k_0}$ with $k_0 \in \mathbb{Z}$, applying (2.1), (2.2) and Hölder's inequality,

$$\int_{B_{k_0}} |H_b f(x)|^p\omega(x)^{1-p}dx = \int_{B_{k_0}}\left|\frac{1}{|x|^n}\int_{|y|<|x|}\big(b(x)-b(y)\big)f(y)dy\right|^p\omega(x)^{1-p}dx$$

$$\le C\sum_{k=-\infty}^{k_0}\int_{C_k}\left|\frac{1}{|x|^n}\int_{B_k}|b(x)-b_{B_k}||f(y)|dy\right|^p\omega(x)^{1-p}dx$$

$$+ C\sum_{k=-\infty}^{k_0}\int_{C_k}\left|\frac{1}{|x|^n}\int_{B_k}|b(y)-b_{B_k}||f(y)|dy\right|^p\omega(x)^{1-p}dx = I + II.$$

We firstly prove I,

$$I \le C\sum_{k=-\infty}^{k_0}2^{-knp}\int_{B_k}|b(x)-b_{B_k}|^p\omega(x)^{1-p}dx\left|\sum_{i=-\infty}^{k}\int_{B_i}|f(y)|dy\right|^p$$

$$\le C\|b\|^p_{Lip_{\beta,\omega}}\sum_{k=-\infty}^{k_0}2^{-knp}\omega(B_k)^{1+\frac{\beta p}{n}}\left|\sum_{i=-\infty}^{k}\left(\int_{B_i}|f(y)|^p\omega(y)dy\right)^{\frac{1}{p}}\left(\int_{B_i}\omega(y)^{1-p'}dy\right)^{\frac{1}{p'}}\right|^p$$

$$\le C\|b\|^p_{Lip_{\beta,\omega}}\|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)}\sum_{k=-\infty}^{k_0}\omega(B_k)^{1+\lambda_1 p}\left|\sum_{i=-\infty}^{k}2^{(i-k)n(1+\lambda)}\right|^p$$

$$\le C\|b\|^p_{Lip_{\beta,\omega}}\|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)}\omega(B_{k_0})^{1+\lambda_1 p}.$$

Breaking II into two parts:

$$\begin{aligned}
II &\leq C \sum_{k=-\infty}^{k_0} 2^{-knp} \int_{B_k} \left| \sum_{i=-\infty}^{k} \int_{B_i} |b(y) - b_{B_k}||f(y)|dy \right|^p \omega(x)^{1-p} dx \\
&\leq C \sum_{k=-\infty}^{k_0} 2^{-knp} \int_{B_k} \left| \sum_{i=-\infty}^{k} \int_{B_i} |b(y) - b_{B_i}||f(y)|dy \right|^p \omega(x)^{1-p} dx \\
&\quad + C \sum_{k=-\infty}^{k_0} 2^{-knp} \int_{B_k} \left| \sum_{i=-\infty}^{k} \int_{B_i} |b_{B_k} - b_{B_i}||f(y)|dy \right|^p \omega(x)^{1-p} dx = II' + II''.
\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}
II' &\leq C \sum_{k=-\infty}^{k_0} 2^{-knp} \int_{B_k} \left| \sum_{i=-\infty}^{k} \left(\int_{B_i} |b(y) - b_{B_i}|^{p'} \omega(y)^{1-p'} dy \right)^{\frac{1}{p'}} \right. \\
&\qquad \times \left. \left(\int_{B_i} |f(y)|^p \omega(y) dy \right)^{\frac{1}{p}} \right|^p \omega(x)^{1-p} dx \\
&\leq C\|b\|^p_{Lip_{\beta,\omega}} \|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)} \sum_{k=-\infty}^{k_0} 2^{-knp} \int_{B_k} \omega(x)^{1-p} dx \left| \sum_{i=-\infty}^{k} \omega(B_i)^{1+\lambda_1} \right|^p \\
&\leq C\|b\|^p_{Lip_{\beta,\omega}} \|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)} \sum_{k=-\infty}^{k_0} \omega(B_k)^{1+\lambda_1 p} \leq C\|b\|^p_{Lip_{\beta,\omega}} \|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)} \omega(B_{k_0})^{1+\lambda_1 p}.
\end{aligned}$$

On the other hand, applying $|b_{B_k} - b_{B_i}| \leq C(k-i)\|b\|_{Lip_{\beta,\omega}} \omega(B_k)^{\frac{\beta}{n}} \frac{\omega(B_i)}{|B_i|}$,

$$\begin{aligned}
II'' &= C \sum_{k=-\infty}^{k_0} 2^{-knp} \int_{B_k} \left| \sum_{i=-\infty}^{k} \int_{B_i} |b_{B_k} - b_{B_i}||f(y)|dy \right|^p \omega(x)^{1-p} dx \\
&\leq C\|b\|^p_{Lip_{\beta,\omega}} \sum_{k=-\infty}^{k_0} 2^{-knp} \omega(B_k)^{\frac{\beta p}{n}} \int_{B_k} \left| \sum_{i=-\infty}^{k} (k-i) \frac{\omega(B_i)}{|B_i|} \left(\int_{B_i} |f(y)|^p \omega(y) dy \right)^{\frac{1}{p}} \right. \\
&\quad \times \left. \left(\int_{B_i} \omega(y)^{1-p'} dy \right)^{\frac{1}{p'}} \right|^p \omega(x)^{1-p} dx \\
&\leq C\|b\|^p_{Lip_{\beta,\omega}} \|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)} \sum_{k=-\infty}^{k_0} \omega(B_k)^{1-p+\frac{\beta p}{n}} \left| \sum_{i=-\infty}^{k} (k-i) \omega(B_i)^{1+\lambda} \right|^p \\
&\leq C\|b\|^p_{Lip_{\beta,\omega}} \|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)} \sum_{k=-\infty}^{k_0} \omega(B_k)^{1+\lambda_1 p} \leq C\|b\|^p_{Lip_{\beta,\omega}} \|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)} \omega(B_{k_0})^{1+\lambda_1 p}.
\end{aligned}$$

By slightly modifying Theorem 3.1, we can get the proof of $[H^*, b]$. Here, we omit its proof.

Remark 4.1. Since $\|b\|_{\dot{C}^{p,\lambda}(\omega)} \leq \|b\|_{C^{p,\lambda}(\omega)}$ and $\|b\|_{C^{p,\lambda}(\omega)} \sim \|b\|_{Lip_{\beta,\omega}}$ when $\omega \in A_1$ and $0 < \lambda < 1/n$, $b \in Lip_{\beta,\omega}$ is a sufficient condition for the boundedness of the $[b, H]$ and $[b, H^*]$ rather than a necessary condition. However, if $b \in \dot{C}^{p,\lambda}(\omega)$, it is still a necessary and sufficient condition.

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