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MULTIPLICITY OF SOLUTIONS FOR A FRACTIONAL LAPLACIAN EQUATION INVOLVING A PERTURBATION

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Abstract. A fractional Laplacian equation involving a perturbation is investigated. Under certain conditions, we obtain at least two solutions to this equation.

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1. Introduction

Fractional Laplacian equations have been applied to many subjects, such as, anomalous diffusion, elliptic problems with measure data, gradient potential theory, minimal surfaces, non-uniformly elliptic problems, optimization, phase transitions, quasigeostrophic flows, singular set of minima of variational functionals, and water waves (see [2]-[11] and the references therein). Fractional Brezis-Nirenberg problems had been investigated by many researchers (such as [2, 10]).

$$\begin{cases} (-\Delta)^s u + \lambda u = |u|^{2_s^*-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $0 < s < 1, N > 2s$, $2_s^* := \frac{2N}{N-2s}$ is the fractional Sobolev critical exponent, Ω is an open bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, and the fractional Laplacian is defined by

$$-(-\Delta)^s u(x) = \frac{C_{N,s}}{2}\int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}\mathrm{d}y, \quad x \in \mathbb{R}^N,$$

$$C_{N,s} = \left(\int_{\mathbb{R}^N} \frac{1-\cos(\zeta_1)}{|\zeta|^{N+2s}}\mathrm{d}\zeta\right)^{-1}. \tag{1.1}$$

Define Hilbert space $D^s(\Omega)$ as the completion of $C_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{D^s}$ induced by the following scalar product

$$\langle u, v\rangle_{D^s} := \frac{C_{N,s}}{2}\int_{\mathbb{R}^{2N}} \frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y.$$

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If Ω is an open bounded Lipschitz domain, then $D^s(\Omega)$ coincides with the Sobolev space

$$X_0 := \{f \in X : f = 0 \text{ a.e. in } \Omega^c\},$$

where X is a linear space of Lebesgue measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that the restriction to Ω of any function f in X belongs to $L^2(\Omega)$ and the map $(x,y) \mapsto \big(f(x) - f(y)\big)|x-y|^{-\frac{N}{2}+s}$ is in $L^2\left(\mathbb{R}^{2N} \setminus (\Omega^c \times \Omega^c),\ \mathrm{d}x\mathrm{d}y\right)$, and Ω^c is the complement of Ω in $\mathbb{R}^N$. Consider fractional Sobolev space

$$H^s(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{2}+s}} \in L^2(\mathbb{R}^{2N}) \right\},$$

equiped the Gagliardo seminorm

$$[u]^2_{H^s(\mathbb{R}^N)} := \frac{C_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y.$$

The fractional Laplacian operator can be defined by

$$\begin{aligned}
(-\Delta)^s u(x) &= C_{N,s}\mathrm{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \mathrm{d}y \\
&= C_{N,s} \lim_{\varepsilon \to 0^+} \int_{B^c_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \mathrm{d}y \\
&= -\frac{1}{2} C_{N,s} \int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}} \mathrm{d}y,
\end{aligned}$$

where $C_{N,s}$ is given by (1.1) and P.V. is the principle value defined by the latter formula. Define the fractional Sobolev space

$$H^s(\Omega) := \left\{ x \in H^s(\mathbb{R}^N) : u = 0 \ \text{ a.e. in } \Omega^c \right\},$$

equipped with the seminorm

$$\|u\|_{H^s(\Omega)} := \left(\lambda \int_\Omega |u|^2 \mathrm{d}x + \frac{C_{N,s}}{2} \int_{\mathbb{R}^{2N} \setminus (\Omega^c \times \Omega^c)} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}},$$

which was introduced in [10]. From $u = 0$ a.e. in Ω^c, it is easy to see that

$$|u|_2^2 := \int_\Omega |u|^2 \mathrm{d}x = \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x,$$

$$\int_{\mathbb{R}^{2N} \setminus (\Omega^c \times \Omega^c)} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y.$$

Hence, we just denote $\|u\|_{H^s(\Omega)}$ by

$$\|u\|_{H^s} := \left(\lambda \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x + \frac{C_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}}.$$

It follows from Lemma 7 in [8] that $\big(H^s(\Omega), \|\cdot\|_{H^s}\big)$ is a Hilbert space.

In present paper, we study the following fractional Laplacian equation involving a perturbation

$$
(1.2)\qquad \begin{cases} (-\Delta)^s u + \lambda u = |u|^{p-2}u + h(x) & \text{in} \quad \Omega, \\ u = 0 & \text{in} \quad \Omega^c, \end{cases}
$$

where $0 < s < 1$, λ is a real parameter, $p \in (2, 2_s^*)$, $h \in L^2(\Omega)$, and $\Omega \subset \mathbb{R}^N$ is an open bounded Lipschitz domain. Via classic methods (see [1] for example), we obtain multiplicity of solutions for fractional Laplacian equation (1.2). The solutions of equation (1.2) coincide with the critical points of the following energy functional

$$
\begin{aligned}
J(u) &= \frac{1}{2}\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y + \frac{\lambda}{2}\int_\Omega |u|^2 \mathrm{d}x - \frac{1}{p}\int_\Omega |u|^p \mathrm{d}x - \int_\Omega hu\mathrm{d}x \\
&= \frac{1}{2}\|u\|_{H^s}^2 - \frac{1}{p}|u|_p^p - \int_\Omega hu\mathrm{d}x, \quad \forall u \in H^s(\Omega).
\end{aligned}
$$

If $h \equiv 0$, then equation (1.2) becomes

$$
(1.3)\qquad \begin{cases} (-\Delta)^s + \lambda u = |u|^{p-2}u & \text{in} \quad \Omega, \\ u = 0 & \text{in} \quad \Omega^c. \end{cases}
$$

Define the energy functional of equation (1.3) and corresponding Nehari manifold as follows:

$$
I(u) = \frac{1}{2}\|u\|_{H^s}^2 - \frac{1}{p}|u|_p^p, \quad \forall u \in H^s(\Omega),
$$

and

$$
\mathcal{N} = \{u \in H^s(\Omega) : u \neq 0, I'(u)u = 0\} = \left\{u \in H^s(\Omega) : u \neq 0, \|u\|_{H^s}^2 = |u|_p^p\right\}.
$$

Our main result reads as follows.

Theorem 1.1. *There exists $\epsilon > 0$ such that for every $h \in L^2(\Omega)$ with $|h|_2 \leqslant \epsilon$, equation (1.2) has at least two solutions.*

2. The proof of Theorem 1.1

We need the following fractional Sobolev embedding results, which was proved in [8].

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded Lipschitz domain. Then $H^s(\Omega) \hookrightarrow L^q(\Omega)$ for every $q \in [1, 2_s^*]$, and $H^s(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for every $q \in [1, 2_s^*)$.*

From Lemma 2.1, we can define a constant S_p.

$$
S_p := \inf\{C > 0 : |u|_p \leqslant C\|u\|_{H^s}, \forall u \in H^s(\Omega)\}.
$$

Next, we give some numbers which will be used in the proof.

$$
a_1 = \left(\frac{1}{(p-1)S_p^p}\right)^{\frac{1}{p-2}}, a_2 = \left(\frac{1}{2}a_1^2\right)^{\frac{1}{p}}, a_3 = \frac{1}{2}\min\left\{a_1, \frac{a_2}{S_p}\right\}.
$$

It is easy to find that $a_3 < a_1$.

Lemma 2.2. *There exists $\epsilon_1 > 0$ such that for every $h \in L^2(\Omega)$ with $|h|_2 \leqslant \epsilon_1$ and for every $u \in H^s(\Omega)$, if*

$$\|u\|_{H^s}^2 = \int_\Omega |u|^p dx + \int_\Omega hu dx = |u|_p^p + \int_\Omega hu dx, \tag{2.1}$$

then either $\|u\|_{H^s} > a_1$ and $|u|_p \geqslant a_2$ or $\|u\|_{H^s} < a_3$.

Proof. It follows from (2.1) that

$$\|u\|_{H^s}^2 \leqslant S_p^p \|u\|^p + |h|_2 |u|_2.$$

By Lemma 2.1, we get $|u|_2 \leqslant C_1 \|u\|_{H^s}$. Then, $\|u\|_{H^s}^2 \leqslant S_p^p \|u\|^p + C_1 |h|_2 \|u\|_{H^s}$. If $u \neq 0$ in $H^s(\Omega)$, then

$$\|u\|_{H^s} - S_p^p \|u\|^{p-1} - C_1 |h|_2 \leqslant 0.$$

For calculation convenience, we define function ϕ: $[0, +\infty) \to \mathbb{R}$ by

$$\phi(t) = t - S_p^p t^{p-1} - C_1 |h|_2.$$

Since $\phi'(t) = 1 - (p-1) S_p^p t^{p-2}$, we get the maxinmum point of ϕ as $a_1 = \left((p-1) S_p^p\right)^{-\frac{1}{p-2}}$. It is easy to see that ϕ is strictly increasing on $(0, a_1)$, strictly decreasing on $(a_1, +\infty)$ and $\phi(0) < 0$, $\lim_{t \to +\infty} \phi(t) = -\infty$.

In order to observe the characteristics of the function ϕ, we calculate the maximum value of ϕ,

$$\begin{aligned}
\phi(a_1) &= \left(\frac{1}{(p-1) S_p^p}\right)^{\frac{1}{p-2}} - S_p^p \left(\frac{1}{(p-1) S_p^p}\right)^{\frac{p-1}{p-2}} - C|h|_2 \\
&= \left(\frac{1}{p-1}\right)^{\frac{1}{p-2}} \left(\frac{1}{S_p^p}\right)^{\frac{1}{p-2}} - \left(\frac{1}{p-1}\right)^{1+\frac{1}{p-2}} \left(\frac{1}{S_p^p}\right)^{\frac{1}{p-2}} - C|h|_2 \\
&= \left(\frac{1}{p-1}\right)^{\frac{1}{p-2}} \left(\frac{1}{S_p^p}\right)^{\frac{1}{p-2}} \left(1 - \frac{1}{p-1}\right) - C|h|_2 \\
&= \left(\frac{1}{p-1}\right)^{\frac{1}{p-2}} \left(\frac{1}{S_p^p}\right)^{\frac{1}{p-2}} \frac{p-2}{p-1} - C|h|_2 =: \alpha_1 - C|h|_2,
\end{aligned}$$

and if we take $|h|_2 \leqslant \frac{\alpha_1}{2C}$, then

$$\phi(a_1) \geqslant \alpha_1 - C\frac{\alpha_1}{2C} = \alpha_1 - \frac{\alpha_1}{2} = \frac{\alpha_1}{2} > 0,$$

which means the function ϕ has two zeros t_1, t_2 and $t_1 < a_1 < t_2$. Then $\phi(t) > 0$ for all $t \in (t_1, t_2)$, while $\phi(t) < 0$ for all $t \in [0, t_1) \cup (t_2, +\infty)$. Substituting t_1 into the function ϕ, we get that

$$C|h|_2 = t_1 - S_p^p t_1^{p-1} = t_1 \left(1 - S_p^p t_1^{p-2}\right).$$

Since $t_1 < a_1$, we have

$$C|h|_2 \geqslant t_1 \left(1 - S_p^p a_1^{p-2}\right) = t_1 \left(1 - \frac{1}{p-1}\right) = t_1 \frac{p-2}{p-1},$$

i.e., $t_1 \leqslant \frac{p-1}{p-2} C|h|_2$. If we take

$$|h|_2 < \frac{p-1}{p-1} \frac{a_3}{C},$$

then

$$t_1 < \frac{p-1}{p-2} \frac{p-2}{p-1} \frac{a_3}{C} = a_3.$$

In summary, for

$$|h|_2 < \min \left\{ \frac{p-2}{p-1} \frac{a_3}{C}, \frac{\alpha_1}{2C} \right\},$$

we get $\phi \leqslant 0$ implies $t < a_3$ or $t > a_1$. If (2.1) hold and $\|u\|_{H^s} > a_1$, we get

$$|u|_p^p = \|u\|_{H^s}^2 - \int_\Omega hu\mathrm{d}x \geqslant a_1^2 - |h|_2 |u|_2 \geqslant a_1^2 - a|h|_2 |u|_p,$$

where $a = |\Omega|^{\frac{p-2}{2p}}$. Namely

$$|u|_p^p + a|h|_2 |u|_p - a_1^2 \geqslant 0. \tag{2.2}$$

Regarding $|u|_p$ as a variable, we get a function $\gamma : [0, +\infty) \to \mathbb{R}$, defined by

$$\gamma(t) = t^p + a|h|_2 t - a_1^2.$$

Since $\gamma'(t) = pt^{p-1} + a|h|_2 > 0$, for all $t > 0$, γ is strictly increasing. Therefore, if

$$|h|_2 < \frac{a_1^2}{2aa_2},$$

then

$$\begin{aligned} \gamma(a_2) &= a_2^p + a|h|_2 a_2 - a_1^2 = \frac{1}{2} a_1^2 + a|h|_2 a_2 - a_1^2 \\ &= a|h|_2 a_2 - \frac{1}{2} a_1^2 < a \frac{a_1^2}{2aa_2} a_2 - \frac{1}{2} a_1^2 = 0. \end{aligned}$$

We see that $\gamma(t) < 0$ for $t \in [0, a_2]$. By (2.2) we derive that $|u|_p \geqslant a_2$.

Summing up, if we choose

$$\epsilon_1 = \min \left\{ \frac{p-2}{p-1} \frac{a_3}{C}, \frac{\alpha_1}{2C}, \frac{a_1^2}{2aa_2} \right\},$$

then Lemma 2.2 holds. □

In the sequel, we always assume $|h|_2 < \epsilon_1$. Now define

$$\begin{aligned} \mathcal{N}_h &:= \{ u \in H^s(\Omega) : J'(u) u = 0, \|u\|_{H^s} > a_1 \} \\ &= \left\{ u \in H^s(\Omega) : \|u\|_{H^s}^2 = |u|_p^p + \int_\Omega hu\mathrm{d}x, \|u\|_{H^s} > a_1 \right\}, \end{aligned}$$

and $m_h = \inf\limits_{u \in \mathcal{N}_h} J(u)$. Notice that $\mathcal{N}_h$ is a subset of Nehari mainfold and for $u \in \mathcal{N}_h$, we have

$$J(u) = \left(\frac{1}{2} - \frac{1}{p} \right) \|u\|_{H^s}^2 - \left(1 - \frac{1}{p} \right) \int_\Omega hu\mathrm{d}x.$$

Now, we prove that $\mathcal{N}_h$ is not empty.

Lemma 2.3. *There exists $\epsilon_2 \in (0, \epsilon_1]$ such that for every $h \in L^2(\Omega)$ with $|h|_2 \leqslant \epsilon_2$, there results $\mathcal{N}_h \neq 0$.*

Proof. Consider function

$$t \mapsto J'(tu)\,tu = t^2\|u\|_{H^s}^2 - t^p \int_\Omega |u|^p \mathrm{d}x - t\int_\Omega hu\mathrm{d}x$$
$$= t\left[t\|u\|^2 - t^{p-1}|u|_p^p - \int_\Omega hu\mathrm{d}x\right],$$

where $u \in H^s(\Omega) \setminus \{0\}$, $t \in (0,+\infty)$. Since $t>0$, we only consider the following function

$$\gamma(t) = t\|u\|_{H^s}^2 - t^{p-1}|u|_p^p - \int_\Omega hu\mathrm{d}x,$$

since $p \in (2, 2_s^*)$, the function γ has a global maximum. Solving

$$\gamma'(t) = \|u\|_{H^s}^2 - (p-1)\,t^{p-2}|u|_p^p = 0,$$

we have the function γ has a global maximum at

$$t' = \left(\frac{\|u\|_{H^s}^2}{(p-1)|u|_p^p}\right)^{\frac{1}{p-2}},$$

and

$$\gamma(t') = \frac{\|u\|_{H^s}^{\frac{2(p-1)}{p-2}}}{|u|_p^{\frac{p}{p-2}}}\frac{1}{(p-1)^{\frac{1}{p-2}}} - \int_\Omega hu\mathrm{d}x =: \frac{\|u\|_{H^s}^{\frac{2(p-1)}{p-2}}}{|u|_p^{\frac{p}{p-2}}}\alpha - \int_\Omega hu\mathrm{d}x$$
$$\geqslant \frac{\|u\|_{H^s}^{\frac{2(p-1)}{p-2}}}{\|u\|_p^{\frac{p}{p-2}} S_p^{\frac{p}{p-2}}}\alpha - \int_\Omega hu\mathrm{d}x \geqslant \|u\|_{H^s}\frac{1}{S_p^{\frac{p}{p-2}}}\alpha - C|h|_2\|u\|$$
$$= \|u\|_{H^s}\left(\frac{\alpha}{S_p^{\frac{p}{p-2}}} - C|h|_2\right).$$

Thus, if

$$|h|_2 \leqslant \frac{\alpha}{2CS_p^{\frac{p}{p-2}}},$$

there results $\gamma(t') > 0$. Moreover, $\gamma(t)$ is strictly increasing in $(0,t')$, strictly decreasing in $(t',+\infty)$ and $\lim_{t\to+\infty}\gamma(t) = -\infty$. Then the function γ has at least one zero $t_1 \in (t',+\infty)$. Then there exists $v = t_1 u$ satisfies (2.1). Next, we verify that v satisfies $\|v\|_{H^s} > a_1$, we get $v \in \mathcal{N}_h$. Since

$$\|v\|_{H^s} = \|t_1 u\|_{H^s} = t_1\|u\|_{H^s} > t'\|u\|_{H^s} = \left(\frac{\|u\|_{H^s}^2}{(p-1)\,|u|_p^p}\right)^{\frac{1}{p-2}}$$
$$= \|u\|_{H^s}^{\frac{p}{p-2}}\left(\frac{1}{p-1}\right)^{\frac{1}{p-2}}\left(\frac{1}{|u|_p^p}\right)^{\frac{1}{p-2}}$$
$$\geqslant (|u|_p)^{\frac{p}{p-2}}\left(\frac{1}{S_p}\right)\left(\frac{1}{p-1}\right)^{\frac{1}{p-2}}\left(\frac{1}{|u|_p}\right)^{\frac{p}{p-2}}$$
$$= \left(\frac{1}{p-1}\right)^{\frac{1}{p-2}}\left(\frac{1}{S_p}\right)^{\frac{p}{p-2}} = a_1,$$

the proof is completed with $\epsilon_2 = \min\left\{\epsilon_1, \frac{\alpha}{2CS_p^{\frac{p}{p-2}}}\right\}$. □

We now show that m_h are uniformly bounded from above and below by three Lemmas below.

Lemma 2.4. *Let $\epsilon_3 = \min\{1, \epsilon_2\}$. Then there exists $C > 0$ such that for every $|h|_2 < \epsilon_3$, there results $m_h \leqslant C$.*

Proof. Denote u_0 and m_0 as the solution and the level of the solution of equation (1.2), that is, $u_0 \in \mathcal{N}$, $I(u_0) = \min_{u\in\mathcal{N}} I(u) = m_0$. Due to Lemma 2.3, letting $|h|_2 < \epsilon_3$, there exists $t_h > 0$ such that $t_h u_0 \in \mathcal{N}_h$. Then

$$\|t_h u_0\|_{H^s}^2 = \int_\Omega |t_h u_0|^p \mathrm{d}x + \int_\Omega hu\mathrm{d}x. \tag{2.3}$$

Noticing $u_0 \in \mathcal{N}$, i.e., $\|u_0\|_{H^s}^2 = |u_0|_p^p$, (2.3) is equivalent to

$$\left(t_h^2 - t_h^p\right)\|u_0\|_{H^s}^2 = t_h \int_\Omega hu_0\mathrm{d}x,$$

namely,

$$\left(t_h - t_h^{p-1}\right)\|u_0\|_{H^s}^2 = \int_\Omega hu_0\mathrm{d}x,$$

which implies that

$$\left(t_h - t_h^{p-1}\right)\|u_0\|_{H^s}^2 \geqslant -C_1|h|_2\|u_0\|_{H^s},$$

that is

$$t_h - t_h^{p-1} \geqslant -\frac{C_1|h|_2}{\|u_0\|_{H^s}} \geqslant -\frac{C_1}{\|u_0\|_{H^s}}. \tag{2.4}$$

Consider function $\phi : t \mapsto t - t^{p-1}$. Since $\lim_{t\to+\infty} \phi(t) = -\infty$, there exists $C_2 > 0$ there $t_h \leqslant C_2$, and then

$$\begin{aligned} m_h \leqslant J(t_h u_0) &= \left(\frac{1}{2} - \frac{1}{p}\right)\|t_h u_0\|_{H^s}^2 - \left(1 - \frac{1}{p}\right)\int_\Omega hu\mathrm{d}x \\ &\leqslant \left(\frac{1}{2} - \frac{1}{p}\right)C_2^2\|u_0\|_{H^s}^2 + \left(1 - \frac{1}{p}\right)C_2C_1|h|_2\|u_0\|_{H^s} \\ &\leqslant \left(\frac{1}{2} - \frac{1}{p}\right)C_2^2\|u_0\|_{H^s}^2 + \left(1 - \frac{1}{p}\right)C_2C_1\|u_0\|_{H^s} =: C. \end{aligned}$$

□

To prove that m_h are uniform bound from below, we need a related Lemma.

Lemma 2.5. *For h that satisfies the condition in Lemma 2.4, there exists a normal number C_3 and a minimizing sequence $\{u_k\}_k$ for m_h such that $\|u_k\|_{H^s} \leqslant C_3$, and $|u_k|_p \leqslant S_pC_3$ for all k .*

Proof. Let $\{v_k\}_k$ be a minimizing sequence for m_h, i.e., $v_k \in \mathcal{N}_h$ and $J(v_k) \to m_h$ since $m_h \leqslant C$, there exists k' such that for every $k \geqslant k'$, $J(v_k) \leqslant 2C$. Then

$$\begin{aligned} 2C \geqslant J(v_k) &= \left(\frac{1}{2}-\frac{1}{p}\right)\|v_k\|_{H^s}^2 - \left(1-\frac{1}{p}\right)\int_\Omega h v_k \mathrm{d}x \\ &\geqslant \left(\frac{1}{2}-\frac{1}{p}\right)\|v_k\|_{H^s}^2 - \left(1-\frac{1}{p}\right) C_1 |h|_2 \|v_k\|_{H^s} =: a\|v_k\|_{H^s}^2 - b\|v_k\|_{H^s}. \end{aligned}$$

We get

$$\frac{b+\sqrt{b^2+8ac}}{2a} =: C_3,$$

and $|v_k|_p \leqslant S_p \|v_k\|_{H^s} = S_p C_3$, where $u_k = v_{k'+k}$. □

The preparation work has been completed. Now we prove the boundness from below.

Lemma 2.6. *There exists $\epsilon_4 \in (0, \epsilon_3]$ such that if $|h|_2 < \epsilon_4$, then $m_h \geqslant \frac{1}{2} m_0 > 0$.*

Proof. We consider $\{u_k\}_k$ obtained in Lemma 2.5. Let t_k be such that $t_k u_k \in \mathcal{N}$, which is equivalent to

$$\|t_k u_k\|_{H^s}^2 = \int_\Omega |t_k u_k|^p \mathrm{d}x,$$

namely,

$$t_k^2 \|u_k\|_{H^s}^2 = t_k^p \int_\Omega |u_k|^p \mathrm{d}x,$$

i.e. ,

$$t_k = \left(\frac{\|u_k\|_{H^s}^2}{|u_k|_p^p}\right)^{\frac{1}{p-2}}.$$

Since $u_k \in \mathcal{N}_h$, we have

$$\|u_k\|_{H^s}^2 = |u_k|_p^p + \int_\Omega h u_k \mathrm{d}x.$$

Then

$$t_k = \left(\frac{|u_k|_p^p + \int_\Omega h u_k \mathrm{d}x}{|u_k|_p^p}\right)^{\frac{1}{p-2}} = \left(1 + \frac{\int_\Omega h u_k \mathrm{d}x}{|u_k|_p^p}\right)^{\frac{1}{p-2}},$$

and

$$\begin{aligned} m_0 \leqslant I(t_k u_k) &= \left(\frac{1}{2}-\frac{1}{p}\right) t_k^2 \|u_k\|_{H^s}^2 \\ &= \left(\frac{1}{2}-\frac{1}{p}\right) t_k^2 \|u_k\|_{H^s}^2 - \left(1-\frac{1}{p}\right) t_k^2 \int_\Omega h u_k \mathrm{d}x + \left(1-\frac{1}{p}\right) t_k^2 \int_\Omega h u_k \mathrm{d}x \qquad (2.5) \\ &= t_k^2 J(u_k) + \left(1-\frac{1}{p}\right) t_k^2 \int_\Omega h u_k \mathrm{d}x. \end{aligned}$$

By Lemma 2.2 and Lemma 2.5, we have

$$t_k = \left(1 + \frac{\int_\Omega h u_k \mathrm{d}x}{|u_k|_p^p}\right)^{\frac{1}{p-2}} \leqslant \left(1 + \frac{C_1 |h|_2 \|u_k\|_{H^s}}{|u_k|_p^p}\right)^{\frac{1}{p-2}} \leqslant \left(1 + \frac{C_1 C_3 |h|_2}{a_2^p}\right)^{\frac{1}{p-2}}.$$

If

$$|h|_2 \leqslant \frac{a_2^p}{C_1C_3}\left[\left(\frac{3}{4}\right)^{\frac{p-2}{2}} - 1\right],$$

then

$$t_k \leqslant \left(\frac{C_1C_3}{a_2^p}\frac{a_2^p}{C_1C_3}\left[\left(\frac{3}{4}\right)^{\frac{p-2}{p}} - 1\right]\right)^{\frac{1}{p-2}} = \left(\frac{3}{4}\right)^{\frac{1}{2}}.$$

Now we consider (2.5)

$$\left|(1-\frac{1}{p})t_k^2\int_\Omega hu_k\mathrm{d}x\right| \leqslant \left(1-\frac{1}{p}\right)t_k^2|h|_2C_1\|u_k\|_{H^s} \leqslant \frac{3}{4}|h|_2\left(1-\frac{1}{p}C_1C_3\right).$$

If we take

$$|h|_2 < \frac{4m_0}{9\left(1-\frac{1}{p}\right)C_1C_3},$$

then

$$\left|(1-\frac{1}{p})t_k^2\int_\Omega hu_k\mathrm{d}x\right| \leqslant \frac{m_0}{3}.$$

Then we can write $m_0 \leqslant t_k^2 J\left(u_k\right) + \frac{m_0}{3}$, i.e., $t_k^2 J\left(u_k\right) \geqslant \frac{2}{3}m_0$. Since

$$J\left(u_k\right) > 0,\ t_k^2 \leqslant \frac{4}{3},$$

we get that $\frac{2}{3}m_0 \leqslant t_k^2 J\left(u_k\right) \leqslant \frac{4}{3}J\left(u_k\right)$, i.e.,

$$\frac{1}{2}m_0 \leqslant J\left(u_k\right), as\ k \to \infty, \tag{2.6}$$

which implies that $\frac{1}{2}m_0 \leqslant m_h$. If we choose

$$\epsilon_4 = \min\left\{\epsilon_3, \frac{a_2^p}{C_1C_3}\left[\left(\frac{4}{3}\right)^{\frac{p-2}{2}} - 1\right], \frac{m_0}{a\left(1-\frac{1}{p}\right)C_1C_3}\right\},$$

then Lemma 2.6 holds. □

The next thing to prove is an important part of the theorem, namely the minimum of J on $\mathcal{N}_h$ is attained.

Lemma 2.7. *There exists $\epsilon_5 \in (0, \epsilon_4]$ such that for every $|h|_2 < \epsilon_5$, m_h is attained by some $u \in \mathcal{N}_h$.*

Proof. We consider $\{u_k\}_k$ obtained in Lemma 2.5 and $|h|_2 < \epsilon_4$. Since Ω is bounded, there exists $u \in H^s(\Omega)$ such that $u_k \rightharpoonup u$ in $H^s\left(\Omega\right)$. By Lemma 2.1, we have $u_k \to u$ in $L^p\left(\Omega\right)$ and in $L^2\left(\Omega\right)$. Then we derive that

$$J\left(u\right) \leqslant \liminf_k J\left(u_k\right) = m_h, \tag{2.7}$$

and

$$\|u\|_{H^s}^2 \leqslant |u|_p^p + \int_\Omega hu\mathrm{d}x. \tag{2.8}$$

Consider the case of equal sign in (2.8). From the Lemma 2.1, we have if (2.7) hold, then either $\|u\|_{H^s} > a_1$ or $\|u\|_{H^s} < a_3$. If $\|u\|_{H^s} > a_1$, then $u \in \mathcal{N}_h$ and (2.7) implies that u is the minimum we are looking for. If $\|u\|_{H^s} < a_3$, then

$$|u|_p \leqslant S_p \|u\|_{H^s} \leqslant S_p a_3 < S_p \frac{a_2}{S_p} = a_2,$$

which is a controdiction with $|u|_p \geqslant a_2$ from Lemma 2.2. Next consider the case of strict inequality in (2.8), namly,

$$\|u\|_{H^s}^2 < |u|_p^p + \int_\Omega hu\mathrm{d}x. \tag{2.9}$$

If we can show that (2.9) dose not hold, then (2.8) only holds when the equal sign is taken. At this time, according to the previous proof, u is the minimum we are looking for, and the proof of Lemma 2.7 is completed. So we only need to show that (2.9) can not hold. By (2.9), there exists $t^* > 0$ such that $t^* u \in \mathcal{N}_h$ and $t^* > t'$ according to (2.8), we have

$$\begin{aligned} t' &\leqslant \left(\frac{|u|_p^p + \int_\Omega hu\mathrm{d}x}{(p-1)\,|u|_p^p} \right)^{\frac{1}{p-2}} = \left(\frac{1}{p-1} + \frac{\int_\Omega hu\mathrm{d}x}{(p-1)\,|u|_p^p} \right)^{\frac{1}{p-2}} \\ &\leqslant \left(\frac{1}{p-1} + \frac{|h|_2 C_1 \|u\|_{H^s}}{(p-1)\,|u|_p^p} \right)^{\frac{1}{p-2}} \leqslant \left(\frac{1}{p-1} + \frac{|h|_2 C_1 C_3}{(p-1)\,a_2^p} \right)^{\frac{1}{p-2}}. \end{aligned}$$

If we choose

$$\epsilon_5 = \min\left\{ \frac{(p-2)\,(p-1)\,a_2^p}{2C_1C_3}, \epsilon_4 \right\},$$

then $t' \leqslant 1$.

For the function γ in Lemma 2.5, since $t^* u \in \mathcal{N}_h$, we have $\gamma(t^*) = 0$ and the ineqality (2.9) is equivalent to $\gamma(1) < 0$. Since $t' < 1$ and $t' < t^*$, we see that $t^* < 1$. According to the definition of m_h, we derive that

$$\begin{aligned} m_h \leqslant J(t^* u) &= (t^*)^2 \left(\frac{1}{2} - \frac{1}{p} \right) \|u\|_{H^s}^2 - t^* \left(1 - \frac{1}{p} \right) \int_\Omega hu\mathrm{d}x \\ &\leqslant (t^*)^2 \liminf_k \left(\frac{1}{2} - \frac{1}{p} \right) \|u_k\|_{H^s}^2 - t^* \lim_k \left(1 - \frac{1}{p} \right) \int_\Omega hu\mathrm{d}x \\ &\leqslant (t^*) \liminf_k \left[\left(\frac{1}{2} - \frac{1}{p} \right) \|u_k\|_{H^s}^2 - \left(1 - \frac{1}{p} \right) \int_\Omega hu\mathrm{d}x \right] \\ &= t^* \liminf_k J(u_k) = t^* m_h < m_h. \end{aligned}$$

Observing the first and last two terms of the above inequality, we obtain that $m_h < m_h$, which is impossible, so the inequality (2.9) does not hold. □

Now we prove that u is the critical point of the functional J.

Lemma 2.8. *There exists $\epsilon_6 \in (0, \epsilon_5)$ such that if $|h|_2 < \epsilon_6$, then u satisfies $J'(u)\,v = 0$ for all $v \in H^s(\Omega)$.*

Proof. Fix $v \in H^s(\Omega)$ and consider function $\phi : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ defined by

$$\phi(s,t) := t^2 \|u + sv\|_{H^s}^2 - t^p |u + sv|^p - t \int_\Omega h(u + sv)\, \mathrm{d}x.$$

Since $u \in \mathcal{N}_h$, we have $\phi(0,1) = 0$. So ϕ is a first-order continuous function and

$$\frac{\partial \phi}{\partial t}(0,1) = 2\|u\|_{H^s}^2 - p|u|_p^p - \int_\Omega hu\mathrm{d}x = (2-p)\|u\|^2 + (p-1)\int_\Omega hu\mathrm{d}x.$$

Letting $\frac{\partial \phi}{\partial t}(0,1) = 0$, then

$$\|u\|_{H^s}^2 = \frac{p-1}{p-2}\int_\Omega hu\mathrm{d}x \leqslant \frac{p-1}{p-2}|h|_2 C_1 \|u\|_{H^s},$$

i.e.,

$$\|u\|_{H^s} \leqslant \frac{p-1}{p-2}|h|_2 C_1.$$

If we take

$$|h|_2 < \frac{p-2}{C_1(p-1)} a_1,$$

then

$$\|u\|_{H^s} < \frac{p-1}{p-2}\frac{p-2}{C_1(p-1)} a_1 C_1 = a_1,$$

which contradicts $u \in \mathcal{N}_h$. So for such choices of h, there must be $\frac{\partial \phi}{\partial t}(0,1) \neq 0$. By the Implicit Function Theorem, there exist a number $\delta > 0$ and a C^1 function $t(s) : (-\delta, \delta) \to \mathbb{R}$ such that $\phi(s, t(s)) = 0$ for every $s \in (-\delta, \delta)$ and $t(0) = 1$. Since $\|u\|_{H^s} > a_1$, we can also take δ small enough such that $t(s)(u + sv) > a_1$. We now study the behaivior of the function $\gamma(s) = J(t(s)(u + sv))$. It can be obtained that γ is differentiable and has a local minimum at $s = 0$. Since $u \in \mathcal{N}_h$, we have

$$0 = \gamma'(0) = J'(u)[t'(0)u + t(0)v] = t'(0)J'(u)u + J'(u)v = J'(u)v,$$

which implies that when $\epsilon_6 < \min\left\{\epsilon_5, \frac{(p-2)a_1}{C_1(p-1)}\right\}$, the minimum u satisfies $J'(u)v = 0$ for all $v \in H^s(\Omega)$. □

So far, we have found a solution to equation (1.2). Next, we show that equation (1.2) has other solution.

Lemma 2.9. *For every $\epsilon > 0$, there exists $\delta > 0$ such that if $|h|_2 < \delta$, equation* (1.2) *admits a solution u_h satisfying $\|u_h\|_{H^s} < \epsilon$.*

Proof. Recalling $I(u) = \frac{1}{2}\|u\|_{H^s}^2 - \frac{1}{p}|u|_p^p$, since

$$S_p = \inf\{C > 0 : |u|_p \leqslant C\|u\|_{H^s}, \forall u \in H^s(\Omega)\},$$

we have

$$I(u) \geqslant \frac{1}{2}\|u\|_{H^s}^2 - \frac{S_p^p}{p}\|u\|_{H^s}^p.$$

The function

$$\phi(t) := \frac{1}{2}t^2 - \frac{S_p^p}{p}t^p$$

is continuous, strictly increasing in a right neighborhood of 0, and $\phi(0) = 0$. There exists $\epsilon' \leqslant \epsilon$ such that for all $t \in (0, \epsilon')$, we have $\phi(t) > 0$. Then for any $\eta \in (0, \epsilon')$, we have $I(u) \geqslant \phi(\eta) > 0$ for $\|u\|_{H^s} = \eta$. We also have

$$J(u) = I(u) - \int_\Omega hu\mathrm{d}x \geqslant \phi(\eta) - |h|_2 C_1 \eta.$$

Choosing $\delta = \frac{\phi(\eta)}{2C_1\eta}$ and $|h|_2 < \delta$, we derive that $J(u) \geqslant \frac{\phi(\eta)}{2} > 0$ for $\|u\|_{H^s} = \eta$.

Define

$$B_\eta = \{u \in H^s(\Omega) : \|u\|_{H^s} \leqslant \eta\},$$

and $n_\eta = \inf_{u \in B_\eta} J(u)$. Obviously, $-\infty < n_\eta \leqslant J(0) = 0$. Then we may proved that n_η is achieved by some $u_h \in B_\eta$. Since $J(u_h) = n_\eta \leqslant 0$, it can not be $\|u_h\|_{H^s} = \eta$, which means u_h lies in the interior of the ball B_η and u_h is a local minimum for J, moreover, u_h is a solution of equation (1.2). □

Proof of Theorem 1.1. By Lemma 2.9, choosing $\epsilon = a_1$, we can fix $\delta > 0$ such that for every $|h|_2 < \delta$ there exists a solution u_h of equation (1.2) with $\|u_h\|_{H^s} < a_1$.

If we take $|h|_2 < \varepsilon := \min\{\epsilon_6, \delta\}$, then, by Lemma 2.8, we obtain a different solution u to equation (1.2), satisfying $\|u\|_{H^s} > a_1$. □

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