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SOLUTIONS FOR MASSLESS BOSONS IN THE MANNHEIM-KAZANAS SPACE-TIME

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The present paper deals with the Mannheim-Kazanas metric which is supposed to offer a reliable explanation on the observed galaxy rotation curves. The effective potential and the geodesic deviations are discussed. For a particular form of the metric, depending only on two parameters, the Gordon equation for massless bosons is satisfied by the Heun general functions. A special attention is given to the physically important local Minkowskian region, which has no analogue in the Schwarzschild case.

Keywords: *Mannheim-Kazanas spacetime: black holes: Gordon equation: Heun functions*

1. *Introduction*. In this work, we consider a spacetime described by a metric containing, besides the well-known Schwarzschild solution, a second linear contribution, of the form kr.

For negative k values, the expression corresponds to the Schwarzschild black hole surrounded by quintessential matter, with the state parameter $\omega = -2/3$ [1]. Such a form of the metric is sustained by recent astrophysical observations suggesting that an accelerating expansion of our universe may be explained by the presence of dark energy, one of the highly plausible candidates being the quintessence.

However, the lack of tangible evidence for the existence of quintessence motivates the consideration of alternative gravity theories aiming to achieve similar results, without the need of the dark sector.

One of the most prominent examples is the metric proposed, in 1989, by Mannheim and Kazanas (the MK metric) [2]. The metric function contains a first term similar to the Schwarzschild solution and a second contribution which grows linearly with r, of the form kr. Now, k is a positive constant, whose value is comparable to the inverse of the Hubble length. Even though the MK metric is not a standard solution of General Relativity Equations, since it leads, as we are going to see, to a negative energy density, it has the advantage of solving, in a purely geometric way, the problem of the flat galactic rotational curves of galaxies. Thus, for $k \approx 10^{-26}\ \mathrm{m}^{-1}$, there is a good agreement between the corresponding flat rotational velocity and the experimental data [2].

Since the Schrödinger-type form of the Gordon equation is generally used to study the field scattering by a black hole, a main part of our work is devoted to the Gordon equation for particles evolving in the spacetime described by a particular expression of the MK metric. In the massless case, the solutions are expressed in terms of the Heun general functions [3,4].

Even though these were introduced more than 100 years ago, by Heun [5], they have been brought to the scientific community attention at the Centennial Workshop on Heun Equations, in 1989 [3]. In the last years, the Heun functions have been found as solutions to a wide range of problems in theoretical and applied science.

However, the theory of Heun equations is very complicated, because of the singular points and there are open questions related to the normalization procedure of the solutions, their series expansions or integration [6,7].

As expected, in the local Minkowskian region, the amplitude function is given by the Airy functions.

Finally, special attention is given to the energy spectrum which is associated with the so-called resonant frequencies, which are essential characteristics of black holes [8].

2. *The modified $\lambda = 0$ MK spacetime*. Let us describe the exterior region of an astrophysical object by the MK vacuum solution [2]

$$g_{00} = 1 - 3\beta k - \frac{\beta(2-3\beta k)}{r} + kr - \lambda r^2 , \tag{1}$$

where β, k and λ are integration constants. Obviously, for $\lambda = k = 0$, one can recover the well-known Schwarzschild external solution.

For the spherically symmetric static line element

$$ds^2 = g_{11}(dr)^2 + r^2\left[(d\,\theta)^2 + \sin^2\theta(d\,\varphi)^2\right] - g_{00}(dt)^2 , \tag{2}$$

where the functions g_{00} and g_{11} are depending only on r and satisfy the relation $g_{11} = g_{00}^{-1}$, one can define the pseudo-orthonormal frame

$$E_1 = \sqrt{g_{00}}\,\partial_r , \quad E_2 = \frac{1}{r}\partial_\theta , \quad E_3 = \frac{1}{r\sin\theta}\partial_\theta , \quad E_4 = \frac{1}{\sqrt{g_{00}}}\partial_t , \tag{3}$$

whose corresponding dual base is

$$\omega^1 = \frac{1}{\sqrt{g_{00}}}dr , \quad \omega^2 = r\,d\,\theta , \quad \omega^3 = r\sin\theta\, d\,\varphi , \quad \omega^4 = \sqrt{g_{00}}\,dt , \tag{4}$$

so that $ds^2 = \eta_{ab}\omega^a\omega^b$, with $\eta_{ab} = \text{diag}\left[1,1,1,-1\right]$.

As in [9], we are working in an SO(3, 1)-gauge covariant approach. Thus, the connection coefficients can be easily obtained from the first Cartan's equation,

$$d\,\omega^a = \Gamma^a_{.[bc]}\omega^b \wedge \omega^c\,, \tag{5}$$

with $1 \le b \le c \le 4$ and $\Gamma^a_{.[bc]} = \Gamma^a_{.bc} - \Gamma^a_{.cb}$, and they read:

$$\Gamma_{122} = \Gamma_{133} = -\Gamma_{212} = -\Gamma_{313} = \frac{\sqrt{g_{00}}}{r}\,,$$
$$\Gamma_{233} = -\Gamma_{323} = -\frac{\cot\theta}{r}\,,\quad \Gamma_{144} = -\Gamma_{414} = \frac{g'_{00}}{2\sqrt{g_{00}}}\,, \tag{6}$$

where ( )' means the derivative with respect to r.

From the second Cartan's equation, one can derive the tetradic curvature tensor components

$$R_{1212} = R_{1313} = -\frac{g'_{00}}{2r} = -R_{2424} = -R_{3434}\,;$$
$$R_{2323} = \frac{1-g_{00}}{r^2}\,,\quad R_{1414} = \frac{g'_{00}}{2}\,. \tag{7}$$

Since the value of the positive constant k is comparable to the inverse of the Hubble length and the dimensionless quantity $3k\beta$ is $3k\beta \approx 10^{-12}$ [2], one can write the metric (1), for $\beta = M$, in the particular form

$$g_{00} = 1 - \frac{2M}{r} + kr\,. \tag{8}$$

Thus, we have kept only the Newtonian term, M/r, and the linear term kr and we have neglected the λr^2 contribution which becomes important at cosmological distances [2].

As in the case of the Schwarzschild solution, this metric has a real singularity in $r = 0$.

Now, the components (7) have the expressions:

$$R_{1212} = R_{1313} = -\frac{M}{r^3} - \frac{k}{2r} = -R_{2424} = -R_{3434}\,,$$
$$R_{2323} = \frac{2M}{r^3} - \frac{k}{r}\,,\quad R_{1414} = -\frac{2M}{r^3}\,, \tag{9}$$

and the Einstein's tensor components read:

$$G_{11} = \frac{2k}{r} = -G_{44}\,,\quad G_{22} = G_{33} = \frac{k}{r}\,. \tag{10}$$

Since the proper energy density $T_{44} = G_{44}$ is negative, one may conclude by saying that the astrophysical object described by the metric (8) is surrounded by exotic matter.

The curvature of the metric (8) is negative,

$$R = -\frac{6k}{r}\,,$$

and has a singularity in $r=0$.

However, in the tetradic frame (3), the components of the Weyl tensor have the simple expressions

$$C_{1212}=C_{1313}=C_{2424}=C_{3434}=C_{2424}=C_{34343}=-\frac{M}{r^3},\quad C_{2323}=\frac{2M}{r^3}=-C_{1414}$$

pointing out that, for $M=0$, the metric is conformally flat. For $M\neq 0$, the above components are all vanishing, for $r\to\infty$.

The unique horizon is the positive solution of the equation $g_{00}=0$, namely:

$$r_h=\frac{\sqrt{1+8kM}-1}{2k}\approx 2M(1-4kM). \tag{11}$$

For discussing the timelike geodesics, we write the spacetime line element (2) as

$$ds^2=\frac{(dr)^2}{g_{00}}+r^2\left[(d\theta)^2+\sin^2\theta(d\varphi)^2\right]-g_{00}(dt)^2=-d\tau^2, \tag{12}$$

where τ is the proper time.

If the motion of the particle with zero angular momentum is on the equatorial plane, the relation (12) leads to the simple expression

$$g_{00}\dot{t}^2-\frac{\dot{r}^2}{g_{00}}=1,$$

where dot means the derivatives with respect to τ. Since the metric (8) is static and the Killing vector field ∂_t is orthogonal to the surfaces $t=$const, the energy derived from the Lagrangian

$$-L=\frac{1}{2}\left[\frac{\dot{r}^2}{g_{00}}+r^2\dot{\theta}^2+\sin^2\theta\dot{\varphi}^2-g_{00}\dot{t}^2\right],$$

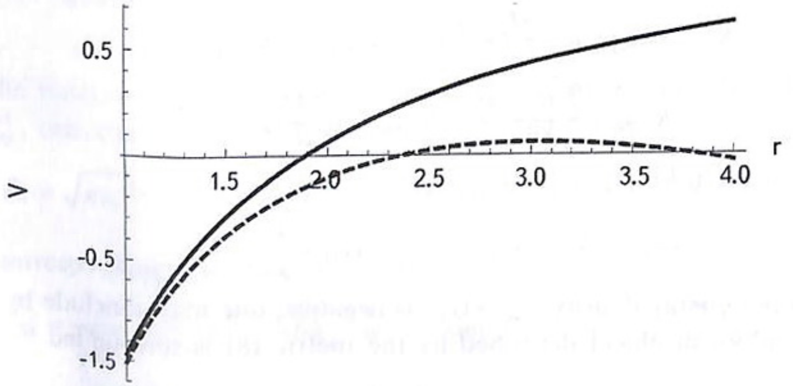

Fig.1. The effective potential (15), represented by the solid line, and (16), represented by the dashed plot. The numerical values of the parameters are: $M=1$, $k=0.03$, $\lambda=1/27$.

as

$$E = \frac{\partial L}{\partial \dot{t}} = g_{00}\dot{t}, \tag{13}$$

is conserved. Thus, we come to the important relation

$$\dot{r}^2 = E^2 - g_{00}, \tag{14}$$

where we identify the effective potential

$$V_{eff} = g_{00} = 1 - \frac{2M}{r} + kr, \tag{15}$$

which is represented in the Fig.1, by the solid plot.

Similarly to the Schwarzschild spacetime, since the effective potential becomes $-\infty$, as $r \to 0$, a particle with suitable total energy can reach the center.

Moreover, the exotic matter brings an additional contribution in the attractive force acting on the particle

$$F = -V'_{eff} = -\frac{2M}{r^2} - k.$$

Let us consider a particle located at the finite distance $r_0 > r_h$, with zero initial velocity, which is falling toward the black hole. The relation

$$E^2 = V_{eff}$$

is leading to the equation

$$kr_0^2 - \varepsilon r_0 - 2M = 0,$$

with $\varepsilon = E^2 - 1 > 0$, whose physical solution is

$$r_0 = \frac{\sqrt{\varepsilon^2 + 8kM} + \varepsilon}{2k}$$

One may easily check that r_0 exceeds the Schwarzschild radius $r_s = 2M$. On the way from r_0 to the horizon, the particle crosses the locally Minkowskian sphere, of radius $r_* = \sqrt{2M/k}$, where $g_{00} = 1$. This is an interesting feature, which has no analogue in the case of the Schwarzschild metric.

The relation (14) can be written as

$$\frac{dr}{d\tau} = -\sqrt{\frac{2M}{r} - kr + \varepsilon},$$

where the negative sign comes from the decreasing of time while r is increasing. With the change of variable $r = r_0 \sin^2 x$, where $x \in [0, \pi/2]$, the above relation leads to

$$d\tau = -2\sqrt{\frac{r_0}{k}} \frac{\sin^2 x dx}{\sqrt{p^2 + \sin^2 x}}, \quad p^2 = \frac{2M}{kr_0^2}.$$

The proper time is expressed in terms of the Elliptic integrals of the first and second kind as [10]

$$\tau = \frac{2}{k}\sqrt{\frac{2M}{r_0}}\left\{ \text{Elliptic}E\left[-\frac{1}{p^2}\right] - \text{Elliptic}K\left[-\frac{1}{p^2}\right]\right\}.$$

For each value of k, one may easily check, using Mathematica, that τ is a positive function, which is increasing with the particle's energy, E. This can be easily understood since, for a particle with a higher energy, r_0 is moving to larger r values and the particle needs more time to reach the $r=0$ singularity.

A circular orbit has the radius R_c solution of the equation [11]

$$V'_{eff} = 0.$$

Since, for the metric (8), $V'_{eff} \neq 0$, there are no circular orbits of the test particles. The situation changes if one is taking the full expression of the MK metric, with $\lambda \neq 0$, and work with the effective potential

$$V_{eff} = 1 - \frac{2M}{r} + kr - \lambda r^2, \tag{16}$$

represented in the Fig.1, by the dashed plot. The horizons are given by the physical solutions of the equation $g_{00}=0$, i.e.

$$-\lambda r^3 + kr^2 + r - 2M = 0.$$

As it is known [12], the cubic equation $ax^3+bx^2+cx+d=0$, has three real distinct solutions if the discriminant $\Delta = 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2$ is positive. This means the relation

$$-108M^2\lambda^2 + 4(1+9kM)\lambda + k^2(1+8kM) > 0,$$

which leads to the upper limit of the model's parameter λ:

$$\lambda \leq \frac{1+9kM+(1+6kM)^{3/2}}{54M^2} \approx \frac{1+9kM}{27M^2}.$$

Among the three roots, given by the intersection of V_{eff} with the horizontal axis, one is negative, while the other two are situated in the allowed r-region, as it can be seen in the Fig.1. We denote the two horizons by r_1 and r_2 and one may notice that the potential has a positive maximum in between. For large r-values, the potential is steeply falling down, to negative values. A particle coming from infinity, with $E^2 = V_{max}$, will follow a circular orbit, with R_c solution of the equation $V'_{eff} = 0$. This orbit is unstable because $V''_{eff} < 0$. Particles with $E^2 < V_{max}$ can follow elliptical orbits, with two turning points.

Finally, using the tetradic components of the Riemann tensor (9), one can work out the equation of geodesic deviation [13]

$$\ddot{\eta}^a + R^a_{bcd}\,\upsilon^b\upsilon^c\eta^d = 0,$$

where v^a is the unit vector tangent to the geodesic and η^a are the components of the connecting vector between two neighboring geodesics.

For the congruence of timelike geodesics with $v_4 v^4 = -1$, the above relation leads to the following system, similar to the one obtained in [14],

$$\ddot{\eta}^1 - \frac{2M}{r^3}\eta^1 = 0, \quad \ddot{\eta}^A + \left[\frac{M}{r^3} + \frac{k}{2r}\right]\eta^A = 0, \tag{17}$$

where $A = 2,\ 3$.

One may noticed that the angular tidal forces, given by the second equation in (17), are affected by the parameter k. With

$$\dot{\eta} = \frac{d\eta}{d\tau} = \dot{r}\,\eta' = \sqrt{E^2 - g_{00}}\,\eta', \quad \ddot{\eta} = \left(E^2 - g_{00}\right)\eta'' - \frac{g_{00}}{2}\eta',$$

we get the explicit form of the geodesic deviation vectors, expressed in terms of the hypergeometric functions, [10], as

$$\begin{gathered} \eta^1 = A_1\sqrt{\frac{2M}{r} - kr} + A_2\left(-1 + \sqrt{1-x}\, F_{21}\left[\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, x\right]\right) \\ \eta^A = B_1 r + B_2\sqrt{r}\, F_{21}\left[-\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, x\right] \end{gathered} \tag{18}$$

where $x = kr^2/2M \in (0,1]$ and $A_{1,2}$ and $B_{1,2}$ are integration constants. For $x \ll 1$, meaning $r \ll r_*$, the expression of η^1 and η^A have the same form as the ones corresponding to the Schwarzschild black hole (SBH):

$$\eta^1 = \frac{C_1}{\sqrt{r}} + C_2 r^2,$$

and

$$\eta^A = D_1 r + D_2\sqrt{r}.$$

3. *The Gordon equation and its Heun solutions*. In order to understand the behavior of a black hole, one can consider a test field evolving in the corresponding manifold. The quantized energy spectrum is expressed in terms of the black hole's parameters and therefore they carry important information.

In the pseudo-orthonormal frame (3), the SO(3, 1) gauge-covariant Klein-Gordon equation, i.e.

$$\eta^{ab}\Phi_{|ab} - \eta^{ab}\Phi_{|c}\Gamma^c_{ab} = \mu^2\Phi,$$

with $\Phi_{|a} = E_a\Phi$, has the explicit form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 g_{00}\frac{\partial\Phi}{\partial r}\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\Phi}{\partial\theta}\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2} - \frac{1}{g_{00}}\frac{\partial^2\Phi}{\partial t^2} - \mu^2\Phi = 0,$$

where the metric function describing the external region of the astrophysical object is (8).

The variables being separated, one can take the wave function as

$$\Phi = F(r) Y_\ell^m(\theta, \varphi) e^{-i\omega t}, \tag{19}$$

where Y_ℓ^m are the spherical functions and the radial equation reads

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2 g_{00}\frac{dF}{dr}\right] + \left[\frac{\omega^2}{g_{00}} - \frac{\ell(\ell+1)}{r^2} - \mu^2\right]F = 0. \tag{20}$$

The standard way for computing the quasinormal modes of the spherically symmetric black holes is to introduce the tortoise coordinate and use the WKB method. We do not insist on the procedure since it has been developed in [15]. However, we want to point out that, introducing the customary amplitude function

$$u(r) = r\sqrt{g_{00}}\,F(r),$$

the equation (20) becomes

$$\frac{d^2u}{dr^2} + \left\{\frac{1}{g_{00}^2}\left[\omega^2 + \frac{g_{00}'^2}{4}\right] - \frac{1}{g_{00}}\left[\frac{k}{r} + \frac{\ell(\ell+1)}{r^2} + \mu^2\right]\right\}u = 0, \tag{21}$$

where one may identify, besides the corrected energy, the Newtonian contribution.

Now, let us watch the field evolution around the locally Minkowskian region $g_{00} \approx 1$, i.e. $r_* = \sqrt{2M/k}$. In terms of the dimensionless space variable $x = r/r_*$, the equation (21) can be written as

$$\frac{d^2u}{dx^2} + \left\{\left(1 - \frac{\sqrt{2kM}}{x} + \sqrt{2kM}\,x\right)^{-2}\left[r_*^2\omega^2 + \frac{kM}{2}\left(\frac{1}{x^2}+1\right)^2\right]\right.$$
$$\left. - \left(1 - \frac{\sqrt{2kM}}{x} + \sqrt{2kM}\,x\right)^{-1}\left[\frac{\sqrt{2kM}}{x} + \frac{\ell(\ell+1)}{x^2} + r_*^2\mu^2\right]\right\}u = 0.$$

For $x = 1+\rho$, to first order in ρ and $\sqrt{2kM}$, the above equation becomes

$$\frac{d^2u}{d\rho^2} + \left\{\left[\frac{2M}{k}\left(\omega^2 - \mu^2\right) - \ell(\ell+1)\right] + \left[2\ell(\ell+1) + \sqrt{2kM}\left(1 - \frac{4M}{k}\left(2\omega^2 - \mu^2\right)\right)\right]\rho\right\}u = 0,$$

and, in the semiclassical linear approximation $\omega \approx \mu + \varepsilon$, with $\omega^2 \approx \mu^2 + 2\mu\varepsilon$, it can be written as

$$\frac{d^2u}{d\rho^2} + \left\{\left[\frac{4\mu\varepsilon M}{k} - \ell(\ell+1)\right] + \left[2\ell(\ell+1) - 4\mu^2 M\sqrt{\frac{2M}{k}}\right]\rho\right\}u = 0. \tag{22}$$

In the particular S-case, corresponding to $\ell = 0$, the above relation turns into the following Schrödinger-like equation for the amplitude function

$$\frac{d^2u}{d\rho^2}+ +2\mu\frac{2M}{k}\left[\varepsilon-\mu\sqrt{2kM}\rho\right]u=0. \tag{23}$$

The solutions are the oscillatory Airy functions [10]

$$u=C_1\,\mathrm{Ai}(y)+C_2\,\mathrm{Bi}(y)$$

of variable

$$y=\left[\frac{2\mu^2}{k^2}\right]^{1/3}\left[\sqrt{2kM}\rho-\frac{\varepsilon}{\mu}\right].$$

To find the solution for the entire range of the variable r, let us go back to the radial equation (20). For the metric (8) written as

$$g_{00}=1-\frac{2M}{k}+kr=\frac{k}{r}\left[r+\frac{a+1}{2k}\right]\left[r-\frac{a-1}{2k}\right], \tag{24}$$

with

$$a=\sqrt{1+8kM},$$

this can be analytically solved only in the massless case. Using Maple, one may find the solution given by the Heun general functions [3,4]

$$F(r)=\left[r-\frac{a-1}{2k}\right]^{i\omega(a-1)/2ka}\left[r+\frac{a+1}{2k}\right]^{-i\omega(a+1)/2ka}\mathrm{Heun}\,G\left[\frac{2a}{a+1},q,\alpha,\beta,\gamma,\delta,1+\frac{2kr}{a+1}\right], \tag{25}$$

whose absolute value is represented in the Fig.2, for $r\geq r_h$, where r_h is given in (11).

One may notice that, just outside the horizon, there is a maximum probability for the particle to exist. For r close to r_h, the probability is sharply decreasing,

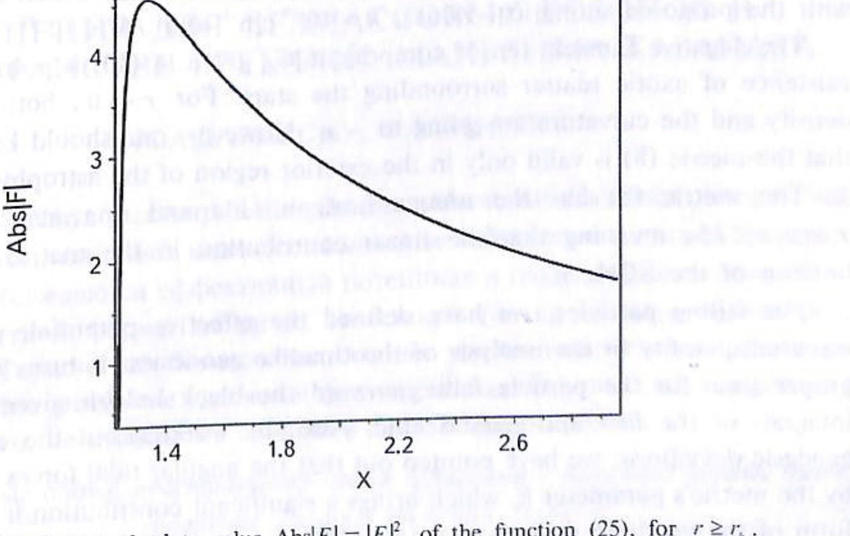

Fig.2. The absolute value $\mathrm{Abs}|F|=|F|^2$, of the function (25), for $r\geq r_h$.

as a clear indication that the particles are not propagating on the horizon.

To have a polynomial form of the Heun general functions, one has to impose that the α parameter

$$\alpha = 1 - \frac{i\omega}{ka} + \frac{\sqrt{k^2 - \omega^2}}{k},$$

satisfies the necessary condition $\alpha = -n$, with n a positive integer [4]. This leads to the following imaginary energy spectrum depending on M and k,

$$\omega_n = \frac{i(n+1)}{8M}\sqrt{1+8kM}\left[1 \pm \sqrt{1+\frac{8kMn(n+2)}{(n+1)^2}}\right]. \tag{26}$$

One may notice that, for $k=0$, we get the equally spaced energy spectrum of the SBH [8],

$$\omega_n^S = \frac{i}{4M}(n+1).$$

To first order in $8kM$, the quantized energy in (26) becomes

$$\omega_n = i\left[\frac{n+1}{4M} + \frac{(3n^2+6n+2)k}{2(n+1)}\right]$$

pointing out an additional contribution depending only on k.

4. *Conclusions*. In this manuscript, we carried out an analysis of the ($\lambda = 0$) MK metric. Even though this metric, with a linear contribution in r, is not a standard solution of Einstein's equations in General Relativity, this has offered an alternative explanation of the observations on the galaxy rotation curves. Throughout this paper, we have worked in the approximation $kM << 1$, which is consistent with the proposed numerical values, $k = 10^{-28}\,\mathrm{cm}^{-1}$ and $kM \approx 10^{-12}$ [2].

The negative Einstein tensor component G_{44} given in (10) is pointing out the existence of exotic matter surrounding the stars. For $r \to 0$, both the energy density and the curvature are going to $-\infty$. However, one should keep in mind that the metric (8) is valid only in the exterior region of the astrophysical object.

The metric (8) has the unique horizon (11) and one may notice that $r_h < r_S = 2M$, meaning that the linear contribution in the metric shrinks the horizon of the SBH.

For falling particles, we have defined the effective potential, which is an essential quantity in the analysis of the timelike geodesics. It turns out that the proper time for the particle falling toward the black hole is given by Elliptic integrals of the first and second kind. Also, by working out the equations of geodesic deviations, we have pointed out that the angular tidal forces are affected by the metric's parameter k, which brings a significant contribution in the explicit form of the geodesic deviation vectors (18).

The final part of the paper is devoted to the analysis of the Gordon equation for massless bosons evolving in the spacetime described by the metric (8). The obtained solution, given in (19) with (25), is valid for any values of k and M and the whole range of r. The graphical description reveals that the absolute value of the radial solution (25) has a maximum just outside the horizon (see Fig.2).

Finally, we would like to mention some interesting features of the local Minkowskian region $g_{00} \approx 1$, which has no analogue in the case of the Schwarzschild metric. For a typical galaxy with

$$2M = \frac{10^{11} M_S G}{c^2} \approx 10^{14}\, m\,,$$

and $k = 10^{-26}\,\mathrm{m}^{-1}$, the terms $2M/r$ and kr are comparable for $r_* = \sqrt{2M/k} \approx 10$ kpc and the potential is constant. Such a potential is leading to approximately constant rotational velocities which agree with the more or less flat observed galactic rotation curves [16].

In the particular case corresponding to $\ell = 0$, the Schrodinger-type equation for the amplitude function (23) is satisfied by the Airy functions, which are well-known solutions to the Schrödinger equation describing particles tunneling through linear potentials.

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РЕШЕНИЯ ДЛЯ БЕЗМАССОВЫХ БОЗОНОВ В ПРОСТРАНСТВЕ-ВРЕМЕНИ МАНГЕЙМА-КАЗАНАСА

М.А.ДАРИЕСКУ, Ч.ДАРИЕСКУ

В данной статье рассматривается метрика Мангейма-Казанаса, которая, как предполагается, может дать надежное объяснение наблюдаемым кривым вращения галактик. Обсуждаются эффективный потенциал и геодезические отклонения. Для конкретной формы метрики, зависящей только от двух параметров, уравнению Гордона для безмассовых бозонов удовлетворяют общие функции Гойна. Особое внимание уделяется физически важному локальному региону Минковского, не имеющему аналога в случае Шварцшильда.

Ключевые слова: *пространство-время Мангейма - Казанаса: черные дыры: уравнение Гордона: функции Гойна*

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