АСТРОФИЗИКА

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TOROIDAL MAGNETIC FIELD IN PULSARS

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The differential rotation of plasma in the core of pulsars ($\Omega_s \neq \Omega_e$) generates convective currents increasing with time which in turn generates the toroidal magnetic field. To avoid difficulties of physical interpretation inherent to the theory of general relativity we have adopted the tetrad approach to discuss the generation of the magnetic field in the core of the neutron stars. The results which we have obtained are in agreement with those obtained earlier.

Key words: *magnetic fields: pulsars*

1. *Introduction.* As it has been shown in paper [1], if in the core of a neutron star neutrons and protons are superfluid while electrons are normal, then the stationary rotating nuclear plasma is unstable with respect to the generation of convective currents and toroidal magnetic fields. This instability occurs only within the framework of Einstein's theory of General Relativity. To establish the existence of this effect it is not only important to know the relativistic form of the hydrodynamic equations but it is also necessary to adopt physically satisfactory definitions of electric and magnetic fields.

The equation of motion of a charged perfect fluid in general relativity in the case of stationary rotation can be written in the following form [1]

$$\frac{\partial P}{\partial x^\alpha} - \left(\rho + \frac{P}{c^2}\right)\frac{\partial \ln u^4}{\partial x^\alpha} + \frac{j^k F_{\alpha k}}{c} = 0, \tag{1}$$

where the latin indices run from 1 to 4 and the greek indices from 1 to 3, the coordinates (x^1, x^2, x^3) denote respectively the spherical coordinates (r, ϑ, φ). ρ and P are respectively the energy density and pressure of the fluid which with the aid of the equation of state can be expressed in terms of the density of particles n_i. The time component u^4 of the unit four velocity of the fluid, $u^l u_l = -1$, can be expressed in the form:

$$u^4 = \left(-g_{44} - 2\frac{\Omega}{c} g_{34} - \frac{\Omega^2}{c^2} g_{33}\right)^{-1/2}. \tag{2}$$

In deriving the above expression we have taken into account the following relation [2]:

$$u^3 = \frac{\Omega}{c} u^4, \tag{3}$$

where Ω is the angular velocity of the fluid. The components of the metric tensor g_{ik} for rotating neutron star can be expressed in the following way

$$-g_{44}=e^{\nu}-\omega^2 e^{\mu}\sin^2\vartheta, \quad g_{11}=e^{\lambda}, \quad g_{22}=e^{\mu}$$
$$g_{33}=e^{\mu}\sin^2\vartheta, \quad g_{34}=2\omega e^{\mu}\sin^2\vartheta, \tag{4}$$

where the functions ν, λ, μ and ω for the neutron star have been obtained in the paper [3].

In section 2 we introduce at every space time event of the domain under consideration a system of 4 mutually orthogonal unit vectors $e_{\bar{a}}(a=1,2,3,4)$ with $e_{\bar{4}}$ timelike and tangent to the time coordinate line and $e_{\bar{\alpha}}(\alpha=1,2,3)$ spacelike with $e_{\bar{1}}$ and $e_{\bar{2}}$ respectively tangent to the coordinate lines x^1 and x^2. Then we obtain the anholonomic form of the equation of motion (1) of the fluid.

In section 3 we derive the expression of the electric field in the core of the neutron star. To obtain the above expression we use the anholonomic form of the equation of motion written for both components of the core plasma: nuclear (superfluid) and electronic (normal). Using the anholonomic formulation of Maxwell equations we obtain the time derivative of the magnetic field expressed in terms of the space derivatives of the electric field. The later expression will permit us to show in section 4, that if the nuclear and electronic components are normal the time derivative of the magnetic field is equal to zero, but if the nuclear component is superfluid while the electronic component is normal, as it takes place in pulsars, then the generation of a toroidal magnetic field in the core of the star is possible.

2. *Relativistic hydrodynamics - Anholonomic formulation.* The gravitational fields of stationarily rotating stars are described by the tensor g_{ik} whose components are independent of the time coordinate and in the general case the nonvanishing components are g_{34} and the diagonal components [2]. If Ω denotes the angular velocity of the star then g_{34} is proportional to Ω and the corresponding corrections to the diagonal components of the metric tensor, due to rotation, are proportional to Ω^2. In order to describe the hydrodynamic properties of a charged fluid in such a gravitational field and to avoid difficulties of physical interpretation we shall adopt the tetrad approach. Let us introduce at every space-time event an orthonormal tetrad $e_{\bar{a}}(a=1,2,3,4)$ whose time like member $e_{\bar{4}}$ is taken tangential to the time coordinate line. The physical meaning of $e_{\bar{4}}$ corresponds to the choice of an observer who is at rest with respect to the fluid rotating with angular velocity Ω. All physical quantities will be defined with respect to this observer. As to the space like components $e_{\bar{\alpha}}(\alpha=1,2,3)$ of the orthonormal tetrad it is conventional to choose $e_{\bar{1}}$, $e_{\bar{2}}$ tangential respectively to the coordinate lines r and ϑ. From the condition of normality one can obtain:

$$e_{\bar{4}}^{l} = \left(0, 0, 0, 1/\sqrt{-g_{44}}\right), \quad e_{\bar{1}}^{l} = \left(1/\sqrt{g_{11}}, 0, 0, 0\right), \quad e_{\bar{2}}^{l} = \left(0, 1/\sqrt{g_{22}}, 0, 0\right). \tag{5}$$

The orthonormality conditions

$$g_{lj} e_{\bar{3}}^{l} e_{\bar{3}}^{j} = 1, \; g_{lj} e_{\bar{3}}^{l} e_{\bar{4}}^{j} = 0 \tag{6}$$

yield for the component $e_{\bar{3}}^{l}$ the following expression:

$$e_{\bar{3}}^{l} = \left\{0, 0, \frac{\sqrt{-g_{44}}}{\sqrt{g_{34}^{2} - g_{33}g_{44}}}, \frac{g_{34}/\sqrt{-g_{44}}}{\sqrt{g_{34}^{2} - g_{33}g_{44}}}\right\}. \tag{7}$$

The anholonomic form of the equation of motion (1) of a charged perfect fluid in stationary rotation may be exhibited in the following form:

$$-\partial_{\bar{\alpha}} \ln u^{4} + \frac{\partial_{\bar{\alpha}} P}{\rho + P/c^{2}} + \frac{enu^{\bar{m}} F_{\bar{\alpha}\bar{m}}}{c\left(\rho + P/c^{2}\right)} = 0\,, \tag{8}$$

where the tetrad components of the electromagnetic field tensor $F_{\bar{a}\bar{b}}$, unit four velocity vector $u^{\bar{a}}$ and Pfaffian derivatives are respectively given by

$$\partial_{\bar{\alpha}} = e_{\bar{\alpha}}^{l}\, \partial_{l}, \quad u^{\bar{m}} = e_{l}^{\bar{m}} u^{l}, \quad F_{\bar{a}\bar{m}} = e_{\bar{a}}^{l} e_{\bar{m}}^{j} F_{lj}\,. \tag{9}$$

Taking into account that all physical quantities in equation (8) do not depend on time and azimuthal coordinate φ, denoting

$$j^{\bar{m}} = enu^{\bar{m}}\,, \tag{10}$$

equation (8) may be expressed in the following form

$$-\partial_{\bar{\alpha}} \ln u^{4} + \frac{\partial_{\bar{\alpha}} P}{\rho + P/c^{2}} + \frac{j^{\bar{m}} F_{\bar{\alpha}\bar{m}}}{c\left(\rho + P/c^{2}\right)} = 0\,, \tag{11}$$

where the index α takes only the values 1 and 2.

3. ***The electric field in the core of the neutron star.*** In the simplest model of a neutron star the core, consisting of nucleons (neutrons and protons) and electrons, is surrounded by a crust consisting of nuclei and electrons. Let us note that in the core of a neutron star the nucleonic component is superfluid and partially charged (protons), while the electronic part is normal and negatively charged. If we describe the friction between these two components by the four vector $N^{\bar{\alpha}}(\alpha = 1, 2)$, equation (11) gives respectively for nucleons and electrons the following equations of motion:

$$-\partial_{\bar{\alpha}} \ln u_{s}^{4} + \frac{\partial_{\bar{\alpha}} P_{s}}{\rho_{s} + P_{s}/c^{2}} + \frac{j_{s}^{\bar{m}} F_{\bar{\alpha}\bar{m}}}{c\left(\rho_{s} + P_{s}/c^{2}\right)} = \frac{N_{\bar{\alpha}}}{\rho_{s} + P_{s}/c^{2}}\,, \tag{12}$$

$$-\partial_{\bar{\alpha}} \ln u_{e}^{4} + \frac{\partial_{\bar{\alpha}} P_{e}}{\rho_{e} + P_{e}/c^{2}} + \frac{j_{e}^{\bar{m}} F_{\bar{\alpha}\bar{m}}}{c\left(\rho_{e} + P_{e}/c^{2}\right)} = -\frac{N_{\bar{\alpha}}}{\left(\rho_{e} + P_{e}/c^{2}\right)}\,. \tag{13}$$

The indices s and e denote respectively the nucleonic and electronic components of the plasma. In writing down these equations we have assumed that the principle of action and reaction do hold (at least up to the term of the first order of v/c). Let us note that $N^{\bar{\alpha}}$ is different from zero only in the presence of electric currents in the plasma. If we denote by $j^{\bar{\alpha}}$ the tetrad components of these currents, then we have

$$j^{\bar{\alpha}} = j_s^{\bar{\alpha}} + j_e^{\bar{\alpha}} . \tag{14}$$

For later investigation it is convenient instead of equations (12) and (13) to use the equations obtained by respectively adding and subtracting equations (12) and (13). Taking into account equation (14) the difference of these two equations will take the following form:

$$\left(\rho_e + P_e/c^2\right)\left(-\partial_{\bar{\alpha}} \ln \frac{u_s^4}{u_e^4}\right) + \left(\partial_{\bar{\alpha}} P_e - \beta \partial_{\bar{\alpha}} P_s\right) + \frac{j^{\bar{4}}}{c} F_{\bar{\alpha}\bar{4}}(1+\beta) + \frac{j^{\bar{\beta}}}{c} F_{\bar{\alpha}\bar{\beta}} = N_{\bar{\alpha}}(1+\beta), \tag{15}$$

where

$$\beta = \frac{\left(\rho_e + P_e/c^2\right)}{\left(\rho_s + P_s/c^2\right)} . \tag{16}$$

As we see β is the ratio of the relativistic energy density of electrons to the energy density of nucleonic matter. This ratio is very small compared to unity, i.e. $\beta \ll 1$, because the density and mass of nucleons are respectively two and three orders of magnitude higher than the corresponding quantities for electrons. Taking into account this fact, equation (15) simplifies and takes the following form:

$$\left(\rho_e + P_e/c^2\right)\left(-\partial_{\bar{\alpha}} \ln \frac{u_s^4}{u_e^4}\right) + \partial_{\bar{\alpha}} P_e + \frac{j^{\bar{4}}}{c} F_{\bar{\alpha}\bar{4}} + \frac{j^{\bar{\beta}}}{c} F_{\bar{\alpha}\bar{\beta}} = N_{\bar{\alpha}} . \tag{17}$$

Now let us multiply respectively equations (12) and (13) by $\rho_s + P_s/c^2$ and $\rho_e + P_e/c^2$ adding them yields:

$$-\partial_{\bar{\alpha}} \ln u_s^4 + \frac{\rho_e + P_e/c^2}{\rho_s + P_s/c^2} \partial_{\bar{\alpha}} \ln u_s^4 + \frac{\partial_{\bar{\alpha}}(P_s + P_e)}{\rho_s + P_s/c^2} + \frac{\left(j_e^{\bar{m}} + j_s^{\bar{m}}\right)}{c\left(\rho_s + P_s/c^2\right)} F_{\bar{\alpha}\bar{m}} = 0 . \tag{18}$$

Let us simplify equation (18). Since in the core of neutron stars densities of degenerate neutrons, protons and electrons satisfy the following conditions: $n_n \gg n_e = n_p = n$ which imply the following inequalities

$$P_s \gg P_e, \quad \rho_s = mc^2 n_s \gg \frac{P_s}{c^2}, \quad j_e^{\bar{\beta}} \gg j_s^{\bar{\beta}} , \tag{19}$$

where m is the mass of the neutron. Taking into account the inequalities (19) equation (18) simplifies and takes the form:

$$\frac{\partial_{\bar{\alpha}} P}{n_s} = mc^2 \partial_{\bar{\alpha}} \ln u_s^4 - \frac{j_e^{\bar{m}}}{c n_s} F_{\bar{\alpha}\bar{m}} . \tag{20}$$

As it is shown in ref. [4], the condition

$$\frac{\partial_{\bar{\alpha}} P}{n_s} = \frac{\partial_{\bar{\alpha}} P_e}{n} \tag{21}$$

is satisfied in the plasma in the core of neutron star. From (20) and (21) we derive the following expression for the spatial Pfaffian derivative of the pressure of the electronic component of the fluid

$$\partial_{\bar{\alpha}} P_e = mc^2 n \partial_{\bar{\alpha}} \ln u_s^4 - \frac{nj^{\bar{\beta}}}{n_s c} F_{\bar{\alpha}\bar{\beta}} . \tag{22}$$

On the other hand, from eq. (17), taking into account the definition (10), we find

$$F_{\bar{\alpha}\bar{4}} = E_{\bar{\alpha}} = \frac{c}{enu_e^{\bar{4}}} \left(\left(\rho_e + P_e/c^2\right) \left(\partial_{\bar{\alpha}} \ln \frac{u_s^4}{u_e^4} \right) - \partial_{\bar{\alpha}} P_e - \frac{j^{\bar{\beta}}}{c} F_{\bar{\alpha}\bar{\beta}} + N_{\bar{\alpha}} \right). \tag{23}$$

Let us recall that $E_{\bar{\alpha}}$ are the components of the electric field as measured by an observer with 4-velocity $e_{\bar{4}}$, i.e. at rest with respect to the fluid rotating with angular velocity Ω.

Inserting eq.(22) into eq.(23) we finally obtain for the components of the electric field the following expression

$$E_{\bar{\alpha}} = \frac{mc^3}{e} \frac{\partial_{\bar{\alpha}} M}{u_e^{\bar{4}}} + \frac{1}{enu_e^{\bar{4}}} \left(1 + \frac{n}{n_s} \right) j^{\bar{\beta}} F_{\bar{\alpha}\bar{\beta}} + \frac{c}{enu_e^{\bar{4}}} N_{\bar{\alpha}} , \tag{24}$$

where

$$M = \ln u_s^4 - \frac{\rho_e + P_e/c^2}{mc^2 n} \ln \frac{u_s^4}{u_e^4} . \tag{25}$$

4. *The time derivative of the magnetic field in the core of neutron star.* The time derivative of magnetic field is derived using the second pair of Maxwell's equation written in the following form

$$F_{\alpha\beta,4} = F_{\alpha 4,\beta} - F_{\beta 4,\alpha} . \tag{26}$$

For the case under consideration only the values 1 and 2 are taken by the indices α, β. Taking into account eq. (5) and (8) the anholonomic form of equation (26) is given by:

$$\partial_4 \left(e_\alpha^{\bar{\alpha}} e_\beta^{\bar{\beta}} F_{\bar{\alpha}\bar{\beta}} \right) = \partial_\beta \left(e_4^{\bar{4}} e_\alpha^{\bar{\alpha}} F_{\bar{\alpha}\bar{4}} \right) - \partial_\alpha \left(e_4^{\bar{4}} e_\beta^{\bar{\beta}} F_{\bar{\beta}\bar{4}} \right). \tag{27}$$

Noting that $e_4^{\bar{4}} = \sqrt{-g_{44}}$ and $e_\alpha^{\bar{\alpha}} = \sqrt{g_{\alpha\alpha}}$ and the fact that the g_{ii}'s are time independent, equation (27) may be written as

$$\partial_4 F_{\bar{\alpha}\bar{\beta}} = \frac{1}{\sqrt{g_{\alpha\alpha} g_{\beta\beta}}} \left(\partial_\beta \left(\sqrt{-g_{44}} \sqrt{g_{\alpha\alpha}} F_{\bar{\alpha}\bar{4}} \right) - \partial_\alpha \left(\sqrt{-g_{44}} \sqrt{g_{\beta\beta}} F_{\bar{\beta}\bar{4}} \right) \right). \tag{28}$$

The equation above connects the time derivative of the magnetic field components to the spatial derivatives of the electric field components. To obtain the final expression of the time derivative of the magnetic field components we have to insert eq. (24) into eq. (28).

Before performing this substitution, let us in the first member of the right hand side of eq. (24) express the Pfaffian derivative and $u_e^{\bar{4}}$ in the terms of the corresponding holonomic components, then instead of eq.(24) we get

$$F_{\bar{\alpha}4}=E_{\bar{\alpha}}=\frac{mc^3}{e}\frac{1}{\sqrt{-g_{44}}\sqrt{g_{\alpha\alpha}}\,u_e^4}\partial_\alpha M-\frac{1}{enu_e^4}\left(1+\frac{n}{n_s}\right)j^{\bar{\beta}}F_{\bar{\alpha}\bar{\beta}}+\frac{c}{enu_e^{\bar{4}}}N_{\bar{\alpha}}. \tag{29}$$

Now let us suppose that at $t=0$ in the core of neutron stars the magnetic field and the currents $j^{\bar{\alpha}}$ generating it are equal to zero. In this case the last two terms of eq.(29) vanish and the electric field is given by

$$E_{\bar{\alpha}}(t=0)=\frac{mc^3}{e}\frac{1}{\sqrt{-g_{44}}\sqrt{g_{\alpha\alpha}}\,u_e^4}\partial_\alpha M \tag{30}$$

the substitution of eq. (30) into eq. (28) yields the following expression time derivative of the magnetic field components at $t=0$

$$\partial_4 F_{\bar{\alpha}\bar{\beta}}(t=0)=\frac{mc^3}{e}\frac{1}{\sqrt{-g_{44}}\sqrt{g_{\alpha\alpha}}}\left(\partial_\alpha\left(\frac{\partial_\beta M}{u_e^4}\right)-\partial_\beta\left(\frac{\partial_\alpha M}{u_e^4}\right)\right). \tag{31}$$

If this derivative is equal to zero, the toroidal magnetic field and the corresponding convective currents will not be generated. On the other hand if this derivative is different from zero, then a toroidal magnetic field will be generating in the core plasma of neutron stars. This means that the plasma is unstable with respect to the generation of toroidal magnetic field in the core of neutron stars. Inserting the full expression for the electric field (29) into (28), we see that the time derivative of the magnetic field will decrease when the magnetic field and corresponding currents increase with time. Let us note that the vanishing of the time derivative defined by (29) determines the distribution of the resulting stationary magnetic field.

We thus see that the determination of the physical condition for the generation of the magnetic field coincide with the conditions setting eq. (31) different from zero.

Let us suppose the superfluid (neutrons and protons) and normal (electrons) components of the neutron star core rotate with the same angular velocity $\Omega_s=\Omega_e=\Omega$ in this case according to (2) $u_s^4=u_e^4$. Taking into account (25) the right hand side of eq.(31) becomes equal to zero. Let us note that the equality $\Omega_s=\Omega_e$, which is a consequence of the stationary rotation, is also satisfied when all components of the core plasma are normal [4]. Thus under the above mentioned conditions there is no generation of toroidal magnetic field in the core of neutron stars. For the generation of toroidal magnetic field the necessary condition is $\Omega_s\neq\Omega_e$.

It is well known that the condition $\Omega_s \neq \Omega_e$ is realized in pulsars. Pulsars are rotating neutron stars whose crust angular velocity Ω_e decreases with time. The time derivative $\dot{\Omega}_e$ has been measured for various pulsars .The theoretical study of the dynamics of rotating pulsars show that the angular velocity Ω_s of the superfluid component is different from the angular velocity Ω_e of the normal component and their difference $\Delta\Omega$ is equal to $\Delta\Omega = \Omega_s - \Omega_e = \tau\dot{\Omega}$, where τ is the relaxation time of superfluid vortices. The relaxation time depends on the friction coefficient between neutron vortices and normal matter. In the core of neutron star this time is large because of strong interaction between electrons and the magnetic field of neutron vortices [5]. So the angular velocity difference $\Delta\Omega$ can be of the order of Ω_s (i.e. $\Omega_s >> \Omega_e$) during the time comparable with the life time of the pulsar. This means that the formulas for computation of the magnetic field in papers [4] and [1] will differ by the factor $\Delta\Omega / \Omega$ which in the case $\Omega_s >> \Omega_e$ will be equal to one. The value of the magnetic field derivative, determined by (31), will be the same as in paper [1].

Thus the differential rotation of the plasma components in the core of pulsars ($\Omega_s \neq \Omega_e$) generate convective currents increasing with time which in turn generates the toroidal magnetic field.

A similar result was obtained in paper [1] using the holonomic formulation of hydrodynamic and electrodynamic equations in general relativity. To avoid difficulties of physical interpretation inherent to the theory of general relativity [4,1], in this paper we have adopted the tetrad approach to discuss the generation of the magnetic field in the core of neutron stars. The results we have obtained are in agreement with those of reference [1].

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ТОРОИДАЛЬНОЕ МАГНИТНОЕ ПОЛЕ В ПУЛЬСАРАХ

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Дифференциальное вращение плазмы в ядре пульсаров ($\Omega_s \neq \Omega_c$) генерирует конвективные токи, возрастающие со временем, которые, в свою очередь, генерируют тороидальное магнитное поле. Для преодоления трудностей физической интерпретации, присущих общей теории относительности, мы использовали тетрадный формализм для обсуждения генерации магнитного поля в ядре нейтронных звезд. Результаты, полученные нами, согласуются с результатами полученными ранее.

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