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NONHOMOGENEOUS DUAL WAVELET FRAMES WITH THE p-REFINABLE STRUCTURE IN $L^2(\mathbb{R}^+)$

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Abstract. In recent years, nonhomogeneous wavelet frames have been widely studied by many researchers, while the ones in $L^2(\mathbb{R}^+)$ have not. Some practical applications indicate that it is desirable to have a nonhomogeneous dual wavelet frame in $L^2(\mathbb{R}^+)$ because of the time variable can not take negative values in signal sampling. In addition, similar to the homogeneous dual wavelet frames, the nonhomogeneous ones derived from refinable functions have fast wavelet algorithms. In view of this, under the setting of $L^2(\mathbb{R}^+)$, we study the properties of nonhomogeneous dual wavelet frames, and obtain a construction of nonhomogeneous dual wavelet frames from a pair of p-refinable functions.

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1. Introduction

The concept of frames was introduced already in 1952 by Duffin and Schaeffer [10] in the study of nonharmonic Fourier series, but the importance of this concept was not recognized by mathematicians until the ground-breaking work of Daubechies et al. [7]. In the past three decades, the theory of frames has attracted many mathematicians and engineers, and has achieved fruitful results (see [5, 6, 27, 28] and many references therein).

An important example about frames is wavelet frames, which are generated by translation and dilation of a finite number of functions. Wavelet frames have many good properties that make them useful in the study of signal processing, image restorations, sampling theory, function spaces [2, 17, 24, 32] and so forth. In order to make the wavelet frames have more applications, several generalized notions of wavelet frames are proposed and studied, namely tight wavelet frames [18], dual wavelet frames [19], (quasi) affine frames and (quasi) affine dual frames [3, 27]. One of the fundamental methods to construct tight wavelet frames from refinable functions is the unitary extension principle (UEP) which was proposed

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by Ron and Shen [27, 28], and then was extended by Daubechies et al. [5] in the form of the oblique extension principle (OEP). They gave sufficient conditions for constructing tight affine frames and affine dual frames from any given refinable functions. From then on, many works along this direction can be found in [1, 4, 25, 34]. Observe that all above works main focus on homogeneous (dual) wavelet frames. In applications, fast wavelet transforms are our main concern, and nonhomogeneous (dual) wavelet frames derived from refinable functions have fast wavelet algorithms. Han in [20–22] comprehensive studied nonhomogeneous (dual) wavelet frames and they connect with homogeneous ones. Similar to the homogeneous dual wavelet frames, the nonhomogeneous ones derived from refinable functions have fast wavelet algorithms, which play an important role in wavelet analysis.

Wavelets and frames have been generalized in many different settings. For example, Lang [23] constructed compactly supported orthogonal wavelets on the locally compact Cantor dyadic group by following the procedure of Daubechies [8] (or see [9]) via scaling filters, and these wavelets turn out to be certain lacunary Walsh series on the real line. Recent works about wavelets and frames on the Cantor group and Vilenkin groups can be found in [12–16]. It is worth noting that the first constructions of wavelet frames on the positive half line with binary addition were proposed by Farkov [11], in which wavelets and frames on the half line $\mathbb{R}^+$ related to the Walsh-Dirichlet kernel and its modification are considered. Shah and Debnath [30] studied Dyadic wavelet frames on a half-line using the Walsh-Fourier transform. Shah in [31] give an explicit construction of tight wavelet frames generated by the Walsh polynomials on positive half-line $\mathbb{R}^+$ using the extension principles, and derive the wavelet frames decomposition and reconstruction formulas.

Intuitively, we can obtain $L^2(\mathbb{R}^+)$ wavelet frames by projection from $L^2(\mathbb{R})$ ones, while it is not the case for $L^2(\mathbb{R}^+)$ since the projections do not have complete affine structure. Furthermore, in many practical problems of nature and physics, the time variable can not take negative values in signal sampling; and in mathematics, $\mathbb{R}^+$ is not closed according to the usual addition "+". As a result, the classical Fourier transform method can not be directly applied to the wavelet frames in $L^2(\mathbb{R}^+)$. However, $\mathbb{R}^+$ is closed in terms of the operation "$\oplus$", and the Walsh-Fourier transform is defined by $\oplus$.

Inspired by the above observation, in this paper we investigate nonhomogeneous dual wavelet frames under the setting of $L^2(\mathbb{R}^+)$. In Section 2 we give some preliminaries and notations. In Section 3 we present some properties of nonhomogeneous dual wavelet frames in $L^2(\mathbb{R}^+)$. Section 4 is devoted to constructing nonhomogeneous dual wavelet frames from a pair of general p-refinable functions.

2. Preliminaries and notations

We first recall some basics of addition "$\oplus$"and subtraction "$\ominus$". We denote by $\mathbb{Z}$, $\mathbb{Z}^+$ and $\mathbb{N}$ the set of integers, the set of nonnegative integers and the set of positive integers, respectively; and by $\mathbb{N}_t$ the set of $\{0, 1, \cdots, t-1\}$ for $t \in \mathbb{N}$. Let $p > 1$ be a fixed integer. For $x, y \in \mathbb{N}_p$, we define the $\oplus$ and $\ominus$ on $\mathbb{N}_p$ respectively by

$$x \oplus y = (x+y)(\text{mod } p) = \begin{cases} x+y, & x+y < p, \\ x+y-p, & x+y > p, \end{cases}$$

and

$$x \ominus y = (x-y)(\text{mod } p) = \begin{cases} x-y, & x > y, \\ x-y+p, & x < y. \end{cases}$$

Given $x \in \mathbb{R}^+$, we denote by $[x]$ its integer part, and by $\{x\}$ its fraction part. Then we have

$$x = \sum_{j=1}^{k_x} x_{-j} p^{j-1} + \sum_{j=1}^{\infty} x_j p^{-j} = [x] + \{x\}, \tag{2.1}$$

where $k_x \in \mathbb{Z}^+, x_j, x_{-j} \in \mathbb{N}_p$ for $j \in \mathbb{N}$, and the sequence $\{x_j\}_{j=1}^{\infty}$ is required to have only finitely many nonzero terms when x is rational. For $y, \omega \in \mathbb{R}^+$, we define y_j, y_{-j} and ω_j, ω_{-j} similarly. Using the above operations on $\mathbb{N}_p$, we define the $\oplus$ and $\ominus$ on $\mathbb{R}^+$ respectively by

$$x \oplus y = \sum_{j=1}^{\infty} (x_j \oplus y_j) p^{j-1} + \sum_{j=1}^{\infty} (x_{-j} \oplus y_{-j}) p^{-j} \tag{2.2}$$

and

$$x \ominus y = \sum_{j=1}^{\infty} (x_j \ominus y_j) p^{j-1} + \sum_{j=1}^{\infty} (x_{-j} \ominus y_{-j}) p^{-j} \tag{2.3}$$

for $x, y \in \mathbb{R}^+$. Note that $z = x \ominus y$ if $z \oplus y = x$, and it is easy to check that $\mathbb{R}^+$ is a group under the operation "$\oplus$". Given $x, \omega \in \mathbb{R}^+$, write

$$\chi(x, \omega) = \exp\left(\frac{2\pi i}{p} \sum_{j=1}^{\infty} (x_j \omega_{-j} + x_{-j} \omega_j) \right). \tag{2.4}$$

For a function $f \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$, its Walsh-Fourier transform is defined by

$$\mathcal{F} f(\cdot) = \int_{\mathbb{R}^+} f(x) \overline{\chi(x, \cdot)} dx,$$

and is extended uniquely to the whole space $L^2(\mathbb{R}^+)$. The details of the Walsh-Fourier transform and Walsh series can be found in [29]. Similarly to the classical Fourier transform, the Walsh-Fourier transform is an unitary operator on $L^2(\mathbb{R}^+)$, and the system $\{\chi(k, \cdot) : k \in \mathbb{Z}^+\}$ is an orthonormal basis for $L^2(\mathbb{T})$ with $\mathbb{T} = [0, 1)$.

We define the dilation operator D and the translation operator $\mathcal{T}_k$ with $k \in \mathbb{Z}^+$ respectively by

$$Df(\cdot) = p^{1/2} f(p\,\cdot) \text{ and } \mathcal{T}_k f(\cdot) = f(\cdot \ominus k) \text{ for } f \in L^2(\mathbb{R}^+).$$

Obviously, they are both unitary operators on $L^2(\mathbb{R}^+)$. And we write

$$f_{j,k} = D^j \mathcal{T}_k f \text{ for } j \in \mathbb{Z} \text{ and } k \in \mathbb{Z}^+.$$

Let $J \in \mathbb{Z}, \psi_0 \in L^2(\mathbb{R}^+)$ and $\Psi = \{\psi_1, \psi_2, \cdots, \psi_L\}$ with $L \in \mathbb{N}$ be a finite subset in $L^2(\mathbb{R}^+)$. We define the homogeneous wavelet system $X(\Psi)$ and the nonhomogeneous wavelet system $X_J(\psi_0,\, \Psi)$ respectively by

$$X(\Psi) = \{\psi_{l,j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^+, 1 \leq l \leq L\} \tag{2.5}$$

and

$$X_J(\psi_0,\, \Psi) = \{\psi_{0,J,k} : k \in \mathbb{Z}^+\} \cup \{\psi_{l,j,k} : j \geq J, k \in \mathbb{Z}^+, 1 \leq l \leq L\}. \tag{2.6}$$

And we write $X_0(\psi_0,\, \Psi) = X(\psi_0,\, \Psi)$ for simplicity. Let $X(\widetilde{\Psi})$ and $X_J(\tilde{\psi}_0,\, \widetilde{\Psi})$ be defined similarly. We say $X(\Psi)$ is a *homogeneous wavelet frame* (HWF) in $L^2(\mathbb{R}^+)$ if there exist two constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^+} |\langle f,\, \psi_{l,j,k}\rangle|^2 \leq B\|f\|^2 \text{ for } f \in L^2(\mathbb{R}^+), \tag{2.7}$$

where A, B are called frame bounds. It is called a Bessel sequence in $L^2(\mathbb{R}^+)$ if only the right-hand side of (2.7) holds, where B is called a Bessel bound. We say $(X(\Psi),\, X(\widetilde{\Psi}))$ is a *homogeneous dual wavelet frame* (HDWF) in $L^2(\mathbb{R}^+)$ if $X(\Psi)$ and $X(\widetilde{\Psi})$ are both Bessel sequences in $L^2(\mathbb{R}^+)$, and the identity

$$\langle f,\, g\rangle = \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^+} \langle f,\, \tilde{\psi}_{l,j,k}\rangle \langle \psi_{l,j,k},\, g\rangle \tag{2.8}$$

holds for $f, g \in L^2(\mathbb{R}^+)$. Similarly, we say $X_J(\psi_0,\, \Psi)$ is a *nonhomogeneous wavelet frame* (NWF) in $L^2(\mathbb{R}^+)$ if there exist two constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{k\in\mathbb{Z}^+} |\langle f,\, \psi_{0,J,k}\rangle|^2 + \sum_{l=1}^{L} \sum_{j=J}^{\infty} \sum_{k\in\mathbb{Z}^+} |\langle f,\, \psi_{l,j,k}\rangle|^2 \leq B\|f\|^2 \text{ for } f \in L^2(\mathbb{R}^+), \tag{2.9}$$

where A, B are called frame bounds. It is called a Bessel sequence in $L^2(\mathbb{R}^+)$ if only the right-hand side of (2.9) holds, where B is called a Bessel bound. We say $\left(X_J(\psi_0; \Psi),\, X_J(\tilde{\psi}_0; \widetilde{\Psi})\right)$ is a *nonhomogeneous dual wavelet frame* (NDWF) in $L^2(\mathbb{R}^+)$ if $X_J(\psi_0; \Psi)$ and $X_J(\tilde{\psi}_0; \widetilde{\Psi})$ are both Bessel sequences in $L^2(\mathbb{R}^+)$, and the identity

$$\langle f,\, g\rangle = \sum_{k\in\mathbb{Z}^+}\langle f,\, \tilde{\psi}_{0,J,k}\rangle\langle \psi_{0,J,k},\, g\rangle + \sum_{l=1}^{L}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^+}\langle f,\, \tilde{\psi}_{l,j,k}\rangle\langle \psi_{l,j,k},\, g\rangle \tag{2.10}$$

holds for $f, g \in L^2(\mathbb{R}^+)$. It is easy to check that both $X_J(\psi_0;\Psi)$ and $X_J(\tilde{\psi}_0;\widetilde{\Psi})$ are frames for $L^2(\mathbb{R}^+)$, and reconstruction formula

$$f = \sum_{k\in\mathbb{Z}^+}\langle f,\, \tilde{\psi}_{0,J,k}\rangle\psi_{0,J,k} + \sum_{l=1}^{L}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^+}\langle f,\, \tilde{\psi}_{l,j,k}\rangle\psi_{l,j,k},$$

or

$$f = \sum_{k\in\mathbb{Z}^+}\langle f,\, \psi_{0,J,k}\rangle\tilde{\psi}_{0,J,k} + \sum_{l=1}^{L}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^+}\langle f,\, \psi_{l,j,k}\rangle\tilde{\psi}_{l,j,k}$$

holds for $f \in L^2(\mathbb{R}^+)$ if $\Big(X_J(\psi_0;\Psi),\, X_J(\tilde{\psi}_0;\widetilde{\Psi})\Big)$ is a NDWF in $L^2(\mathbb{R}^+)$.

Nonhomogeneous (dual) wavelet frames play an important role in frame theory because they are related to filter banks and have a natural relationship with refinable structures as pointed out in [26] where this type of wavelet frames was introduced for the first time. It is worth noting that Han named the term 'nonhomogeneous' for this type of frames and widely studied them in the distribution space and in $L^2(\mathbb{R}^d)$ [21, 22]. In particular, Han proved that if $\Big(X_{J_0}(\psi_0;\Psi),\, X_{J_0}(\tilde{\psi}_0;\widetilde{\Psi})\Big)$ is a NDWF in $L^2(\mathbb{R}^d)$ for some $J_0 \in \mathbb{Z}$, then $\Big(X_J(\psi_0;\Psi),\, X_J(\tilde{\psi}_0;\widetilde{\Psi})\Big)$ is a NDWF in $L^2(\mathbb{R}^d)$ for a general $J \in \mathbb{Z}$, and $(X(\Psi),\, X(\widetilde{\Psi}))$ is a HDWF in $L^2(\mathbb{R}^d)$.

3. Some properties of NDWFs in $L^2(\mathbb{R}^+)$

This section is devoted to some properties of NDWFs in $L^2(\mathbb{R}^+)$. Observe that the dilation operator and the Walsh-Fourier transforms are unitary operator on $L^2(\mathbb{R}^+)$. Let $\{\mathcal{T}_k\psi_0 : k \in \mathbb{Z}^+\}$ and $\{\mathcal{T}_k\tilde{\psi}_0 : k \in \mathbb{Z}^+\}$ be Bessel sequences in $L^2(\mathbb{R}^+)$, define a quasi-interpolatory operator P_J on $L^2(\mathbb{R}^+)$ with $J \in \mathbb{Z}$ by

$$P_J f = \sum_{k\in\mathbb{Z}^+}\langle f,\, \tilde{\psi}_{0,J,k}\rangle\psi_{0,J,k} \text{ for } f \in L^2(\mathbb{R}^+). \tag{3.1}$$

It is not difficult to prove that $\{\psi_{0,J,k} : k \in \mathbb{Z}^+\}$ and $\{\tilde{\psi}_{0,J,k} : k \in \mathbb{Z}^+\}$ are also Bessel sequences for each $J \in \mathbb{Z}$ under the Bessel assumptions of integer translation of ψ_0 and $\tilde{\psi}_0$. Therefore, P_J is a bounded operator by the Cauchy-Schwarz inequality, and is well defined. Also we have next result.

Lemma 3.1. *Given $J \in \mathbb{Z}$, let $\{\mathcal{T}_k\psi_0 : k \in \mathbb{Z}^+\}$ and $\{\mathcal{T}_k\tilde{\psi}_0 : k \in \mathbb{Z}^+\}$ be Bessel sequences in $L^2(\mathbb{R}^+)$, then we have*

$$\lim_{J\to-\infty} P_J f = 0 \text{ for } f \in L^2(\mathbb{R}^+). \tag{3.2}$$

Proof. Fix $f \in L^2(\mathbb{R}^+)$. For an arbitrary $\epsilon > 0$, let $g \in L^2(\mathbb{R}^+)$ with $\mathrm{supp}(g) \subset [0,\, R]$ for some $R > 0$ such that $\|f - g\| < \epsilon$. Then by the above argument, we have

$$\|P_J f\| \le \|P_J(f-g)\| + \|P_J g\| \le C\epsilon + \|P_J g\| \text{ for some constant } C > 0.$$

Next, we prove $\lim_{J\to-\infty} P_J g = 0$ to complete the proof. We estimate

$$\|P_J g\|^2 \le C \sum_{k\in\mathbb{Z}^+} |\langle g,\, \tilde{\psi}_{0,J,k}\rangle|^2 \le C\|g\|^2 \sum_{k\in\mathbb{Z}^+} \int_{[0,\, R]} |\tilde{\psi}_{0,J,k}(x)|^2 dx$$

(3.3)
$$= C\|g\|^2 \sum_{k\in\mathbb{Z}^+} \int_{[0,\, R]} |p^{J/2}\tilde{\psi}_0(p^J x \ominus k)|^2 dx = C\|g\|^2 \int_{\cup_{k\in\mathbb{Z}^+}[0,\, p^J R + k]} |\tilde{\psi}_0(y)|^2 dy,$$

it tends to 0 as $J \to -\infty$ by Lebesgue's dominate convergence theorem, and thus $\lim_{J\to-\infty} P_J g = 0$. □

The following theorem shows that the equivalence of NDWFs between different scale levels, and an NDWF in $L^2(\mathbb{R}^+)$ can derive an HDWF.

Theorem 3.1. *Given an integer J_0. Let $\psi_0 \in L^2(\mathbb{R}^+)$ and $\Psi = \{\psi_1, \psi_2, \cdots, \psi_L\}$ be a finite subset in $L^2(\mathbb{R}^+)$. Suppose $\Big(X_{J_0}(\psi_0;\Psi),\, X_{J_0}(\tilde{\psi}_0;\widetilde{\Psi})\Big)$ is a* NDWF *for $L^2(\mathbb{R}^+)$, then $\Big(X_J(\psi_0;\Psi),\, X_J(\tilde{\psi}_0;\widetilde{\Psi})\Big)$ is a* NDWF *for $L^2(\mathbb{R}^+)$ for all integer J. In particular, $\Big(X(\Psi),\, X(\widetilde{\Psi})\Big)$ is a* HDWF *for $L^2(\mathbb{R}^+)$.*

Proof. For any integer J and $f, g \in L^2(\mathbb{R}^+)$, we have

$$\langle f,\, \tilde{\psi}_{0,J,k}\rangle = \langle D^{J_0-J}f,\, \tilde{\psi}_{0,J_0,k}\rangle,\ \langle \psi_{0,J,k},\, g\rangle = \langle \psi_{0,J_0,k},\, D^{J_0-J}g\rangle \tag{3.4}$$

and

$$\langle f,\, \tilde{\psi}_{l,j,k}\rangle = \langle D^{J_0-J}f,\, \tilde{\psi}_{l,j+J_0-J,k}\rangle,\ \langle \psi_{l,j,k},\, g\rangle = \langle \psi_{l,j+J_0-J,k},\, D^{J_0-J}g\rangle \tag{3.5}$$

due to D is a unitary operator on $L^2(\mathbb{R}^+)$. And thus, we have

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^+} \langle f,\, \tilde{\psi}_{0\,J,k}\rangle\langle \psi_{0,J,k},\, g\rangle + \sum_{l=1}^{L}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^+} \langle f,\, \tilde{\psi}_{l,j,k}\rangle\langle \psi_{l,j,k},\, g\rangle \\
&= \sum_{k\in\mathbb{Z}^+} \langle D^{J_0-J}f,\, \tilde{\psi}_{0\,J_0,k}\rangle\langle \psi_{0,J_0,k},\, D^{J_0-J}g\rangle \\
&+ \sum_{l=1}^{L}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^+} \langle D^{J_0-J}f,\, \tilde{\psi}_{l,j+J_0-J,k}\rangle\langle \psi_{l,j+J_0-J,k},\, D^{J_0-J}g\rangle \\
&= \sum_{k\in\mathbb{Z}^+} \langle D^{J_0-J}f,\, \tilde{\psi}_{0,J_0,k}\rangle\langle \psi_{0,J_0,k},\, D^{J_0-J}g\rangle \\
&\quad + \sum_{l=1}^{L}\sum_{j=J_0}^{\infty}\sum_{k\in\mathbb{Z}^+} \langle D^{J_0-J}f,\, \tilde{\psi}_{l,j,k}\rangle\langle \psi_{l,j,k},\, D^{J_0-J}g\rangle,
\end{aligned} \tag{3.6}$$

it equals to $\langle D^{J_0-J}f,\, D^{J_0-J}g\rangle$, and then equals to $\langle f,\, g\rangle$, since $(X_{J_0}(\psi_0;\Psi), X_{J_0}(\tilde{\psi}_0;\widetilde{\Psi}))$ is a NWDF for $L^2(\mathbb{R}^+)$. So $\Big(X_J(\psi_0;\Psi),\, X_J(\tilde{\psi}_0;\widetilde{\Psi})\Big)$ is a NDWF for $L^2(\mathbb{R}^+)$ for all integer J, and thus

$$\langle P_J f,\, g\rangle + \sum_{l=1}^{L}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^+}\langle f,\, \tilde{\psi}_{l,j,k}\rangle\langle \psi_{l,j,k},\, g\rangle = \langle f,\, g\rangle \text{ for } f,g\in L^2(\mathbb{R}^+). \tag{3.7}$$

Letting $J\to-\infty$ in (3.7) and using Lemma 3.1, we obtain

$$\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^+}\langle f,\, \tilde{\psi}_{l,j,k}\rangle\langle \psi_{l,j,k},\, g\rangle = \langle f,\, g\rangle \text{ for } f,g\in L^2(\mathbb{R}^+). \tag{3.8}$$

Therefore, $\Big(X(\Psi),\, X(\widetilde{\Psi})\Big)$ is a HDWF for $L^2(\mathbb{R}^+)$. The proof is completed. □

Theory 3.1 tells us that the study of NDWFs of the form $\Big(X_{J_0}(\psi_0;\Psi),\, X_{J_0}(\tilde{\psi}_0;\widetilde{\Psi})\Big)$ with general $J_0\in\mathbb{Z}$ can reduces to the study of NDWFs with $J_0=0$. The next theorem characterizes NDWFs in $L^2(\mathbb{R}^+)$ under the general Bessel assumption.

Theorem 3.2. *Let $\psi_0\in L^2(\mathbb{R}^+)$ and $\Psi=\{\psi_1,\psi_2,\cdots,\psi_L\}$ be a finite subset in $L^2(\mathbb{R}^+)$. Suppose $\{\mathcal{T}_k\psi_l : k\in\mathbb{Z}^+, 0\le l\le L\}$ and $\{\mathcal{T}_k\tilde{\psi}_l : k\in\mathbb{Z}^+, 0\le l\le L\}$ are Bessel sequences in $L^2(\mathbb{R}^+)$. Then $\Big(X(\psi_0,\,\Psi),\, X(\tilde{\psi}_0,\,\widetilde{\Psi})\Big)$ is a* NDWF *for $L^2(\mathbb{R}^+)$ if and only if*

$$\lim_{J\to\infty}\langle P_J f,\, g\rangle = \langle f,\, g\rangle \tag{3.9}$$

and

$$\langle P_{J+1} f,\, g\rangle = \langle P_J f,\, g\rangle + \sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^+}\langle f,\, \tilde{\psi}_{l,J,k}\rangle\langle \psi_{l,J,k},\, g\rangle \tag{3.10}$$

for $f,g\in L^2(\mathbb{R}^+)$ and $J\in\mathbb{Z}$, where P_J is defined as in (3.1).

Proof. "⇐": It follows from (3.10) that

$$\langle P_{J+1} f,\, g\rangle = \langle P_0 f,\, g\rangle + \sum_{l=1}^{L}\sum_{j=0}^{J}\sum_{k\in\mathbb{Z}^+}\langle f,\, \tilde{\psi}_{l,j,k}\rangle\langle \psi_{l,j,k},\, g\rangle \tag{3.11}$$

for $f,g\in L^2(\mathbb{R}^+)$ and $J\in\mathbb{Z}$. Letting $J\to\infty$ in (3.11) and using (3.9), we have

$$\langle f,\, g\rangle = \langle P_0 f,\, g\rangle + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^+}\langle f,\, \tilde{\psi}_{l,j,k}\rangle\langle \psi_{l,j,k},\, g\rangle \tag{3.12}$$

for $f,g\in L^2(\mathbb{R}^+)$. Therefore, $\Big(X(\psi_0,\,\Psi),\, X(\tilde{\psi}_0,\,\widetilde{\Psi})\Big)$ is a NDWF for $L^2(\mathbb{R}^+)$.

"⇒": Suppose $\Big(X(\psi_0,\, \Psi),\, X(\tilde{\psi}_0,\, \widetilde{\Psi})\Big)$ is a NDWF for $L^2(\mathbb{R}^+)$, then $(X_J(\psi_0,\, \Psi),$ $X_J(\tilde{\psi}_0,\, \widetilde{\Psi}))$ is a NDWF for $L^2(\mathbb{R}^+)$ for all integer J by Theory 3.1. It follows that

$$\begin{aligned} \langle f,\, g\rangle &= \langle P_{J+1}f,\, g\rangle + \sum_{l=1}^{L}\sum_{j=J+1}^{\infty}\sum_{k\in\mathbb{Z}^+}\langle f,\, \tilde{\psi}_{l,j,k}\rangle\langle\psi_{l,j,k},\, g\rangle \\ &= \langle P_J f,\, g\rangle + \sum_{l=1}^{L}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^+}\langle f,\, \tilde{\psi}_{l,j,k}\rangle\langle\psi_{l,j,k},\, g\rangle \end{aligned} \tag{3.13}$$

for $f, g \in L^2(\mathbb{R}^+)$ and $J \in \mathbb{Z}$, which leads to (3.10), and thus

$$\langle P_{J+1}f,\, g\rangle = \langle P_0 f,\, g\rangle + \sum_{l=1}^{L}\sum_{j=0}^{J}\sum_{k\in\mathbb{Z}^+}\langle f,\, \tilde{\psi}_{l,j,k}\rangle\langle\psi_{l,j,k},\, g\rangle \tag{3.14}$$

for $f, g \in L^2(\mathbb{R}^+)$ and $J \in \mathbb{Z}$. Also, observe that $\Big(X(\psi_0,\, \Psi),\, X(\tilde{\psi}_0,\, \widetilde{\Psi})\Big)$ is a NDWF for $L^2(\mathbb{R}^+)$. Letting $J \to \infty$ in (3.14), we obtain (3.9). The proof is completed. □

4. Refinable functions based construction of NDWFs in $L^2(\mathbb{R}^+)$

This section is devoted to constructing NDWFs from a pair of general refinable functions.

For $f, g \in L^2(\mathbb{R}^+)$, we define

$$[f,\, g](\cdot) = \sum_{k\in\mathbb{Z}^+} f(\cdot \oplus k)\overline{g(\cdot \oplus k)} \quad \text{a.e. on } \mathbb{R}^+, \tag{4.1}$$

then it belongs to $L^1(\mathbb{T})$, and is well defined. And we write

$$\mathcal{D} := \{f \in L^2(\mathbb{R}^+) : \mathcal{F}f \in L^\infty(\mathbb{R}^+) \text{ and } \operatorname{supp}(\mathcal{F}f) \text{ is bounded}\}, \tag{4.2}$$

where $\operatorname{supp}(\mathcal{F}f) = \{\xi \in \mathbb{R}^+ : \mathcal{F}f(\xi) \neq 0\}$ for $f \in L^2(\mathbb{R}^+)$ and is well defined up to a set 0. It is not difficult to verify that $\mathcal{D}$ is dense in $L^2(\mathbb{R}^+)$.

Now, let us make some assumptions:

Assumption 1. $\psi_0, \tilde{\psi}_0 \in L^2(\mathbb{R}^+)$ are p-refinable functions with symbols in $L^\infty(\mathbb{T})$, i.e., there exist $m_0, \widetilde{m}_0 \in L^\infty(\mathbb{T})$ such that

$$\mathcal{F}\psi_0(p\,\cdot) = m_0(\cdot)\mathcal{F}\psi_0(\cdot) \text{ and } \mathcal{F}\tilde{\psi}_0(p\,\cdot) = \widetilde{m}_0(\cdot)\mathcal{F}\tilde{\psi}_0(\cdot) \text{ a.e. on } \mathbb{R}^+. \tag{4.3}$$

Assumption 2. $\lim\limits_{j\to\infty} \mathcal{F}\psi_0(p^{-j}\cdot)\mathcal{F}\tilde{\psi}_0(p^{-j}\cdot) = 1$ a.e. on $\mathbb{R}^+$.

Assumption 3. $[\mathcal{F}\psi_0,\, \mathcal{F}\psi_0], [\mathcal{F}\tilde{\psi}_0,\, \mathcal{F}\tilde{\psi}_0] \in L^\infty(\mathbb{T})$.

Given $L \in \mathbb{N}$, let $m_l, \widetilde{m}_l \in L^\infty(\mathbb{T})$ with $1 \le l \le L$, and define ψ_l and $\tilde{\psi}_l$ by

$$\mathcal{F}\psi_l(p\,\cdot) = m_l(\cdot)\mathcal{F}\psi_0(\cdot) \text{ and } \mathcal{F}\tilde{\psi}_l(p\,\cdot) = \widetilde{m}_l(\cdot)\mathcal{F}\tilde{\psi}_0(\cdot) \text{ a.e. on } \mathbb{R}^+. \tag{4.4}$$

With m_l and $\widetilde{m}_l, l = 0, 1, \cdots, L$ as the framelet symbols, we write

(4.5)

$$\mathcal{M}(\cdot) = \begin{pmatrix} m_0(\cdot) & m_1(\cdot) & \cdots & m_L(\cdot) \\ m_0(\cdot \oplus 1/p) & m_1(\cdot \oplus 1/p) & \cdots & m_L(\cdot \oplus 1/p) \\ \vdots & \vdots & \ddots & \vdots \\ m_0(\cdot \oplus (p-1)/p) & m_1(\cdot \oplus (p-1)/p) & \cdots & m_L(\cdot \oplus (p-1)/p) \end{pmatrix}$$

and

(4.6)

$$\widetilde{\mathcal{M}}(\cdot) = \begin{pmatrix} \widetilde{m}_0(\cdot) & \widetilde{m}_1(\cdot) & \cdots & \widetilde{m}_L(\cdot) \\ \widetilde{m}_0(\cdot \oplus 1/p) & \widetilde{m}_1(\cdot \oplus 1/p) & \cdots & \widetilde{m}_L(\cdot \oplus 1/p) \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{m}_0(\cdot \oplus (p-1)/p) & \widetilde{m}_1(\cdot \oplus (p-1)/p) & \cdots & \widetilde{m}_L(\cdot \oplus (p-1)/p) \end{pmatrix}$$

We will study what m_l, $\widetilde{m}_l \in L^\infty(\mathbb{T})$ with $0 \le l \le L$ are qualified for $(X(\psi_0, \Psi)$, $X(\tilde{\psi}_0, \widetilde{\Psi}))$ to be a NDWF in $L^2(\mathbb{R}^+)$. We begin with some lemmas for latter use.

The following lemma shows that Assumption 3 is equivalent to the fact that $\{\mathcal{T}_k\psi_0 : k \in \mathbb{Z}^+\}$ is a Bessel sequence in $L^2(\mathbb{R}^+)$.

Lemma 4.1. ([33, Theorem 2.1]) *Let $\psi_0 \in L^2(\mathbb{R}^+)$. Then $\{\mathcal{T}_k\psi_0 : k \in \mathbb{Z}^+\}$ is a Bessel sequence in $L^2(\mathbb{R}^+)$ with Bessel bound B if and only if*

$$[\mathcal{F}\psi_0, \mathcal{F}\psi_0](\cdot) \le B \textit{ a.e. on } \mathbb{T}.$$

Observe that $\{\chi(k, \cdot) : k \in \mathbb{Z}^+\}$ is an orthonormal basis for $L^2(\mathbb{T})$ and the Walsh-Fourier transform is a unitary operator on $L^2(\mathbb{R}^+)$.

Lemma 4.2. *Let $k \in \mathbb{Z}^+$ and $f, \psi \in L^2(\mathbb{R}^+)$. Then, $\langle f, \psi_{j,k}\rangle$ is the k-th Walsh Fourier coefficient of $[p^{j/2}\mathcal{F}f(p^j\cdot), \mathcal{F}\psi(\cdot)]$ for each $j \in \mathbb{Z}^+$. In particular, we have*

$$[p^{j/2}\mathcal{F}f(p^j\cdot), \mathcal{F}\psi(\cdot)](\xi) = \sum_{k\in\mathbb{Z}^+} \langle f, \psi_{j,k}\rangle\chi(k, \xi) \textit{ a.e. } \xi \in \mathbb{R}^+, \tag{4.7}$$

if $\{\mathcal{T}_k\psi : k \in \mathbb{Z}^+\}$ is a Bessel sequence in $L^2(\mathbb{R}^+)$.

Proof. Since $f, \psi \in L^2(\mathbb{R}^+)$, we have $\mathcal{F}f(p^j\cdot)\overline{\mathcal{F}\psi(\cdot)} \in L^1(\mathbb{R}^+)$, and thus

$$\begin{aligned} \int_{\mathbb{T}} [p^{j/2}\mathcal{F}f(p^j\cdot), \mathcal{F}\psi(\cdot)](\xi)\chi(k, \xi)d\xi =& p^{j/2}\int_{\mathbb{R}^+} \mathcal{F}f(p^j\xi)\overline{\mathcal{F}\psi(\xi)}\chi(k, \xi)d\xi \\ =& p^{-j/2}\int_{\mathbb{R}^+} \mathcal{F}f(\xi)\overline{\mathcal{F}\psi(p^{-j}\xi)}\chi(k, p^{-j}\xi)d\xi \\ =& \int_{\mathbb{R}^+} \mathcal{F}f(\xi)\overline{[\mathcal{F}(\psi_{j,k})(\cdot)](\xi)}d\xi = \langle f, \psi_{j,k}\rangle, \end{aligned} \tag{4.8}$$

so $\langle f, \psi_{j,k}\rangle$ is the k-th Walsh-Fourier coefficient of $[p^{j/2}\mathcal{F}f(p^j\cdot), \mathcal{F}\psi(\cdot)]$ for each $j \in \mathbb{Z}^+$.

If $\{\mathcal{T}_k\psi : k \in \mathbb{Z}^+\}$ is a Bessel sequence in $L^2(\mathbb{R}^+)$, then $\{D^j\mathcal{T}_k\psi : k \in \mathbb{Z}^+\}$, that is, $\{\psi_{j,k} : k \in \mathbb{Z}^+\}$ is a Bessel sequence in $L^2(\mathbb{R}^+)$ for each $j \in \mathbb{Z}^+$ due to D^j being unitary, it follows that $\{\langle f,\, \psi_{j,k} : k \in \mathbb{Z}^+\rangle\} \in \ell^2(\mathbb{Z}^+)$, and thus (4.7) holds. □

As an application of Lemma 4.2, we have the following lemma immediately

Lemma 4.3. *Let $\psi_0, \tilde{\psi}_0 \in L^2(\mathbb{R}^+)$ satisfy Assumption 3. Then we have*

$$\langle P_n f,\, g\rangle = p^n \int_{\mathbb{T}} [\mathcal{F}f(p^n\cdot),\, \mathcal{F}\tilde{\psi}_0(\cdot)](\xi)[\mathcal{F}\psi_0,\, \mathcal{F}g(p^n\cdot)](\xi)d\xi$$

for $f, g \in L^2(\mathbb{R}^+)$ and $n \in \mathbb{Z}$, where P_n is defined as in (3.1).

The following two lemmas are necessary for us to prove the main result.

Lemma 4.4. *Let $\psi_0, \tilde{\psi}_0 \in L^2(\mathbb{R}^+)$ satisfy Assumptions 2 and 3. Then*

$$\lim_{n\to\infty} \langle P_n f,\, g\rangle = \langle f,\, g\rangle \text{ for } f, g \in \mathcal{D},$$

where $\mathcal{D}$ is defined as in (4.2).

Proof. By Lemma 4.3, we have

$$\langle P_n f,\, g\rangle = p^n \int_{[0,\,1]} [\mathcal{F}f(p^n\cdot),\, \mathcal{F}\tilde{\psi}_0(\cdot)](\xi)[\mathcal{F}\psi_0,\, \mathcal{F}g(p^n\cdot)](\xi)d\xi.$$

Since $p > 1$ and $\mathrm{supp}(\mathcal{F}f)$ and $\mathrm{supp}(\mathcal{F}g)$ are bounded, then there exists $N > 0$ such that $\mathrm{supp}(\mathcal{F}f(p^n\cdot)), \mathrm{supp}(\mathcal{F}g(p^n\cdot)) \subset [0,\, 1)$ when $n > N$, and thus

$$[\mathcal{F}f(p^n\cdot),\, \mathcal{F}\tilde{\psi}_0(\cdot)](\xi) = \mathcal{F}f(p^n\xi)\overline{\mathcal{F}\tilde{\psi}_0(\xi)}$$

and

$$[\mathcal{F}\psi_0(\cdot),\, \mathcal{F}g(p^n\cdot)](\xi) = \overline{\mathcal{F}g(p^n\xi)}\mathcal{F}\psi_0(\xi)$$

for a.e. $\xi \in (0,\, 1)$ and $n > N$. So

$$\begin{aligned}\langle P_n f,\, g\rangle &= p^n \int_{[0,\,1]} \mathcal{F}f(p^n\xi)\overline{\mathcal{F}g(p^n\xi)\mathcal{F}\tilde{\psi}_0(\xi)}\mathcal{F}\psi_0(\xi)d\xi \\ &= \int_{\mathbb{R}^+} \mathcal{F}f(\xi)\overline{\mathcal{F}g(\xi)\mathcal{F}\tilde{\psi}_0(p^{-n}\xi)}\mathcal{F}\psi_0(p^{-n}\xi)\chi_{[0,\,1]}(p^{-n}\xi)d\xi \qquad (4.9)\end{aligned}$$

when $n > N$. By Assumption 3 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\left|\overline{\mathcal{F}\tilde{\psi}_0(\cdot)}\mathcal{F}\psi_0(\cdot)\right| &\le \sum_{l\in\mathbb{Z}^+} \left|\overline{\mathcal{F}\tilde{\psi}_0(\cdot \oplus l)}\mathcal{F}\psi_0(\cdot \oplus l)\right| \\ &\le \left([\mathcal{F}\tilde{\psi}_0,\, \mathcal{F}\tilde{\psi}_0](\cdot)\right)^{1/2} \left([\mathcal{F}\psi_0,\, \mathcal{F}\psi_0](\cdot)\right)^{1/2} \le C\end{aligned}$$

for some constant $C > 0$. Therefore, the integrand in (4.9) is dominated in module by $C|\mathcal{F}f(\cdot)\mathcal{F}g(\cdot)|$, which belongs to $L^1(\mathbb{R}^+)$. Applying the Lebesgue dominated convergence theorem to (4.9), we obtain

$$\lim_{n\to\infty} \langle P_n f,\, g\rangle = \langle f,\, g\rangle$$

by Assumption 2. □

Lemma 4.5. *Let $\psi_0, \tilde{\psi}_0 \in L^2(\mathbb{R}^+)$ satisfy Assumptions 1 and 3. Assume that $m_l, \widetilde{m}_l \in L^\infty(\mathbb{T})$ with $1 \le l \le L$, are such that*

$$\mathcal{M}(\cdot)\widetilde{\mathcal{M}}^*(\cdot) = I_p \text{ a.e. on } \mathbb{T}, \tag{4.10}$$

where $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are defined as in (4.5) and (4.6). Define $\psi_l, \tilde{\psi}_l, 1 \le l \le L$ as in (4.4). Then

$$\langle P_{n+1} f,\, g\rangle = \langle P_n f,\, g\rangle + \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}^+} \langle f,\, \tilde{\psi}_{l,n,k}\rangle \langle \psi_{l,n,k},\, g\rangle \tag{4.11}$$

for $f, g \in L^2(\mathbb{R}^+)$ and $n \in \mathbb{Z}$.

Proof. First, we claim that (4.11) is equivalent to

$$\langle P_1 f,\, g\rangle = \langle P_0 f,\, g\rangle + \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}^+} \langle f,\, \tilde{\psi}_{l,0,k}\rangle \langle \psi_{l,0,k},\, g\rangle \tag{4.12}$$

for $f, g \in L^2(\mathbb{R}^+)$. Indeed, if (4.12) holds, we can get (4.11) by replacing f by $D^{-n} f$ and g by $D^{-n} g$ in (4.12), respectively. And, by Lemma 4.3, (4.12) can be written as

$$p \int_{\mathbb{T}} [\mathcal{F}f(p\,\cdot),\, \mathcal{F}\tilde{\psi}_0(\cdot)](\xi)[\mathcal{F}\psi_0(\cdot),\, \mathcal{F}g(p^n\cdot)](\xi)d\xi = \int_{\mathbb{T}} \sum_{l=0}^{L} [\mathcal{F}f,\, \mathcal{F}\tilde{\psi}_l](\xi)[\mathcal{F}\psi_l,\, \mathcal{F}g](\xi)d\xi \tag{4.13}$$

for $f, g \in L^2(\mathbb{R}^+)$.

Next, we prove (4.13) to complete the proof. Note that, $m_l, \widetilde{m}_l, 1 \le l \le L$ are 1-periodic functions. By the definitions of $\tilde{\psi}_l, 1 \le l \le L$ and Assumption 1, we have

$$\begin{aligned}
[\mathcal{F}f,\, \mathcal{F}\tilde{\psi}_l](\xi) &= \sum_{k \in \mathbb{Z}^+} \mathcal{F}f(\xi \oplus k)\overline{\widetilde{m}_l(p^{-1}(\xi \oplus k))\mathcal{F}\tilde{\psi}_0(p^{-1}(\xi \oplus k))} \\
&= \sum_{i=0}^{p-1} \overline{\widetilde{m}_l(p^{-1}(\xi \oplus i/p))} \sum_{k \in \mathbb{Z}^+} \mathcal{F}f(\xi \oplus i/p \oplus pk)\mathcal{F}\tilde{\psi}_0(p^{-1}(\xi \oplus i/p) \oplus k) \\
&= \sum_{i=0}^{p-1} \overline{\widetilde{m}_l(p^{-1}(\xi \oplus i/p))}[\mathcal{F}f(p\,\cdot),\, \mathcal{F}\tilde{\psi}_0(\cdot)](p^{-1}(\xi \oplus i/p)) \qquad (4.14)
\end{aligned}$$

for $0 \le l \le L$. Similarly, we have

$$[\mathcal{F}\psi_l,\, \mathcal{F}g](\xi) = \sum_{i'=0}^{p-1} m_l(p^{-1}(\xi \oplus i'/p))[\mathcal{F}\psi_0(\cdot),\, \mathcal{F}g(p\,\cdot)](p^{-1}(\xi \oplus i'/p)) \tag{4.15}$$

for $0 \le l \le L$. By a simple computation, we obtain

$$\begin{aligned}
\sum_{l=0}^{L} [\mathcal{F}f,\, \mathcal{F}\tilde{\psi}_l](\xi)[\mathcal{F}\psi_l,\, \mathcal{F}g](\xi) &= \sum_{i=0}^{p-1} [\mathcal{F}f(p\,\cdot),\, \mathcal{F}\tilde{\psi}_0(\cdot)](p^{-1}(\xi \oplus i/p)) \times \\
&\times \sum_{i'=0}^{p-1} \left(\mathcal{M}\widetilde{\mathcal{M}}^*(p^{-1}\xi)\right)_{i,i'} [\mathcal{F}\psi_0(\cdot),\, \mathcal{F}g(p\,\cdot)](p^{-1}(\xi \oplus i'/p)), \qquad (4.16)
\end{aligned}$$

where $\left(\mathcal{M}\widetilde{\mathcal{M}}^*(\cdot)\right)_{i,i'}$ denotes the $(i,\, i')$-entry of $\mathcal{M}\widetilde{\mathcal{M}}^*(\cdot), 0 \le i, i' \le p-1$. By (4.10), (4.13) therefore follows that

$$\begin{aligned}
&\int_{\mathbb{T}} \sum_{l=0}^{L} [\mathcal{F}f,\, \mathcal{F}\tilde{\psi}_l](\xi)[\mathcal{F}\psi_l,\, \mathcal{F}g](\xi) d\xi \\
&= \int_{\mathbb{T}} \sum_{i=0}^{p-1} [\mathcal{F}f(p\,\cdot),\, \mathcal{F}\tilde{\psi}_0(\cdot)](p^{-1}(\xi \oplus i/p))[\mathcal{F}\psi_0(\cdot),\, \mathcal{F}g(p\,\cdot)](p^{-1}(\xi \oplus i/p)) d\xi \\
&= p \sum_{i=0}^{p-1} \int_{p^{-1}(\mathbb{T}+i/p)} [\mathcal{F}f(p\,\cdot),\, \mathcal{F}\tilde{\psi}_0(\cdot)](\xi)[\mathcal{F}\psi_0(\cdot),\, \mathcal{F}g(p\,\cdot)](\xi) d\xi \\
&= p \int_{\mathbb{T}} [\mathcal{F}f(p\,\cdot),\, \mathcal{F}\tilde{\psi}_0(\cdot)](\xi)[\mathcal{F}\psi_0(\cdot),\, \mathcal{F}g(p^n\cdot)](\xi) d\xi. \qquad (4.17)
\end{aligned}$$

Therefore, (4.13) holds. The proof is completed. □

The following theorem gives a sufficient condition for $\left(X(\psi_0,\, \Psi),\, X(\tilde{\psi}_0,\, \widetilde{\Psi})\right)$ to be a NDWF in $L^2(\mathbb{R}^+)$.

Theorem 4.1. *Let $\psi_0, \tilde{\psi}_0 \in L^2(\mathbb{R}^+)$ satisfy Assumptions* 1-3*. Assume that $m_l, \widetilde{m}_l, \in L^\infty(\mathbb{T})$ with $1 \le l \le L$, are such that*

$$\mathcal{M}(\cdot)\widetilde{\mathcal{M}}^*(\cdot) = I_p \ \textit{a.e. on } \mathbb{T}. \qquad (4.18)$$

where $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are defined as in (4.5) *and* (4.6)*. Define ψ_l and $\tilde{\psi}_l, 1 \le l \le L$ as in* (4.4)*. Then $\left(X(\psi_0,\, \Psi),\, X(\tilde{\psi}_0,\, \widetilde{\Psi})\right)$ is a* NDWF *for $L^2(\mathbb{R}^+)$.*

Proof. Since $m_l, \widetilde{m}_l, \in L^\infty(\mathbb{T})$ for $1 \le l \le L$, by Lemma 4.1 and Assumptions 1 and 3, then we have

$$\{\mathcal{T}_k\psi_l : k \in \mathbb{Z}^+, 1 \le l \le L\} \quad \text{and} \quad \{\mathcal{T}_k\tilde{\psi}_l : k \in \mathbb{Z}^+, 1 \le l \le L\}$$

are Bessel sequences in $L^2(\mathbb{R}^+)$. Therefore, the conclusion follows directly by Theory 3.2, Lemmas 4.4 and 4.5. The proof is completed. □

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