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EMBEDDING OF BESOV SPACES INTO TENT SPACES AND APPLICATIONS

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Abstract. In this paper, we study the boundedness and compactness of the inclusion mapping from Besov spaces to tent spaces. Meanwhile, the boundedness, compactness and essential norm of Volterra integral operators from Besov spaces to general function spaces are also investigated.

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1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the class of functions analytic in $\mathbb{D}$. The Hardy space H^p $(0 < p < \infty)$ is the set of all $f \in H(\mathbb{D})$ with (see [4])

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

Let H^∞ be the space of all bounded analytic functions with the supremum norm $\|f\|_{H^\infty} = \sup_{z\in\mathbb{D}} |f(z)|$.

For $1 < p < \infty$, the Besov space, denoted by B_p, is the space of all functions $f \in H(\mathbb{D})$ satisfy

$$\|f\|_{B_p}^p = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2} dA(z) < \infty.$$

Let $0 < p < \infty$, $-2 < q < \infty$ and $0 \le s < \infty$. The space $F(p, q, s)$ is the space consisting of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{F(p,q,s)}^p = |f(0)|^p + \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q (1 - |\varphi_a(z)|^2)^s dA(z) < \infty,$$

where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. This space was first introduced by Zhao in [22]. $F(2, 0, s)$ is the Q_s space (see [18]). $F(2, 0, 1)$ is the $BMOA$ space. $F(p, \alpha, 0)$ is called the Dirichlet type space, denoted by $\mathcal{D}^p_\alpha$. In particular, $F(p, p-2, 0)$ is the Besov space

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B_p. $F(p,p,0)$ is just the classical Bergman space A^p. When $s>1$, from [22] we see that $F(p,p-2,s)$ is equivalent to the Bloch space, denoted by $\mathcal{B}$, which consisting of all $f\in H(\mathbb{D})$ such that $\|f\|_{\mathcal{B}}=|f(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|<\infty$.
The Volterra integral operator T_g was introduced by Pommerenke in [13]. Pommerenke showed that T_g is bounded on H^2 if and only if $g\in BMOA$, where

$$T_gf(z)=\int_0^z f(w)g'(w)dw, \qquad f\in H(\mathbb{D}).$$

The companion operator I_g induced by $g\in H(\mathbb{D})$ is defined by

$$I_gf(z)=\int_0^z f'(w)g(w)dw, \qquad f\in H(\mathbb{D}).$$

The multiplication operator M_g is defined by $M_gf(z)=f(z)g(z)$. It is easy to see that $M_gf(z)=f(0)g(0)+I_gf(z)+T_gf(z)$. Recently, much attention has been paid to the operators T_g and I_g.

See [1, 2], [5]-[9], [11]-[16], [20, 21] and the references therein for more study of the operators T_g and I_g.

For any arc $I\subseteq\partial\mathbb{D}$, the boundary of $\mathbb{D}$, let $|I|=\frac{1}{2\pi}\int_I|d\zeta|$ denote the normalized length of I and $S(I)$ be the Carleson box defined by

$$S(I)=\{z\in\mathbb{D}:1-|I|\le|z|<1, \quad z/|z|\in I\}.$$

Let $0\le s<\infty, 0<q<\infty$ and μ be a positive Borel measure on $\mathbb{D}$. Let $T_s^q(\mu)$ be the space of all μ-measurable functions f such that (see, e.g., [12])

$$\sup_{I\subseteq\partial\mathbb{D}}\frac{1}{|I|^s}\int_{S(I)}|f(z)|^q d\mu(z)<\infty.$$

Let $0\le\alpha<\infty$, $0<s<\infty$ and μ be a positive Borel measureon $\mathbb{D}$. We say that μ is a α-logarithmic s-Carleson measure if (see [21])

$$\|\mu\|_{LCM_{\alpha,s}}:=\sup_{I\subseteq\partial\mathbb{D}}\frac{(\log\frac{2}{|I|})^\alpha\mu(S(I))}{|I|^s}<\infty.$$

When $\alpha=0$, it gives the s-Carleson measure. When $\alpha=0, s=1$, it gives the classical Carleson measure. μ is said to be a vanishing α-logarithmic s-Carleson measure if (see [11])

$$\lim_{|I|\to0}\frac{(\log\frac{2}{|I|})^\alpha\mu(S(I))}{|I|^s}=0.$$

The Carleson measure is very useful in the theory of function spaces and operator theory. The famous embedding theorem say that the inclusion mapping $i:H^p\to L^p(d\mu)$ is bounded if and only if μ is a Carleson measure (see [4]). See [3] for the study of Carleson measure for the Besov space B_p. In [5], Girela and Peláez studied the Carleson measure for Dirichlet type spaces. Among others, under the assumption that $0<p<q<\infty$, they showed that the inclusion mapping $i:B_p\to$

$L^q(d\mu)$ is bounded if and only if μ is $q(1-\frac{1}{p})$-logarithmic 0-Carleson measure. In [20], Xiao proved that the inclusion mapping $i: Q_s \to T_s^2(\mu)$ is bounded if and only if μ is 2-logarithmic s-Carleson measure. In [10], Liu and Lou showed that the inclusion mapping $i: \mathcal{L}^{2,s} \to T_s^2(\mu)$ is bounded if and only if μ is a Carleson measure, where $\mathcal{L}^{2,s}$ is the Morrey space. The main ideas and methods used in [10] more or less are motivated by the three sections 3.2, 4.3, 6.4 of [19]. In [12], Pau and Zhao showed that the inclusion mapping $i: F(p,p-2,s) \to T_s^p(\mu)$ is bounded if and only if μ is p-logarithmic s-Carleson measure. In [7], Li, Liu and Yuan proved that the inclusion mapping $i: \mathcal{D}_{p-1}^p \to T_s^p(\mu)$ is bounded if and only if μ is a $(s+1)$-Carleson measure by using the Carleson embedding theorem for Bergman spaces.

Motivated by [5, 7, 10, 12, 20], in this paper, we study the boundedness and compactness of the inclusion mapping from B_p into $T_s^q(\mu)$. More precisely, we show that the inclusion mapping $i: B_p \to T_s^q(\mu)$ is bounded (resp. compact) if and only if μ is a $q(1-\frac{1}{p})$-logarithmic s-Carleson measure (resp. vanishing $q(1-\frac{1}{p})$-logarithmic s-Carleson measure) under the assumption that $1 < p < q < \infty$ and $0 < s < \infty$. Moreover, we study the boundedness, compactness and essential norm of the operators T_g and I_g acting from B_p to $F(q,q-2,s)$.

In this paper, the symbol $f \approx g$ means that $f \lesssim g \lesssim f$. We say that $f \lesssim g$ if there exists a constant C such that $f \leq Cg$.

2. Embedding from Besov spaces B_p to $T_s^q(\mu)$

We need the following equivalent description of p-logarithmic s-Carleson measure, see Lemma 2.2 in [12].

Lemma 2.1. *Let $0 \leq \alpha < \infty, 0 < s, t < \infty$ and μ be a positive Borel measure on $\mathbb{D}$. Then μ is a α-logarithmic s-Carleson measure if and only if*

$$\sup_{a\in\mathbb{D}} \left(\log \frac{2}{1-|a|^2}\right)^{\alpha} \int_{\mathbb{D}} \frac{(1-|a|^2)^t}{|1-\bar{a}z|^{s+t}} d\mu(z) < \infty.$$

Moreover,

$$\|\mu\|_{LCM_{\alpha,s}} \approx \sup_{a\in\mathbb{D}} \left(\log \frac{2}{1-|a|^2}\right)^{\alpha} \int_{\mathbb{D}} \frac{(1-|a|^2)^t}{|1-\bar{a}z|^{s+t}} d\mu(z).$$

Using [23, Lemma 3.10], we can easily obtain the following result.

Lemma 2.2. *Let $1 < p < \infty$ and $w \in \mathbb{D}$. Set*

$$f_w(z) = \left(\frac{1}{\log \frac{2}{1-|w|^2}}\right)^{1/p} \log \frac{2}{1-\overline{w}z}, \qquad F_w(z) = \frac{1-|w|^2}{\overline{w}(1-\overline{w}z)}, \qquad z \in \mathbb{D}.$$

Then $f_w, F_w \in B_p$.

Lemma 2.3. *Let $1 < p \le q < \infty$, $0 < s < \infty$ and μ be a positive Borel measure on $\mathbb{D}$. Suppose that $f \in B_p$ and μ is a $q(1-\frac{1}{p})$-logarithmic s-Carleson measure. Then*

$$\int_{\mathbb{D}} |f(z)|^q d\mu(z) \lesssim \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2+s} (\log \frac{2}{1-|z|^2})^{\frac{q}{p}} dA(z).$$

Proof. Suppose that $f \in B_p$. For any fixed q, s, let α be big enough such that $q\alpha - s > 0$ and $q\alpha + 2 - q - 2s > 0$. From the proof of [12, Lemma 3.2] we have

$$|f(z)|^q \lesssim \int_{\mathbb{D}} \frac{|f'(w)|^q (1-|w|^2)^{q\alpha}}{|1-\overline{w}z|^{q\alpha+2-q}} \left(\log \frac{2}{1-|w|^2}\right)^q dA(w).$$

Since μ is a $q(1-\frac{1}{p})$-logarithmic s-Carleson measure, combining with Lemma 2.1 and the fact that $B_p \subseteq \mathcal{B}$, we obtain

$$\int_{\mathbb{D}} |f(z)|^q d\mu(z) \lesssim \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f'(w)|^q (1-|w|^2)^{q\alpha}}{|1-\overline{w}z|^{q\alpha+2-q}} \left(\log \frac{2}{1-|w|^2}\right)^q dA(w) d\mu(z)$$

$$\lesssim \int_{\mathbb{D}} |f'(w)|^q (1-|w|^2)^{q-2+s} (\log \frac{2}{1-|w|^2})^{\frac{q}{p}} \left(\left(\log \frac{2}{1-|w|^2}\right)^{q(1-\frac{1}{p})} \times \right.$$

$$\left. \times \int_{\mathbb{D}} \frac{(1-|w|^2)^s}{|1-\overline{w}z|^{2s}} d\mu(z) \right) dA(w) \lesssim \int_{\mathbb{D}} |f'(w)|^p (1-|w|^2)^{p-2+s} (\log \frac{2}{1-|w|^2})^{\frac{q}{p}} dA(w).$$

The proof is complete. □

Theorem 2.1. *Let $1 < p < q < \infty$, $0 < s < \infty$ and μ be a positive Borel measure on $\mathbb{D}$. Then the inclusion mapping $i : B_p \to T_s^q(\mu)$ is bounded if and only if μ is a $q(1-\frac{1}{p})$-logarithmic s-Carleson measure.*

Proof. First we assume that $i : B_p \to T_s^q(\mu)$ is bounded. For any given arc $I \subseteq \partial\mathbb{D}$, set $a = (1-|I|)\eta$ and η is the center point of I. It is easy to see that

$$|1-\overline{a}z| \approx 1-|a|^2 \approx |I|, \quad z \in S(I).$$

Let

$$f_a(z) = \left(\frac{1}{\log \frac{2}{1-|a|^2}} \right)^{1/p} \log \frac{2}{1-\overline{a}z}.$$

By Lemma 2.2, we see that $f_a \in B_p$. From the boundedness of $i : B_p \to T_s^q(\mu)$, we have

$$\|f_a\|^q_{T_s^q(\mu)} = \sup_{I \subseteq \partial\mathbb{D}} \frac{1}{|I|^s} \int_{S(I)} |f_a(z)|^q d\mu(z) < \infty.$$

By the fact that $|f_a(z)| \approx (\log \frac{2}{|I|})^{1-\frac{1}{p}}$ when $z \in S(I)$, we get

$$\sup_{I \subseteq \partial\mathbb{D}} \frac{(\log \frac{2}{|I|})^{q(1-\frac{1}{p})} \mu(S(I))}{|I|^s} < \infty.$$

Hence μ is a $q(1-\frac{1}{p})$-logarithmic s-Carleson measure.

Conversely, assume that μ is a $q(1-\frac{1}{p})$-logarithmic s-Carleson measure. Let $f\in B_p$. For any given arc $I\subseteq\partial\mathbb{D}$, set $w=(1-|I|)\eta$ and η is the center point of I. Then

$$\begin{aligned}\frac{1}{|I|^s}\int_{S(I)}|f(z)|^q d\mu(z)\lesssim&\frac{1}{|I|^s}\int_{S(I)}|f(z)-f(w)|^q d\mu(z)+\frac{1}{|I|^s}\int_{S(I)}|f(w)|^q d\mu(z)\\ =&A+B,\end{aligned}$$

where

$$A=\frac{1}{|I|^s}\int_{S(I)}|f(z)-f(w)|^q d\mu(z),\qquad B=\frac{1}{|I|^s}\int_{S(I)}|f(w)|^q d\mu(z).$$

Since

$$|f(w)|\lesssim\left(\log\frac{2}{1-|w|^2}\right)^{1-\frac{1}{p}}\|f\|_{B_p}\lesssim\left(\log\frac{2}{|I|}\right)^{1-\frac{1}{p}}\|f\|_{B_p},$$

we get

$$B\lesssim\frac{(\log\frac{2}{|I|})^{q(1-\frac{1}{p})}\mu(S(I))}{|I|^s}\|f\|_{B_p}^q\lesssim\|f\|_{B_p}^q.$$

By Lemma 2.3, we have

$$\begin{aligned}A\lesssim&(1-|w|^2)^s\int_{S(I)}\left|\frac{f(z)-f(w)}{(1-\overline{w}z)^{\frac{2s}{q}}}\right|^q d\mu(z)\\ \lesssim&(1-|w|^2)^s\int_{\mathbb{D}}\left|\left(\frac{f(z)-f(w)}{(1-\overline{w}z)^{\frac{2s}{q}}}\right)'\right|^p(1-|z|^2)^{p-2+s}(\log\frac{2}{1-|z|^2})^{\frac{q}{p}}dA(z).\end{aligned}$$

Since

$$\left(\frac{f(z)-f(w)}{(1-\overline{w}z)^{\frac{2s}{q}}}\right)'=\frac{f'(z)(1-\overline{w}z)^{\frac{2s}{q}}+\overline{w}(\frac{2s}{q})(f(z)-f(w))(1-\overline{w}z)^{\frac{2s}{q}-1}}{(1-\overline{w}z)^{\frac{4s}{q}}},$$

we deduce that $A\lesssim W_1+W_2$, where

$$W_1=(1-|w|^2)^s\int_{\mathbb{D}}\frac{|f'(z)|^p}{|1-\overline{w}z|^{\frac{2ps}{q}}}(1-|z|^2)^{p-2+s}(\log\frac{2}{1-|z|^2})^{\frac{q}{p}}dA(z)$$

and

$$W_2=(1-|w|^2)^s\int_{\mathbb{D}}\frac{|f(z)-f(w)|^p}{|1-\overline{w}z|^{\frac{2ps}{q}+p}}(1-|z|^2)^{p-2+s}(\log\frac{2}{1-|z|^2})^{\frac{q}{p}}dA(z).$$

Since $p<q$ and $\sup_{z\in\mathbb{D}}(1-|z|^2)^{2s(1-\frac{p}{q})}(\log\frac{2}{1-|z|^2})^{\frac{q}{p}}<\infty$, we get that

$$W_1\lesssim\|f\|_{B_p}^p.$$

Let $0<\epsilon<\min\{\frac{p}{2},s,2s(1-\frac{p}{q})\}$. Combining with the fact that $\sup_{z\in\mathbb{D}}(1-|z|^2)^{\epsilon}(\log\frac{2}{1-|z|^2})^{\frac{q}{p}}<\infty$, we obtain

$$W_2=(1-|w|^2)^s\int_{\mathbb{D}}\frac{|f(z)-f(w)|^p}{|1-\overline{w}z|^{\frac{2ps}{q}+p}}(1-|z|^2)^{p-2+s-\epsilon}dA(z).$$

Making the change of variable $\eta=\varphi_w(z)$ and combining with [23, Proposition 4.2], we have

$$
\begin{aligned}
W_2 = (1-|w|^2)^s \int_{\mathbb{D}} & \frac{|(f\circ\varphi_w)(\eta)-(f\circ\varphi_w)(0)|^p}{|1-\overline{w}\varphi_w(\eta)|^{\frac{2ps}{q}+p}}(1-|\varphi_w(\eta)|^2)^{p-2+s-\epsilon} \\
& \times \frac{(1-|w|^2)^2}{|1-\overline{w}\eta|^4} dA(\eta) \\
=&(1-|w|^2)^{2s-\frac{2ps}{q}-\epsilon}\int_{\mathbb{D}} |(f\circ\varphi_w)(\eta)-(f\circ\varphi_w)(0)|^p \frac{(1-|\eta|^2)^{p-2+s-\epsilon}}{|1-\overline{w}\eta|^{p+2s-\frac{2ps}{q}-2\epsilon}} dA(\eta) \\
\lesssim&(1-|w|^2)^{2s-\frac{2ps}{q}-\epsilon}\int_{\mathbb{D}} |(f\circ\varphi_w)'(\eta)|^p \frac{(1-|\eta|^2)^{2p-2+s-\epsilon}}{|1-\overline{w}\eta|^{p+2s-\frac{2ps}{q}-2\epsilon}} dA(\eta) \\
\lesssim&(1-|w|^2)^{2s-\frac{2ps}{q}-\epsilon}\int_{\mathbb{D}} |f'(\varphi_w(\eta))|^p(1-|\varphi_w(\eta)|^2)^p \frac{(1-|\eta|^2)^{p-2+s-\epsilon}}{|1-\overline{w}\eta|^{p+2s-\frac{2ps}{q}-2\epsilon}} dA(\eta) \\
\lesssim&(1-|w|^2)^{2s-\frac{2ps}{q}-\epsilon}\int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^p \frac{(1-|\varphi_w(z)|^2)^{p-2+s-\epsilon}}{|1-\overline{w}\varphi_w(z)|^{p+2s-\frac{2ps}{q}-2\epsilon}} \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4} dA(z) \\
\lesssim&(1-|w|^2)^{s}\int_{\mathbb{D}} |f'(z)|^p \frac{(1-|z|^2)^{2p-2+s-\epsilon}}{|1-\overline{w}z|^{p+\frac{2ps}{q}}} dA(z) \lesssim \|f\|_{B_p}^p.
\end{aligned}
$$

Therefore,

$$
\sup_{I\subseteq\partial\mathbb{D}} \frac{1}{|I|^s}\int_{S(I)} |f(z)|^q d\mu(z) \lesssim \|f\|_{B_p}^p,
$$

which implies the desired result. The proof is complete. □

We say that the inclusion mapping $i: B_p\to T_s^q(\mu)$ is compact if

$$
\lim_{n\to\infty} \frac{1}{|I|^s}\int_{S(I)} |f_n(z)|^q d\mu(z) = 0
$$

whenever $I\subseteq\partial\mathbb{D}$ and $\{f_n\}$ is a bounded sequence in B_p that converges to 0 uniformly on compact subsets of $\mathbb{D}$.

Theorem 2.2. *Let $1<p<q<\infty$, $0<s<\infty$. Let μ be a nonnegative Borel measure on $\mathbb{D}$ such that point evaluation is a bounded functional on $T_s^q(\mu)$. Then the inclusion mapping $i: B_p\to T_s^q(\mu)$ is compact if and only if μ is a vanishing $q(1-\frac{1}{p})$-logarithmic s-Carleson measure.*

Proof. First we assume that $i: B_p\to T_s^q(\mu)$ is compact. Let $\{I_k\}$ be a sequence arcs with $\lim_{k\to\infty}|I_k|=0$. Set $a_k=(1-|I_k|)\eta_k$, where η_k is the midpoint of arc I_k. Take

$$
f_k(z) = \left(\frac{1}{\log\frac{2}{1-|a_k|^2}}\right)^{1/p} \log\frac{2}{1-\overline{a_k}z}.
$$

We see that $f_k \in B_p$ and $\{f_k\}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$ when $k \to \infty$. Then we get

$$\frac{\left(\log \frac{2}{|I_k|}\right)^{q(1-\frac{1}{p})} \mu(S(I_k))}{|I_k|^s} \lesssim \frac{1}{|I_k|^s} \int_{S(I_k)} |f_k(z)|^q d\mu(z) \to 0,$$

as $k \to \infty$, which implies that μ is a vanishing $q(1-\frac{1}{p})$-logarithmic s-Carleson measure.

Conversely, assume that μ is a vanishing $q(1-\frac{1}{p})$-logarithmic s-Carleson measure. From [12] we see that

$$\|\mu - \mu_r\|_{LCM_{q(1-\frac{1}{p}),s}} \to 0, r \to 1.$$

Here $\mu_r(z) = \mu(z)$ for $|z| < r$ and $\mu_r(z) = 0$ for $r \le |z| < 1$. Let $\|f_k\|_{B_p} \lesssim 1$ and $\{f_k\}$ converge to 0 uniformly on compact subsets of $\mathbb{D}$. Then

$$\begin{aligned}\frac{1}{|I|^s}\int_{S(I)} |f_k(z)|^q d\mu(z) &\lesssim \frac{1}{|I|^s}\int_{S(I)} |f_k(z)|^q d\mu_r(z) + \frac{1}{|I|^s}\int_{S(I)} |f_k(z)|^q d(\mu-\mu_r)(z)\\ &\lesssim \frac{1}{|I|^s}\int_{S(I)} |f_k(z)|^q d\mu_r(z) + \|\mu-\mu_r\|_{LCM_{q(1-\frac{1}{p}),s}} \|f_k\|_{B_p}^q\\ &\lesssim \frac{1}{|I|^s}\int_{S(I)} |f_k(z)|^q d\mu_r(z) + \|\mu-\mu_r\|_{LCM_{q(1-\frac{1}{p}),s}}.\end{aligned}$$

Letting $k \to \infty$ and then $r \to 1$, we have $\lim_{k\to\infty} \|f_k\|_{T_s^q(\mu)} = 0$. Therefore $i : B_p \to T_s^q(\mu)$ is compact. □

3. The operators T_g and I_g from B_p to $F(q, q-2, s)$

In this section, we consider the boundedness, compactness and essential norm of operators T_g and I_g from B_p to $F(q, q-2, s)$. Before we state our results in this section, let us recall some definitions.

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and $T : X \to Y$ be a bounded linear operator. The essential norm of $T : X \to Y$ is defined by

$$\|T\|_{e,X\to Y} = \inf_K \{\|T-K\|_{X\to Y} : K \text{ is compact from } X \text{ to } Y\}.$$

Let Φ be a closed subspace of X. Given $f \in X$, the distance from f to Φ, denoted by $\mathrm{dist}_X(f, \Phi)$, is defined by $\mathrm{dist}_X(f,\Phi) = \inf_{g\in\Phi} \|f-g\|_X$.

Suppose that $0 \le \alpha < \infty, 0 < q, s < \infty$. The space $F_L(q, q-2, s, \alpha)$ is the space consisting of all $f \in H(\mathbb{D})$ such that

$$\|f\|_L^q = \sup_{a\in\mathbb{D}} \left(\log \frac{2}{1-|a|^2}\right)^{\alpha} \int_{\mathbb{D}} |f'(z)|^q (1-|z|^2)^{q-2} (1-|\varphi_a(z)|^2)^s dA(z) < \infty.$$

It is easy to check that $F_L(q, q-2, s, \alpha)$ is a Banach space under the norm $\|f\|_{F_L(q,q-2,s,\alpha)}^q = |f(0)|^q + \|f\|_L^q$ when $q \ge 1$. When $\alpha = 0$, $F_L(q, q-2, s, 0)$ is just the space

$F(q,q-2,s)$. Let $F^0_L(q,q-2,s,\alpha)$ denote the space of all $f\in F_L(q,q-2,s,\alpha)$ such that

$$\lim_{|a|\to 1}\left(\log\frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|f'(z)|^q(1-|z|^2)^{q-2}(1-|\varphi_a(z)|^2)^s dA(z)=0.$$

By Lemma 2.1 we easily obtain the following equivalent characterization of the space $F_L(q,q-2,s,\alpha)$.

Lemma 3.1. *Let $0\le\alpha<\infty, 0<q,s<\infty$. Then $f\in F_L(q,q-2,s,\alpha)$ if and only if*

$$\sup_{I\subseteq\partial\mathbb{D}}\frac{\left(\log\frac{2}{|I|}\right)^{\alpha}}{|I|^s}\int_{S(I)}|f'(z)|^q(1-|z|^2)^{q-2+s}dA(z)<\infty.$$

Moreover,

$$\|f\|^q_{F_L(q,q-2,s,\alpha)}\approx\sup_{I\subseteq\partial\mathbb{D}}\frac{\left(\log\frac{2}{|I|}\right)^{\alpha}}{|I|^s}\int_{S(I)}|f'(z)|^q(1-|z|^2)^{q-2+s}dA(z).$$

Lemma 3.2. *Let $0\le\alpha<\infty, 0<q,s<\infty$. If $g\in F_L(q,q-2,s,\alpha)$, then*

$$\limsup_{|a|\to 1}\left(\left(\log\frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|g'(z)|^q(1-|z|^2)^{q-2}(1-|\varphi_a(z)|^2)^s dA(z)\right)^{1/q}$$
$$\approx \mathrm{dist}_{F_L(q,q-2,s,\alpha)}(g,F^0_L(q,q-2,s,\alpha))\approx\limsup_{r\to 1^-}\|g-g_r\|_{F_L(q,q-2,s,\alpha)}.$$

Here $g_r(z)=g(rz)$, $0<r<1, z\in\mathbb{D}$.

Proof. For any given $g\in F_L(q,q-2,s,\alpha)$, then $g_r\in F^0_L(q,q-2,s,\alpha)$ and

$$\|g_r\|_{F_L(q,q-2,s,\alpha)}\lesssim\|g\|_{F_L(q,q-2,s,\alpha)}.$$

Let $\delta\in(0,1)$. We choose $a\in(0,\delta)$. Then $\varphi_a(z)$ lies in a compact subset of $\mathbb{D}$. So $\lim_{r\to 1}\sup_{z\in\mathbb{D}}|g'(\varphi_a(z))-rg'(r\varphi_a(z))|=0$. Making a change of variables, we have

$$\begin{aligned}
&\lim_{r\to 1}\sup_{|a|\le\delta}\left(\log\frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|g'(z)-g'_r(z)|^q(1-|z|^2)^{q-2}(1-|\varphi_a(z)|^2)^s dA(z)\\
=&\lim_{r\to 1}\sup_{|a|\le\delta}\left(\log\frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|g'(\sigma_a(z))-g'_r(\sigma_a(z))|^q(1-|z|^2)^{q+s-2}|\varphi'_a(z)|^q dA(z)\\
=&\lim_{r\to 1}\sup_{|a|\le\delta}\sup_{z\in\mathbb{D}}|g'(\varphi_a(z))-g'_r(\varphi_a(z))|^q\left(\log\frac{2}{1-|a|^2}\right)^{\alpha}\times\\
\times&\int_{\mathbb{D}}(1-|z|^2)^{q+s-2}|\varphi'_a(z)|^q dA(z)=0.
\end{aligned}$$

By the definition of distance, we obtain

$$\begin{aligned}
&\mathrm{dist}_{F_L(q,q-2,s,\alpha)}(g,F^0_L(q,q-2,s,\alpha))=\inf_{f\in F^0_L(q,q-2,s,\alpha)}\|g-f\|_{F_L(q,q-2,s,\alpha)}\\
\le&\lim_{r\to 1}\|g-g_r\|_{F_L(q,q-2,s,\alpha)}
\end{aligned}$$

$$
\begin{aligned}
= & \lim_{r\to 1}\left(\sup_{|a|>\delta}\left(\log\frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|g'(z)-g_r'(z)|^q(1-|z|^2)^{q-2}(1-|\varphi_a(z)|^2)^s dA(z)\right)^{\frac{1}{q}}\\
+ & \lim_{r\to 1}\left(\sup_{|a|\le\delta}\left(\log\frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|g'(z)-g_r'(z)|^q(1-|z|^2)^{q-2}(1-|\varphi_a(z)|^2)^s dA(z)\right)^{\frac{1}{q}}\\
\lesssim & \left(\sup_{|a|>\delta}\left(\log\frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|g'(z)|^q(1-|z|^2)^{q-2}(1-|\varphi_a(z)|^2)^s dA(z)\right)^{\frac{1}{q}}\\
+ & \lim_{r\to 1}\left(\sup_{|a|>\delta}\left(\log\frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|g_r'(z)|^q(1-|z|^2)^{q-2}(1-|\varphi_a(z)|^2)^s dA(z)\right)^{\frac{1}{q}}.
\end{aligned}
$$

Let $\psi_{r,a}(z)=\varphi_{ra}\circ r\varphi_a(z)$. Then $\psi_{r,a}$ is an analytic self-map of $\mathbb{D}$ and $\psi_{r,a}(0)=0$. Making a change variable of $z=\varphi_a(z)$ and applying the Littlewood's subordination theorem (see Theorem 1.7 of [4]), we have

$$
\begin{aligned}
& \left(\log\frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|g_r'(z)|^q(1-|z|^2)^{q-2}(1-|\varphi_a(z)|^2)^s dA(z)\\
= & \left(\log\frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|g_r'(\varphi_a(z))|^q(1-|\varphi_a(z)|^2)^q(1-|z|^2)^{s-2} dA(z)\\
\le & \left(\log\frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|g'\circ\varphi_{ra}\circ\psi_{r,a}(z)|^q(1-|\varphi_{ra}\circ\psi_{r,a}(z)|^2)^q(1-|z|^2)^{s-2} dA(z)\\
\le & \left(\log\frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|g'\circ\varphi_{ra}\circ\psi_{r,a}(z)|^q(1-|\varphi_{ra}\circ\psi_{r,a}(z)|^2)^q(1-|z|^2)^{s-2} dA(z)\\
\le & \left(\log\frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|g'\circ\varphi_{ra}(z)|^q(1-|\varphi_{ra}(z)|^2)^q(1-|z|^2)^{s-2} dA(z)\\
\le & \left(\log\frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|g'(z)|^q(1-|z|^2)^{q-2}(1-|\varphi_{ra}(z)|^2)^s dA(z).
\end{aligned}
$$

Since δ is arbitrary, we get

$$
\begin{aligned}
& \mathrm{dist}_{F_L(q,q-2,s,\alpha)}(g,F_L^0(q,q-2,s,\alpha))\\
\lesssim & \limsup_{|a|\to 1}\left(\left(\log\frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|g'(z)|^q(1-|z|^2)^{q-2}(1-|\varphi_a(z)|^2)^s dA(z)\right)^{1/q}.
\end{aligned}
$$

On the other hand, for any $g\in F_L(q,q-2,s,q(1-\frac{1}{p}))$,

$$
\begin{aligned}
& \mathrm{dist}_{F_L(q,q-2,s,\alpha)}(g,F_L^0(q,q-2,s,\alpha))=\inf_{f\in F_L^0(q,q-2,s,\alpha)}\|g-f\|_{F_L(q,q-2,s,\alpha)}\\
\gtrsim & \limsup_{|a|\to 1}\left(\left(\log\frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|g'(z)|^q(1-|z|^2)^{q-2}(1-|\varphi_a(z)|^2)^s dA(z)\right)^{1/q},
\end{aligned}
$$

implies the desired result. □

Lemma 3.3. *Let $1 < p < q < \infty$, $0 < s < \infty$. If $0 < r < 1$ and $g \in F_L(q, q-2, s, q(1-\frac{1}{p}))$, then $T_{g_r} : B_p \to F(q, q-2, s)$ is compact.*

Proof. Given $\{f_k\} \subset B_p$ such that $\{f_k\}$ converges to zero uniformly on any compact subset of $\mathbb{D}$ and $\sup_k \|f_k\|_{B_p} \leq 1$. For each $a \in \mathbb{D}$,

$$
\begin{aligned}
&\|T_{g_r} f_k\|^q_{F(q,q-2,s)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_k(z)|^q |g_r'(z)|^q (1-|z|^2)^{q-2} (1-|\varphi_a(z)|^2)^s dA(z) \\
&\lesssim \frac{\|g\|^q_{F_L(q,q-2,s,q(1-\frac{1}{p}))}}{\left(\log \frac{2}{1-r^2}\right)^{q(1-\frac{1}{p})} (1-r^2)^q} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_k(z)|^q (1-|z|^2)^{q-2} (1-|\varphi_a(z)|^2)^s dA(z) \\
&\lesssim \frac{\|g\|^q_{F_L(q,q-2,s,q(1-\frac{1}{p}))}}{\left(\log \frac{2}{1-r^2}\right)^{q(1-\frac{1}{p})} (1-r^2)^q} \int_{\mathbb{D}} |f_k(z)|^q (1-|z|^2)^{q-2} dA(z) \\
&\lesssim \frac{\|g\|^q_{F_L(q,q-2,s,q(1-\frac{1}{p}))}}{\left(\log \frac{2}{1-r^2}\right)^{q(1-\frac{1}{p})} (1-r^2)^q} \int_{\mathbb{D}} |f_k'(z)|^q (1-|z|^2)^{q-2} dA(z) \\
&\lesssim \frac{\|g\|^q_{F_L(q,q-2,s,q(1-\frac{1}{p}))}}{\left(\log \frac{2}{1-r^2}\right)^{q(1-\frac{1}{p})} (1-r^2)^q} \|f_k\|^q_{B_p} \int_{\mathbb{D}} 1 dA(z).
\end{aligned}
$$

By the dominated convergence theorem, we get

$$
\begin{aligned}
\lim_{k\to\infty} \|T_{g_r} f_k\|^q_{F(q,q-2,s)} &\lesssim \lim_{k\to\infty} \int_{\mathbb{D}} |f_k(z)|^q (1-|z|^2)^{q-2} dA(z) \\
&\lesssim \int_{\mathbb{D}} \lim_{k\to\infty} |f_k(z)|^q (1-|z|^2)^{q-2} dA(z) = 0,
\end{aligned}
$$

as desired. The proof is complete. □

The following result is very useful to study the essential norm of operators on some analytic function spaces, see [17].

Lemma 3.4. *Let X, Y be two Banach spaces of analytic functions on $\mathbb{D}$. Suppose that*

(1) *The point evaluation functionals on Y are continuous.*
(2) *The closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets.*
(3) *$T : X \to Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets.*

Then, T is a compact operator if and only if for any bounded sequence $\{f_n\}$ in X such that $\{f_n\}$ converges to zero uniformly on every compact set of $\mathbb{D}$, then the sequence $\{Tf_n\}$ converges to zero in the norm of Y.

Theorem 3.1. *Let $1 < p < q < \infty$, $0 < s < \infty$ and $g \in H(\mathbb{D})$. Then $T_g : B_p \to F(q, q-2, s)$ is bounded if and only if $g \in F_L(q, q-2, s, q(1-\frac{1}{p}))$.*

Proof. Suppose that $f \in B_p$ and $g \in F_L(q, q-2, s, q(1-\frac{1}{p}))$. From Lemma 3.2 we see that $d\mu_g(z) = |g'(z)|^q(1-|z|^2)^{q-2+s}dA(z)$ is a $q(1-\frac{1}{p})$-logarithmic s-Carleson measure. By Theorem 1, for any $I \subseteq \partial\mathbb{D}$ we deduce that

$$\begin{aligned}&\frac{1}{|I|^s}\int_{S(I)} |(T_g f)'(z)|^q(1-|z|^2)^{q-2+s}dA(z)\\ =&\frac{1}{|I|^s}\int_{S(I)} |f(z)|^q|g'(z)|^q(1-|z|^2)^{q-2+s+\frac{q}{p}}dA(z)\\ =&\frac{1}{|I|^s}\int_{S(I)} |f(z)|^q d\mu_g(z) \lesssim \|f\|^q_{B_p}\|g\|^q_{F_L(q,q-2,s)} < \infty,\end{aligned}$$

which implies that $T_g : B_p \to F(q, q-2, s)$ is bounded by Lemma 3.1 again.

Conversely, suppose that $T_g : B_p \to F(q, q-2, s)$ is bounded. For any $I \subseteq \partial\mathbb{D}$, let $a = (1-|I|)\zeta$, where ζ is the center of I. Then $1-|a| \approx |1-\overline{a}z| \approx |I|, \ \ z \in S(I)$. Let f_a be defined as in Lemma 2.2. We have

$$\begin{aligned}&\frac{\left(\log\frac{2}{|I|}\right)^{q(1-\frac{1}{p})}}{|I|^s}\int_{S(I)} |g'(z)|^q(1-|z|^2)^{q-2+s}dA(z)\\ \lesssim&\frac{1}{|I|^s}\int_{S(I)} |f_a(z)|^q|g'(z)|^q(1-|z|^2)^{q-2+s}dA(z)\\ \lesssim&\frac{1}{|I|^s}\int_{S(I)} |(T_g f_a)'(z)|^q(1-|z|^2)^{q-2+s}dA(z)\\ \lesssim&\|T_g f_a\|^q_{F(q,q-2,s)} < \infty,\end{aligned}$$

which implies that $g \in F_L(q, q-2, s, q(1-\frac{1}{p}))$ by Lemma 3.1. □

Theorem 3.2. *Let $1 < p < q < \infty$, $0 < s < \infty$ and $g \in H(\mathbb{D})$. Then $I_g : B_p \to F(q, q-2, s)$ is bounded if and only if $g \in H^\infty$.*

Proof. Let $f \in B_p$ and $g \in H^\infty$. By the fact that $B_p \subset \mathcal{B}$, we get

$$\begin{aligned}&\int_{\mathbb{D}} |(I_g f)'(z)|^q(1-|z|^2)^{q-2}\left(1-|\varphi_a(z)|^2\right)^s dA(z)\\ =&\int_{\mathbb{D}} |f'(z)|^q|g(z)|^q(1-|z|^2)^{q-2}\left(1-|\varphi_a(z)|^2\right)^s dA(z)\\ =&\|g\|^q_{H^\infty}\|f\|^{q-p}_{\mathcal{B}}\int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p-2}dA(z) \lesssim \|g\|^q_{H^\infty}\|f\|^q_{B_p} < \infty,\end{aligned}$$

which implies that $I_g : B_p \to F(q, q-2, s)$ is bounded.

Conversely, assume that $I_g : B_p \to F(q, q-2, s)$ is bounded. For $a \in \mathbb{D}$ and $r > 0$, let $\mathbb{D}(a, r) = \{z \in \mathbb{D} : \beta(a, z) < r\}$ denote the Bergman metric disk centered at a with radius r. Here $\beta(a, z)$ is the Bergman metric between z and a. For any

$w \in \mathbb{D}$, let F_w be defined as in Lemma 2.2. Using the subharmonic property of $|g|^q$ and the fact that (see [23])

$$\frac{(1-|w|^2)^2}{|1-\bar{w}z|^4} \approx \frac{1}{(1-|z|^2)^2} \approx \frac{1}{(1-|w|^2)^2} \approx \frac{1}{|\mathbb{D}(w,r)|}, \qquad z \in \mathbb{D}(w,r),$$

where $|\mathbb{D}(w,r)|$ denotes the area of the Bergman disk $\mathbb{D}(w,r)$, we have

$$\begin{aligned}
\infty >& \|I_g F_w\|^q_{F(q,q-2,s)} \\
\gtrsim & \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |F_w'(z)|^q |g(z)|^q (1-|z|^2)^{q-2} \left(1-|\varphi_a(z)|^2\right)^s dA(z) \\
\gtrsim & \int_{\mathbb{D}} |F_w'(z)|^q |g(z)|^q (1-|z|^2)^{q-2} \left(1-|\varphi_w(z)|^2\right)^s dA(z) \\
\gtrsim & \int_{\mathbb{D}(w,r)} |g(z)|^q (1-|z|^2)^{-2} \left(1-|\varphi_w(z)|^2\right)^s dA(z) \\
\gtrsim & \frac{1}{(1-|w|^2)^2} \int_{\mathbb{D}(w,r)} |g(z)|^q dA(z) \gtrsim |g(w)|^q,
\end{aligned}$$

which implies

$$\infty > \|I_g F_w\|^q_{F(q,q-2,s)} \gtrsim \|g\|^q_{H^\infty},$$

as desired. The proof is complete. □

Remark. Let $1 < p < q < \infty$, $0 < s < \infty$ and $g \in H(\mathbb{D})$. From the fact that

$$M_g f(z) = f(0)g(0) + I_g f(z) + T_g f(z),$$

we see that $M_g : B_p \to F(q,q-2,s)$ is bounded if and only if

$$g \in F_L(q, q-2, s, q(1-\frac{1}{p})) \cap H^\infty.$$

Theorem 3.3. *Let $1 < p < q < \infty$, $0 < s < \infty$ and $g \in H(\mathbb{D})$. If $T_g : B_p \to F(q,q-2,s)$ is bounded, then*

$$\|T_g\|_{e, B_p \to F(q,q-2,s)} \approx \mathrm{dist}_{F_L(q,q-2,s,q(1-\frac{1}{p}))}(g, F_L^0(q, q-2, s, q(1-\frac{1}{p}))).$$

Proof. Let $\{a_k\}$ be a sequence in $\mathbb{D}$ such that $\lim_{k\to\infty} |a_k| = 1$. For each k, set

$$f_{a_k}(z) = \left(\frac{1}{\log \frac{2}{1-|a_k|^2}} \right)^{1/p} \log \frac{2}{1-\overline{a_k} z}.$$

Then $\{f_{a_k}\}$ is bounded in B_p and $\{f_{a_k}\}$ converges to zero uniformly on every compact subset of $\mathbb{D}$. For any given compact operator $K : B_p \to F(q,q-2,s)$, by

Lemma 3.4 we have $\lim_{k\to\infty}\|Kf_{a_k}\|_{F(q,q-2,s)}=0$. So

$$\begin{aligned}
&\|T_g-K\|\gtrsim \limsup_{k\to\infty}\|(T_g-K)f_{a_k}\|_{F(q,q-2,s)}\\
&\gtrsim \limsup_{k\to\infty}\Big(\|T_gf_{a_k}\|_{F(q,q-2,s)}-\|Kf_{a_k}\|_{F(q,q-2,s)}\Big)\\
&=\limsup_{k\to\infty}\|T_gf_{a_k}\|_{F(q,q-2,s)}\\
&\geq \limsup_{k\to\infty}\left(\int_{\mathbb{D}}|f_{a_k}(z)|^q|g'(z)|^q(1-|z|^2)^{q-2}(1-|\varphi_{a_k}(z)|^2)^s dA(z)\right)^{\frac{1}{q}}\\
&\gtrsim \limsup_{|a_k|\to 1}\left(\left(\log\frac{2}{1-|a_k|^2}\right)^{q(1-\frac{1}{p})}\int_{\mathbb{D}}|g'(z)|^q(1-|z|^2)^{q-2}(1-|\varphi_{a_k}(z)|^2)^s dA(z)\right)^{\frac{1}{q}}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\|T_g\|_{e,B_p\to F(q,q-2,s)}\\
&\gtrsim \limsup_{k\to\infty}\left(\left(\log\frac{2}{1-|a_k|^2}\right)^{q(1-\frac{1}{p})}\int_{\mathbb{D}}|g'(z)|^q(1-|z|^2)^{q-2}(1-|\varphi_{a_k}(z)|^2)^s dA(z)\right)^{\frac{1}{q}}.
\end{aligned}$$

By Lemma 3.2 and the arbitrariness of $\{a_k\}$, we get that

$$\|T_g\|_{e,B_p\to F(q,q-2,s)}\gtrsim \mathrm{dist}_{F_L(q,q-2,s,q(1-\frac{1}{p}))}(g,F_L^0(q,q-2,s,q(1-\frac{1}{p}))).$$

On the other hand, by Lemma 3.3, $T_{g_r}:B_p\to F(q,q-2,s)$ is compact. Then

$$\|T_g\|_{e,B_p\to F(q,q-2,s)}\leq\|T_g-T_{g_r}\|=\|T_{g-g_r}\|\approx\|g-g_r\|_{F_L(q,q-2,s,q(1-\frac{1}{p}))}.$$

Using Lemma 3.2 again, we get

$$\begin{aligned}
\|T_g\|_{e,B_p\to F(q,q-2,s)}&\lesssim \limsup_{r\to 1^-}\|g-g_r\|_{F_L(q,q-2,s,q(1-\frac{1}{p}))}\\
&\approx \mathrm{dist}_{F_L(q,q-2,s,q(1-\frac{1}{p}))}(g,F_L^0(q,q-2,s,q(1-\frac{1}{p}))).
\end{aligned}$$

The proof is complete. □

By the well-known result that $T:X\to Y$ is compact if and only if $\|T\|_{e,X\to Y}=0$, we get the following result by Theorem 3.3 directly.

Corollary 3.1. *Let $1<p<q<\infty$ and $0<s<\infty$. If $g\in H(\mathbb{D})$, then $T_g:B_p\to F(q,q-2,s)$ is compact if and only if*

$$g\in F_L^0(q,q-2,s,q(1-\frac{1}{p})).$$

Theorem 3.4. *Let $1<p<q<\infty$ and $0<s<\infty$. If $g\in H(\mathbb{D})$ such that $I_g:B_p\to F(q,q-2,s)$ is bounded, then*

$$\|I_g\|_{e,B_p\to F(q,q-2,s)}\approx\|g\|_{\mathcal{H}^\infty}.$$

Proof. Let $\{a_k\}$ and K be defined as in the proof of Theorem 3.3. Set

$$F_{a_k}(z) = \frac{1-|a_k|^2}{\overline{a_k}(1-\overline{a_k}z)}, \quad z \in \mathbb{D}.$$

By Lemma 2.2 we see that $F_{a_k} \in B_p$. By Lemma 3.4 we get $\lim_{k\to\infty} \|KF_{a_k}\|_{F(q,q-2,s)} = 0$. Hence,

$$\begin{aligned}\|I_g - K\| &\gtrsim \limsup_{k\to\infty} \|(I_g - K)F_{a_k}\|_{F(q,q-2,s)} \\ &\gtrsim \limsup_{k\to\infty} \left(\|I_g F_{a_k}\|_{F(q,q-2,s)} - \|KF_{a_k}\|_{F(q,q-2,s)}\right) \\ &= \limsup_{k\to\infty} \|I_g F_{a_k}\|_{F(q,q-2,s)},\end{aligned}$$

which implies

$$\|I_g\|_{e,B_p\to F(q,q-2,s)} \gtrsim \limsup_{k\to\infty} \|I_g F_{a_k}\|_{F(q,q-2,s)}.$$

Similarly to the proof of Theorem 3.2 we get that $\|I_g F_{a_k}\|_{F(q,q-2,s)} \gtrsim |g(a_k)|$, which implies that

$$\|I_g\|_{e,B_p\to F(q,q-2,s)} \gtrsim \|g\|_{H^\infty}.$$

On the other hand, by Theorem 3.2 we obtain

$$\|I_g\|_{e,B_p\to F(q,q-2,s)} = \inf_K \|I_g - K\| \le \|I_g\| \lesssim \|g\|_{H^\infty}.$$

The proof is complete. □

From Theorem 3.4 we get the following result.

Corollary 3.2. *Let $1 < p < q < \infty$ and $0 < s < \infty$. If $g \in H(\mathbb{D})$, then $I_g : B_p \to F(q, q-2, s)$ is compact if and only if $g = 0$.*

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