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ON THE FERMAT-TYPE DIFFERENCE EQUATION $f^3(z) + [c_1 f(z+c) + c_0 f(z)]^3 = e^{\alpha z+\beta}$

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Abstract. In this article, we deal with the solutions of the difference analogue of Fermat-type equation of the form $f^3(z) + [c_1 f(z+c) + c_0 f(z)]^3 = e^{\alpha z+\beta}$ and prove a result generalizing a result of Han and Lü [J. Contm. Math. Anal. 2019] and Ma *et al.* [J. Func. Spaces, Vol. 2020, Article ID 3205357]. Furthermore, we explore the class of functions satisfying the Fermat-type difference equation. A considerable number of examples have been exhibited throughout the paper pertinent with the different issues. We characterized all possible non-constant solutions of the Fermat-type difference equation $f^2(z) + f^2(z+c) = e^{\alpha z+\beta}$.

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1. Introduction

The so called Fermat's Last Theorem, which was proved by Wiles [30], Taylor and Wiles [29] in 1995, states that there do not exist non-zero rational numbers x and y and an integer $n \geq 3$, for which $x^n + y^n = 1$. There is a close relationship between Fermat's Last Theorem and family of solutions (f, g) of the following functional equation

$$f^n + g^n = 1. \tag{1.1}$$

For $n = 1$, finding the solution is effortless, and for $n = 2$, it is easy to see that the pairs $(\sin(\alpha), \cos(\alpha))$ and

$$\left(\frac{1}{\sqrt{2}}[\sin(\alpha) \pm \cos(\alpha)], \frac{1}{\sqrt{2}}[\sin(\alpha) \mp \cos(\alpha)]\right)$$

always solves the equation for an entire function α. For $n \geq 2$, Gross [8] proved that all the meromorphic solutions are of the form

$$f(z) = \frac{2\beta(z)}{1+\beta^2(z)} \quad \text{and} \quad g(z) = \frac{1-\beta^2(z)}{1+\beta^2(z)}.$$

For $n \geq 3$, it has no transcendental entire solutions proved in [Gauthier-Villars, Paris, (1927), 135—136] but meromorphic solutions exists which is confirmed by

Gross in [8] and one such solution is

$$f(z) = 4^{-1/6}(\wp')^{-1}\left(1 + 3^{-1/2} \cdot 4^{1/3}\wp\right)$$
$$g(z) = 4^{-1/6}(\wp)^{-1}\left(1 - 3^{-1/2} \cdot 4^{1/3}\wp\right),$$

where $\wp$ is a Weierstrass $\wp$-function. For $n \geq 4$, it has no transcendental meromorphic solutions confirmed in [8]. No other solutions of the equation (1.1) exist which is confirmed by Gross in [9].

It has been determined for which positive integers n, the equation (1.1) has non-constant solutions f and g in each of the following four function classes (i) meromorphic functions, (ii) rational functions, (iii) entire functions, and (iv) polynomials; (see [11, 12]. The study of the functions analogous to the Fermat-type diophantine equations $x^n + y^n = 1$ was initiated by Gross [8] and Baker [2]. They actually proved that the equation

$$f^n + g^n = 1 \tag{1.2}$$

does not admit any non-constant meromorphic solutions in the complex plane $\mathbb{C}$ if $n > 3$, and does not admit any entire solutions if $n > 2$. For the possible non-constant meromorphic solutions of (1.2), they also characterized it in the case of when $n = 2, 3$. In fact, for the case $n = 3$, Gross [8] and Baker [2] proved that the following pair (f, g), where

$$f(z) = \left(\frac{1}{2} + \frac{\wp'(z)}{2\sqrt{3}}\right)/\wp(z) \tag{1.3}$$

and

$$g(z) = \left(\frac{1}{2} - \frac{\wp'(z)}{2\sqrt{3}}\right)/\wp(z), \tag{1.4}$$

are meromorphic solution of equation (1.2), where $\wp$ is Weierstrass $\wp$-function.

It is worth to observe that the equation $x^3 + y^3 = 1$ defines an algebraic function whose Reimann surface has genus 1, and there is accordingly a uniformization by Weierstrass elliptic function. Weierstrass elliptic function $\wp(z) := \mathcal{P}(z, \omega_1, \omega_2)$ is a doubly periodic meromorphic function with periods ω_1 and ω_2, and this function is defined by

$$\wp(z, \omega_1, \omega_2) = \frac{1}{z^2} + \sum_{\substack{\mu,\nu\in\mathbb{Z} \\ \mu^2+\nu^2\neq 0}} \left(\frac{1}{(z + \mu\omega_1 + \nu\omega_2)^2} - \frac{1}{(\mu\omega_1 + \nu\omega_2)^2}\right),$$

which is even and satisfies, after appropriate choosing ω_1 and ω_2,

$$(\wp')^2 = 4\wp^3 - 1. \tag{1.5}$$

In the same paper, Gross conjectured that every meromorphic solutions of $f^3 + g^3 = 1$ are necessarily elliptic function of entire functions. Later, Baker [2] confirmed the conjecture and established the following result.

Theorem A. [2] *Each pair of meromorphic solutions f and g to the following equation*

$$f^3(z) + g^3(z) = 1 \tag{1.6}$$

over $\mathbb{C}$ must be of the form $f = f_1(h(z))$ and $g(z) = \omega g_1(h(z)) = \omega f_1(-h(z))$, where h is an entire function in $\mathbb{C}$ and ω is a cube root of unity.

In this paper, a meromorphic function will always be non-constant and meromorphic in the complex plane $\mathbb{C}$, unless specifically stated otherwise. In what follows, we assume that the reader is familiar with the elementary Nevanlinna theory (see [7, 33, 35]). In particular, for a meromorphic function f, we denote $\mathcal{S}(f)$ the family of all meromorphic function ω for which $T(r, \omega) = S(r, f) = o(T(r, f))$, where $r \to \infty$ outside of a possible set of finite logarithmic measure. For convenience, we agree that $\mathcal{S}(f)$ includes all constant functions and $\overline{\mathcal{S}}(f) := \mathcal{S}(f) \cup \{\infty\}$. Here, the order $\rho(f)$ of a meromorphic function is defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

In 2016, Lü and Han [20] proved that the equation $f(z) + f'(z) = 1$ has the general solution $f(z) = 1 - ae^{-z}$ for $a \in \mathbb{C}$ and $f^2(z) + (f'(z))^2 = 1$ has the general solution $f(z) = \pm\sin(z + b)$ for some $b \in \mathbb{C}$. Nevertheless, $f^n(z) + (f'(z))^n = 1$ can not have any non-constant meromorphic solution when $n > 2$.
Below, we recall a well-known facts about the order of composite meromorphic functions which have been established by Edrei and Fuchs [6], and by Bergweiler [4].

Theorem B. *Let f be a meromorphic functions and h be an entire function in $\mathbb{C}$. When $0 < \rho(f), \rho(h) < \infty$, then $\rho(f \circ h) < \infty$, and h is transcendental, then $\rho(f) = 0$.*

In the recent years, Nevanlinna characteristic of $f(z+c)$ ($c \in \mathbb{C} \setminus \{0\}$), the value distribution theory of difference polynomials, Nevanlinna theory of the difference operator and the difference analogue of the lemma of the logarithmic derivative has been established (see [5, 14, 15]). Due to this development of theories, there has been a recent study on whether the derivative f' of f can be replaced by the shift $f(z+c)$ or difference operator $\Delta_c f$. The difference analogues of the Fermat type functional equations have been investigated in a number of papers (see [19, 24, 26, 27, 31, 34]).

For a meromorphic function f, we define its difference operators by

$$\Delta_c f = f(z+c) - f(z)$$
$$\Delta_c^n f = \Delta_c^{n-1}(\Delta_c f),\ n \in \mathbb{N},\ n \geq 2.$$

In 2016, Lü and Han [20] described a property of meromorphic solutions to the equation (1.6) with $g(z) := f(z+c)$, for $c \in \mathbb{C} \setminus \{0\}$ as the following.

Theorem C. [20] *The difference equation $f^3(z) + f^3(z+c) = 1$ does not have meromorphic solutions of finite order.*

For $n \geq 4$ and $\gamma \not\equiv 0$, if we consider the meromorphic solution of the equations $f^n(z)+(f')^n = \gamma^n$, then by the Proposition 1.1 in [16] we see that both the functions f/γ and f'/γ must be constants. Therefore, if we assume $f = c_1\gamma$ and $f' = c_2\gamma$, then a simple computation shows that $c_1^n + c_2^n = 1$. Observe that $c_1 \neq 0$, otherwise $f \equiv 0$, hence $\gamma = 0$. Similarly, $c_2 \neq 0$, otherwise, f and γ will be constants. Therefore, when $c_1c_2 \neq 0$, then γ cannot have any zeros and poles. Hence $\gamma^n(z) = e^{\alpha z+\beta}$ where $\alpha = nc_2/c_1$.

Motivated by the above observations, Han and Lü [16] have investigated the above equation with $f(z+c)$ in the place of $f'(z)$ for the case $n = 3$ and proved the following interesting result.

Theorem D. [16] *The difference equation $f^3(z)+f^3(z+c) = e^{\alpha z+\beta}$, where $\alpha,\ \beta \in \mathbb{C}$, does not have meromorphic solutions of finite order.*

Regarding existence of solutions of the difference equation $f^n(z) + [\Delta_c f]^n = 1$ for a positive integer n, we have the following note.

Remark 1.1. A simple computation shows that the difference equation $f(z) + \Delta_c f = 1$ has no non-constant meromorphic solutions. Following the proof of Theorem 1.5 of Liu *et al.* in [18, Theorem 1.5], one can observe that there does not exist any non-constant meromorphic solutions of the difference equation $f^2(z) + [\Delta_c f]^2 = 1$.

Therefore, a natural question arises as the following.

Question 1.1. *Does there exist any non-constant meromorphic solutions of the difference equation $f^3(z) + [\Delta_c f]^3 = 1$?*

Recently, Ma *et al.* [21] have investigated Theorem B by considering the difference operator $\Delta_c f$ and proved the following result which answers Question 1.1.

Theorem E. [21] *The difference equation $f^3(z) + [\Delta_c f(z)]^3 = 1$ does not have meromorphic solutions of finite order.*

In the same paper, Han and Lü [16] proved the next result by producing a complete characterization of the solutions.

Theorem F. [16] *The meromorphic solutions f of the following differential equation*

$$f^n(z) + [f'(z)]^n = e^{\alpha z+\beta} \tag{1.7}$$

must be entire functions and the following assertions hold.

(i) *For $n = 1$, the general solution of (1.7) are $f(z) = e^{\alpha z+\beta}/(\alpha + 1)$, when $\alpha \neq -1$, and $f(z) = ze^{-z+\beta} + ae^{-z}$.*

(ii) *For $n = 2$, either $\alpha = 0$, and the general solution of (1.7) are $f(z) = e^{\beta/2}\sin(z + b)$, or $f(z) = de^{(\alpha z+\beta)/2}$.*

(iii) *For $n \geq 3$, the general solution of (1.7) is $f(z) = de^{(\alpha z+\beta)/n}$,*

where $a, b, d, \alpha, \beta \in \mathbb{C}$ with $d^n \left(1 + (\alpha/n)^n\right) = 1$, for $n \geq 2$.

The paper is organized as follows. In Section 2, we prove a result generalizing the Theorem D and Theorem E. In Subsection 2.1, the characterization of the solutions of $f^2(z) + f^2(z + c) = e^{\alpha z+\beta}$ is discussed and a result is proved. In Section 3, the claim of Han and Lü in [16, page 102] is disproved exhibiting several counter examples. Section 4 is devoted mainly to prove the main results of this paper. Future course of work on the results of this paper has been discussed in Section 5.

2. Main result

Motivating from Remark 1.1, we are interested to investigate for the non-constant meromorphic solutions of general difference equations. Henceforth, we recall here $L_c(f)$ defined by the present author in [1] as $L_c(f) := c_1 f(z + c) + c_0 f(z)$, $c_1(\neq 0)$, $c_0 \in \mathbb{C}$. It is easy to see that the shift $f(z + c)$ and difference operator $\Delta_c f$ are the particular cases of $L_c(f)$. With this setting, in this paper, our aim is to investigate Theorems D and E further to establish a combined result. Before state the main result of this paper, we have the following remark.

Remark 2.1. The equation $f^n(z)+[L_c(f)]^n = e^{\alpha z+\beta}$, may consists of non-constant entire as well as meromorphic solutions for $n = 1$ and $n = 2$, from the following examples we ensure this fact.

Example 2.1. *Let*

$$f(z) = \left(-\frac{c_0 + 1}{c_1}\right)^{z/c} h(z) + \delta e^{\alpha z+\beta},$$

where h is c-periodic finite order entire functions like $h(z) = \sin(2\pi z/c)$ or $\cos(2\pi z/c)$ or $e^{2\pi iz/c}$ etc. and their linear combinations and c be such that $e^{\alpha c} = (1 - \delta(c_0 + 1))/c_1\delta$. It is easy to verify that $f(z)$ solves the equation $f(z) + L_c(f(z)) = e^{\alpha z+\beta}$.

Example 2.2. *Let*

$$f(z) = \left(-\frac{c_0 + 1}{c_1}\right)^{z/c} \frac{g(z) + 1}{g(z) - 1} + \delta e^{\alpha z+\beta}.$$

where g is c-periodic finite order entire or meromorphic functions like in Example 2.1 and c be such that $e^{\alpha c} = (1 - \delta(c_0 + 1))/c_1\delta$. It is easy to see that $f(z)$ solves the equation $f(z) + L_c(f(z)) = e^{\alpha z+\beta}$.

Example 2.3. *Let $f(z) = (1/2)e^{(\alpha z+\beta)/3}\left(e^{(\alpha z+\beta)/3} + 1\right)$. We choose $c \in \mathbb{C}$ such that $e^{\alpha c/3} \neq 1$. Let*

$$L_c(f) = \frac{2i}{e^{\frac{\alpha c}{3}}\left(e^{\frac{\alpha c}{3}} - 1\right)} f(z + c) + \frac{i\left(e^{\frac{\alpha c}{3}} + 1\right)}{1 - e^{\frac{\alpha c}{3}}} f(z).$$

Clearly, $f(z)$ solves the equation $f^2(z) + [L_c(f(z))]^2 = e^{\alpha z+\beta}$.

Example 2.4. *Let*

$$f(z) = \frac{1}{2}\left(e^{\gamma(\alpha z+\beta)} \sin\left(\frac{2\pi z}{c}\right) + \frac{e^{(1-\gamma)(\alpha z+\beta)}}{\sin\left(\frac{2\pi z}{c}\right)}\right) \text{ where } \gamma \in \mathbb{C} \setminus \left\{\frac{1}{2}\right\}.$$

Let

$$L_c(f) = \frac{2}{i\left(e^{(1-\gamma)\alpha c} - e^{\gamma\alpha c}\right)} f(z + c) + \frac{i\left(e^{(1-\gamma)\alpha c} + e^{\gamma\alpha c}\right)}{\left(e^{(1-\gamma)\alpha c} - e^{\gamma\alpha c}\right)} f(z).$$

It is easy to verify that $f(z)$ solves the equation $f^2(z) + [L_c(f(z))]^2 = e^{\alpha z+\beta}$.

The observations from the above examples motivate us to establish a single result combining the results of Lü and Han [16], and Ma *et al.* [21] (*i.e.*, for the case $n = 3$). Therefore, the following question is inevitable.

Question 2.1. *Does there exist any non-constant meromorphic solution of the equation of $f^3(z) + [L_c(f(z))]^3 = e^{\alpha z+\beta}$?*

In this paper, with the help of some ideas of [16], we establish Theorem 2.1 which answers Question 2.1 completely.

Theorem 2.1. *The difference equation*

$$f^3(z) + [L_c(f(z))]^3 = e^{\alpha z+\beta} \tag{2.1}$$

does not have infinite order meromorphic solutions.

Remark 2.2. In case of meromorphic function of infinite order, the next example evidents that (2.1) may admit solution.

Example 2.5. *Let $f(z)$ be given by (4.2) with $h(z) = e^z$. Therefore, we have $\rho(f) = \infty$ and for $c = \pi i$, each α with $e^{c\alpha/3} = \{1,\ \omega,\ \omega^2\}$ where ω is a non-real cube root of unity. It is easy to see that $f^3(z) + [L_c(f(z))]^3 = e^{\alpha z+\beta}$.*

Our aim is to generalize Theorem F for general setting of the equation. In order to generalize Theorem F, we would like to explore the meromorphic solutions of the following Fermat-type differential equation

$$f^n(z) + \left(f^{(k)}(z)\right)^n = e^{\alpha z+\beta} \ \text{ for } \ k \in \mathbb{N}. \tag{2.2}$$

Henceforth, to this end, we denote θ by $\theta = \cos(3\pi/k) + i\sin(3\pi/k)$ where k is a positive integer such that $\theta^k = -1$.

Theorem 2.2. *Let k be any positive integer. Then the meromorphic solutions f of the differential equation*

$$f^n(z) + [f^{(k)}(z)]^n = e^{\alpha z+\beta} \tag{2.3}$$

must be entire functions. Furthermore,

(i) *When $n = 1$, the general solution of (2.3) is*

$$f(z) = \begin{cases} \displaystyle\sum_{j=1}^{k} a_j e^{\theta^j z} + \frac{ze^{\alpha z+\beta}}{\alpha^k+1}, & \text{for } \alpha \neq \theta, \theta^2, \ldots, \theta^{k-1} \\ \displaystyle\sum_{j=1}^{k} a_j e^{\theta^j z} + \frac{ze^{\alpha z+\beta}}{k\alpha^{(k-1)}}, & \text{for } \alpha \in \{\theta, \theta^2, \ldots, \theta^{k-1}\}, \\ \displaystyle\sum_{j=1}^{k} a_j e^{\theta^j z} + \frac{ze^{-z+\beta}}{k}, & \text{for } \alpha = -1 \text{ and } k \text{ is odd,} \\ \displaystyle\sum_{j=1}^{k} a_j e^{\theta^j z} - \frac{ze^{-z+\beta}}{k}, & \text{for } \alpha = -1 \text{ and } k \text{ is even,} \end{cases}$$

(ii) *When $n = 2$, one of the following holds: Either*

  (a) *$\alpha = 0$, and the general solution of (2.3) are $f(z) = e^{\beta/2}\sin(z+b)$, only when k is odd but when k is even, then f must be constant, $e^{\beta/2}$, or*

  (b) *$f(z) = de^{(\alpha z+\beta)/2}$.*

(iii) *When $n \geq 3$, the general solution of (2.3) is $f(z) = de^{(\alpha z+\beta)/n}$,*

where $a, b, d, \alpha, \beta \in \mathbb{C}$ are such that $d^n\left(1 + (\alpha/n)^{nk}\right) = 1$, for $n \geq 2$.

2.1. Characterization of the solutions of $f^2(z)+f^2(z+c) = e^{\alpha z+\beta}$.

In contrast to Theorem 2.1 in [16], Han and Lü have shown that even though the existence of finite or infinite order meromorphic solutions of the difference equation

$$f^2(z) + f^2(z+c) = e^{\alpha z+\beta} \tag{2.4}$$

can be described but they could not prove a result finding the general solution of (2.4). Therefore, it is interesting to seek the possible general meromorphic solutions of the difference equation (2.4). In this paper, we take this opportunity to find out the possible general meromorphic solutions of the above Fermat-type difference equation. Consequently, we prove the following result which may give a complete characterization of the solutions of the difference equation (2.4).

Theorem 2.3. *The general meromorphic solutions of the Fermat-type difference equation* $f^2(z) + f^2(z+c) = e^{\alpha z+\beta}$ *are the following:*

(i) *If f is a non-constant entire function, then*

$$f(z) = \begin{cases} de^{\frac{\alpha z+\beta}{2}}, \ \text{where } d \neq \pm 1,\ d^2 = \frac{1}{e^{\alpha c}+1} \text{ with } e^{\alpha c} \neq -1, & \text{when order of } f \text{ is finite.} \\ e^{\frac{\alpha z+\beta}{2}} \sin\left(\frac{(4k+1)\pi z}{2c} + \eta\right), & \text{when order of } f \text{ is finite,} \\ e^{\frac{\alpha z+\beta}{2}} \sin\left(\frac{(4k+1)\pi z}{2c} + \mathcal{H}(z)\right), & \text{when order of } f \text{ is infinite.} \end{cases}$$

(ii) *If f is a non–constant meromorphic function, then*

$$f(z) = \begin{cases} \dfrac{e^{\frac{1}{4}(\alpha z+\beta)}}{2}\left(g(z) + \dfrac{e^{\frac{1}{2}(\alpha z+\beta)}}{g(z)}\right), \\ \dfrac{e^{\frac{1}{4}(\alpha z+\beta)}}{2}\left(e^{\frac{1}{2}(\alpha z+\beta)}g(z) + \dfrac{1}{g(z)}\right), \end{cases}$$

where g is a meromorphic function, $\mathcal{H}$ is a c-periodic entire function, η is a complex number and $e^{\alpha c} = 1$.

Remark 2.3. If g is a constant or an exponential function, then the solution becomes transcendental entire.

Example 2.6. *Let*

(i) $f_1(z) = \dfrac{1}{9}e^{\frac{\alpha z+\beta}{2}}$, *with* $e^{\alpha c} = 8$, $\rho(f) \leq 1$,

(ii) $f_2(z) = e^{\frac{\alpha z+\beta}{2}} \cos\left(\dfrac{\pi z}{2c} + 1\right)$, $\rho(f) \leq 1$,

(iii) $f_3(z) = e^{\frac{\alpha z+\beta}{2}} \sin\left(\dfrac{\pi z}{2c} - 1\right)$, $\rho(f) \leq 1$,

(iv) $f_4(z) = \dfrac{e^{\frac{1}{4}(\alpha z+\beta)}}{2}\left(\dfrac{1}{3} + 3e^{\frac{1}{2}(\alpha z+\beta)}\right)$, *with* $e^{\alpha c} = 1$, $\rho(f) \leq 1$,

(v) $f_5(z) = \dfrac{1}{2}\left(e^{\frac{3}{4}(\alpha z+\beta)+\frac{2\pi i z}{c}} + e^{\frac{1}{4}(\alpha z+\beta)-\frac{2\pi i z}{c}}\right)$, *with* $e^{\alpha c} = 1$, $\rho(f) \leq 1$,

(vi) $f_6(z) = e^{\frac{\alpha z+\beta}{2}} \sin\left(e^{\frac{2\pi i z}{c}} + \dfrac{\pi z}{2c} + 1\right)$, *with* $e^{\alpha c} = 1$, $\rho(f) = \infty$,

It is easy to verify that $f_j^2(z) + f_j^2(z+c) = e^{\alpha z+\beta}$ *for all* $j = 1,\ 2,\ \ldots, 6$.

3. Remarks on the general solution of Fermat-type difference equations

In their paper, Han and Lü [16] have discussed briefly about the meromorphic solutions of the difference equation

$$f(z)+f(z+c)=e^{\alpha z+\beta}. \tag{3.1}$$

In [16, page 102], Han and Lü claimed that the general solution of the difference equation (3.1) is either of the form $f(z)=\delta(z)+de^{\alpha z+\beta}$ or $f(z)=\delta(z)-(z/c)e^{\alpha z+\beta}$, where $\delta(z)$ is a meromorphic function satisfying $\delta(z+c)=-\delta(z)$.

In this paper, after a careful investigation on the functional equation (3.1), we found the following list of counter examples confirming that $f(z)=\delta(z)+de^{\alpha z+\beta}$ or $f(z)=\delta(z)-(z/c)e^{\alpha z+\beta}$ are not the general solution rather some particular solutions of the difference equation $f(z)+f(z+c)=e^{\alpha z+\beta}$.

Example 3.1. *Let*

$$f(z)=\frac{e^{\frac{\pi i z}{c}}}{\sin\left(\frac{2\pi z}{c}\right)-1}+e^{\alpha z+\beta}\cos^2\left(\frac{\pi z}{2c}\right),$$

where c be so chosen that $e^{\alpha c}=1$. We verify that $f(z)$ solves the equation $f(z)+f(z+c)=e^{\alpha z+\beta}$ and f is neither in the specific forms suggested by Lü and Han.

Example 3.2. *Let*

$$f(z)=e^{\frac{\pi i z}{c}}\frac{g(z)+1}{g(z)-1}+e^{\alpha z+\beta}\sin^2\left(\frac{\pi z}{2c}\right),$$

where c be such that $e^{\alpha c}=1$, and g is any c-periodic finite order entire or meromorphic functions like $g(z)=\sin(2\pi z/c)$ or $\cos(2\pi z/c)$ or $\tan(\pi z/c)$ or $\cot(\pi z/c)$ etc. Evidently, $f(z)+f(z+c)=e^{\alpha z+\beta}$ and f is neither in the specific forms claimed by Lü and Han.

Remark 3.1. In connection with the existence of solutions, we see that, in page 148, Liu *et al.* [18] have investigated to find non-constant solutions of the difference equation

$$f^n(z)+f^m(z+c)=1$$

for different range of values of m and n, where $m,\ n\in\mathbb{N}$. But in particular, when $m=1=n$, Liu *et al.* have claimed that the general entire solutions are of the form $f(z)=1/2+e^{\pi iz/c}h(z)$, where h is a c-periodic entire function. In the following, we construct examples to show that the general solution is not always of that form. Therefore, we consider the function $g(z)=\sin z$ or $\cos z$.

Example 3.3. *Let* $f(z) = g^2(\pi z/2c) + e^{\pi iz/c}h(z)$, *where* h *is a* c*-periodic entire function. We see that although* $f(z)$ *solves the equation* $f(z) + f(z+c) = 1$ *but not in the said form.*

Example 3.4. *Let* $f(z) = (3/5)g^2(\pi z/2c) + 1/5$. *Clearly,* $f(z)$ *solves the equation* $f(z) + f(z+c) = 1$ *without being of the said form.*

4. Proof of the main result

Proof of Theorem 2.1. The difference equation $f^3(z) + [L_c(f)]^3 = e^{\alpha z+\beta}$ of the theorem, can be expressed as

$$\left(\frac{f(z)}{e^{\frac{\alpha z+\beta}{3}}}\right)^3 + \left(\frac{L_c(f)}{e^{\frac{\alpha z+\beta}{3}}}\right)^3 = 1.$$

By the Proposition 1.1 in [16], it is known that the only non-constant meromorphic solutions of $F^3(z) + G^3(z) = 1$ are

$$F(z) = \frac{1}{2\wp(h)}\left(1 + \frac{1}{\sqrt{3}}\wp'(h)\right) \text{ and } G(z) = \frac{\omega}{2\wp(h)}\left(1 - \frac{1}{\sqrt{3}}\wp'(h)\right),$$

where h is an entire function, ω is a cube root of unity and $\wp$ denotes the Weierstrass $\wp$-function. Therefore, in view of the Proposition 1.1, we obtain

$$f(z) = \frac{1}{2\wp(h)}\left(1 + \frac{1}{\sqrt{3}}\wp'(h)\right)e^{\frac{\alpha z+\beta}{3}} \tag{4.1}$$

and

$$L_c(f) = \frac{\omega}{2\wp(h)}\left(1 - \frac{1}{\sqrt{3}}\wp'(h)\right)e^{\frac{\alpha z+\beta}{3}}. \tag{4.2}$$

From (4.2), we obtain

$$f(z+c) = \frac{\dfrac{\omega - c_0}{2} - \dfrac{\omega + c_0}{2\sqrt{3}}\wp'(z)}{c_1\wp(h(z))}e^{\frac{\alpha z+\beta}{3}}. \tag{4.3}$$

A routine computation using (4.1) and (4.3) shows that

$$\frac{(\omega - c_0) - \dfrac{\omega + c_0}{\sqrt{3}}\wp'(h(z))}{\wp(h(z))} = \frac{c_1\left(1 + \dfrac{\wp'(h(z+c))}{\sqrt{3}}\right)}{\wp(h(z+c))}e^{\frac{\alpha c}{3}}. \tag{4.4}$$

Equation (4.1) can be written as

$$\frac{\wp'(h(z))}{\sqrt{3}} = 2\wp(h(z))f(z) - 1. \tag{4.5}$$

Assuming $\rho(f) < \infty$, then in view of (1.5) and (4.5), we obtain

$$\frac{3f^2(z)\wp^2(h(z))}{e^{\frac{2}{3}(\alpha z+\beta)}} - \frac{3f(z)\wp(h(z))}{e^{\frac{1}{3}(\alpha z+\beta)}} + 1 = \wp^3(h(z)). \tag{4.6}$$

We recall here the estimate (2.7) of Bank and Langley [3] which states that

$$T(r,\wp) = \frac{\pi r^2}{A}(1 + o(1)) \text{ and } \rho(\wp) = 2, \tag{4.7}$$

where A is the area of the parallelogram $\mathfrak{P}$ with the vertices $0,\ \omega_1,\ \omega_2$ and $\omega_1+\omega_2$. Therefore, taking into account that $T\left(r, e^{\alpha z}\right)=(\alpha r / \pi)(1+O(1))$, combining (4.5) and (4.7), we obtain

$$T(r, \wp(h)) \leq 2 T(r, f)+\frac{2}{3} T\left(r, e^{\alpha z}\right)+O(1), \tag{4.8}$$

and hence $\rho(\wp(h))<\infty$ as well.

By Corollary 1.2 of Edrei and Fuchs [6] (see also Theorem of Bergweiler [4]), h must be a polynomial.

Actually, we have $T(r, \wp(h))=O\left(r^{2 q}\right)$, for $q \geq 1$. It is easy to see that if $\wp\left(z_0\right)=0$, then from (1.5), we obtain $\left(\wp^{\prime}\left(z_0\right)\right)^2=-1$ which shows that $\wp^{\prime}\left(z_0\right)= \pm i$. We now denote $\left\{z_n\right\}_{n \in \mathbb{N}}$ by all the zeros of $\wp(z)$ that satisfy $z_n \rightarrow \infty$ when $n \rightarrow \infty$ and assume that $h\left(a_{n, k}\right)=z_n$, for $k=1,\ 2, \ldots, \operatorname{deg}(h)$. Thus we have $\left(\wp^{\prime}\right)^2\left(h\left(a_{n, k}\right)\right)=\left(\wp^{\prime}\right)^2\left(z_n\right)=-1$. Suppose there is a sub-sequence $\left\{a_{n, k}\right\}_{n \in \mathbb{N}}$ with respect to n such that $\wp\left(h\left(a_{n, k}+c\right)\right)=0$. We denote this sub-sequence still by $\left\{a_{n, k}\right\}_{n \in \mathbb{N}}$ and fixed the index k below. Therefore, we have $\left(\wp^{\prime}\right)^2\left(h\left(a_{n, k}+c\right)\right)=-1$. Differentiating both sides of (4.4), we obtain

$$\begin{aligned}
&\left(-\frac{\omega+c_0}{\sqrt{3}} \wp^{\prime \prime}(h(z)) h^{\prime}(z)\right) \wp(h(z+c)) \\
&+\left(\left(\omega-c_0\right)-\frac{\omega+c_0}{\sqrt{3}} \wp^{\prime}(h(z))\right) \wp^{\prime}(h(z+c)) h^{\prime}(z+c) \\
=&\left(\frac{c_1}{\sqrt{3}} \wp^{\prime \prime}(h(z+c)) h^{\prime}(z+c)\right) \wp(h(z)) e^{\frac{\alpha c}{3}}+c_1\left(1+\frac{\wp^{\prime}(h(z+c))}{\sqrt{3}}\right) \wp^{\prime}(h(z)) h^{\prime}(z) e^{\frac{\alpha c}{3}} .
\end{aligned} \tag{4.9}$$

Substituting $a_{n, k}$ (for sufficiently large n) into the equation (4.9) and by using $\wp\left(h\left(a_{n, k}+c\right)\right)=0$ and $\wp\left(h\left(a_{n, k}\right)\right)=0$, we obtain

$$\begin{aligned}
&\left(\left(\omega-c_0\right)-\frac{\omega+c_0}{\sqrt{3}} \wp^{\prime}\left(h\left(a_{n, k}\right)\right)\right) \wp^{\prime}\left(h\left(a_{n, k}+c\right)\right) h^{\prime}\left(a_{n, k}+c\right) \\
&=c_1\left(1+\frac{\wp^{\prime}\left(h\left(a_{n, k}+c\right)\right)}{\sqrt{3}}\right) \wp^{\prime}\left(h\left(a_{n, k}\right)\right) h^{\prime}\left(a_{n, k}\right) e^{\frac{\alpha c}{3}} .
\end{aligned} \tag{4.10}$$

Noting that $\wp^{\prime}\left(h\left(a_{n, k}\right)\right)= \pm i$ and $\wp^{\prime}\left(h\left(a_{n, k}+c\right)\right)= \pm i$, without any loss of generality, together with (4.4), we assume that there exists a sub-sequence $\left\{a_{n, k}\right\}_{n \in \mathbb{N}}$ (here we still denote it by $\left\{a_{n, k}\right\}_{n \in \mathbb{N}}$) such that the following four possible cases may occur.

Case 1. If $\wp^{\prime}\left(h\left(a_{n, k}\right)\right)=i$ and $\wp^{\prime}\left(h\left(a_{n, k}+c\right)\right)=i$, then in view of (4.10), we obtain

$$\left(\omega-c_0-\frac{\omega+c_0}{\sqrt{3}} i\right) h^{\prime}\left(a_{n, k}+c\right)=c_1\left(1+\frac{i}{\sqrt{3}}\right) h^{\prime}\left(a_{n, k}\right) e^{\frac{\alpha c}{3}} . \tag{4.11}$$

Case 2. If $\wp'(h(a_{n,k})) = -i$ and $\wp'(h(a_{n,k}+c)) = i$, then we get from (4.10),

$$\left(\omega - c_0 + \frac{\omega + c_0}{\sqrt{3}} i\right) h'(a_{n,k}+c) = -c_1 \left(1 + \frac{i}{\sqrt{3}}\right) h'(a_{n,k}) e^{\frac{\alpha c}{3}}. \tag{4.12}$$

Case 3. If $\wp'(h(a_{n,k})) = i$ and $\wp'(h(a_{n,k}+c)) = -i$, then we obtain from (4.10),

$$\left(\omega - c_0 - \frac{\omega + c_0}{\sqrt{3}} i\right) h'(a_{n,k}+c) = -c_1 \left(1 - \frac{i}{\sqrt{3}}\right) h'(a_{n,k}) e^{\frac{\alpha c}{3}}. \tag{4.13}$$

Case 4. If $\wp'(h(a_{n,k})) = i$ and $\wp'(h(a_{n,k}+c)) = i$, then (4.10) yields

$$\left(\omega - c_0 + \frac{\omega + c_0}{\sqrt{3}} i\right) h'(a_{n,k}+c) = c_1 \left(1 - \frac{i}{\sqrt{3}}\right) h'(a_{n,k}) e^{\frac{\alpha c}{3}}. \tag{4.14}$$

Since $h(z)$ and $h(z+c)$ are polynomials of same degree with same leading coefficient and there are infinitely many $a_{n,k}$ (with $|a_{n,k}| \to \infty$), we would have to conclude

$$\begin{cases} \left(\omega - c_0 - \frac{\omega + c_0}{\sqrt{3}} i\right) h'(z+c) = c_1 \left(1 + \frac{i}{\sqrt{3}}\right) h'(z) e^{\frac{\alpha c}{3}} \\ \left(\omega - c_0 + \frac{\omega + c_0}{\sqrt{3}} i\right) h'(z+c) = -c_1 \left(1 + \frac{i}{\sqrt{3}}\right) h'(z) e^{\frac{\alpha c}{3}} \\ \left(\omega - c_0 - \frac{\omega + c_0}{\sqrt{3}} i\right) h'(z+c) = -c_1 \left(1 - \frac{i}{\sqrt{3}}\right) h'(z) e^{\frac{\alpha c}{3}} \\ \left(\omega - c_0 + \frac{\omega + c_0}{\sqrt{3}} i\right) h'(z+c) = c_1 \left(1 - \frac{i}{\sqrt{3}}\right) h'(z) e^{\frac{\alpha c}{3}} \end{cases}$$

This is possible only when

$$e^{\dfrac{\alpha c}{3}} = \begin{cases} -\dfrac{2c_0+1}{2c_1} + \dfrac{\sqrt{3}}{2c_1} i,\ \dfrac{2c_0+1}{2c_1} - \dfrac{\sqrt{3}}{2c_1} i,\ -\dfrac{2+c_0}{2c_1} - \dfrac{\sqrt{3}c_0}{2c_1} i, \\ \dfrac{2c_0+1}{2c_1} + \dfrac{\sqrt{3}}{2c_1} i,\ \dfrac{1-2c_0}{2c_1} + \dfrac{\sqrt{3}}{2c_1} i,\ \dfrac{c_0+1}{2c_1} - \dfrac{\sqrt{3}(c_0+1)}{2c_1} i, \\ \dfrac{c_0+1}{2c_1} + \dfrac{\sqrt{3}(1-c_0)}{2c_1} i,\ -\dfrac{c_0+1}{c_1},\ -\dfrac{c_0+1}{2c_1} - \dfrac{\sqrt{3}}{2c_1} i, \\ \dfrac{c_0+1}{2c_1} - \dfrac{\sqrt{3}(c_0-1)}{2c_1} i,\ \dfrac{c_0+1}{2c_1} + \dfrac{\sqrt{3}(c_0+1)}{2c_1} i,\ \dfrac{1-2c_0}{2c_1} - \dfrac{\sqrt{3}}{2c_1} i, \end{cases}$$

since $\omega = 1$, $\omega = -\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2} i$.

Therefore, there exists a positive integer m_0 satisfying $P(h(a_n + c)) \neq 0$ for $n > m_0$.

When this is true, one has uniformly following the above set of equations (which are in terms of $h'(z+c)$ and $h'(z)$) that $h(z) = az + b$ for $ac \neq 0$. Again we know that the function $\wp(z)$ has two distinct zeros in $\mathfrak{P}$, and hence in each associated lattice, we see that all the zeros $\{z_n\}_{n\in\mathbb{N}}$ of $\wp(z)$ are transferred to each other through (an integral multiple) of ac. Therefore, for the simplicity, we can consider two cases: either $ac = \omega_1,\ \omega_2,\ \omega_1 + \omega_2$ or $ac \neq \omega_1,\ \omega_2,\ \omega_1 + \omega_2$ and $ac \in \mathfrak{P}$. It is worth noticing that the former cannot occur in view of (4.4) and the periodicity of $\wp(z)$ and $\wp'(z)$, and the later cannot occur either $\wp(z)$ has a unique double pole in each lattice. We now substitute $z_\infty = -(b/a)$ into (4.4), and obtain the following

$$\infty = \frac{(\omega - c_0) - \frac{\omega + c_0}{\sqrt{3}}\wp'(0)}{\wp(0)} = \frac{c_1\left(1 + \frac{\wp'(ac)}{\sqrt{3}}\right)}{\wp(ac)} e^{\frac{\alpha c}{3}} < \infty$$

which leads to a contradiction.

It is easy to see that $\wp(h(a_{n,k}+c)) = 0$ may occur only for finitely $a_{n,k}$'s. Without loss of generality, we assume that $\wp(h(a_{n,k}+c)) \neq 0$ for $k = 1,\ 2,\ \ldots, \deg(h)$ and all $n > N$, with N being a sufficiently large positive integer. Again since $\wp(h(a_{n,k})) = 0$ and $(\wp')^2(h(a_{n,k})) = -1$, hence by (4.4) we must have $\wp(h(a_{n,k})) = \infty$ for $n > N$. This implies that the zeros of $\wp(h(z))$ are the poles of $\wp(h(z+c))$ except for finitely many points. We observe that $O(\log r) = S(r, \wp(h))$, and hence we can write

$$\begin{aligned} N\left(r, \frac{1}{\wp(h(z))}\right) &\leq \bar{N}\left(r, \frac{1}{\wp(h(z))}\right) + 2N\left(r, \frac{1}{h'(z)}\right) \\ &\leq \bar{N}\left(r, \wp(h(z+c))\right) + 2T(r, h'(z)) + O(\log r) \\ &\leq \bar{N}\left(r, \wp(h(z+c))\right) + S\left(r, \wp(h(z))\right). \end{aligned} \tag{4.15}$$

In view of equation (4.1) and the estimate in (4.27), we obtain

$$\begin{aligned} T(r, f) &\leq T(r, \wp(h)) + T(r, \wp'(h)) + \frac{1}{3}T\left(r, e^{\alpha z}\right) + O(1) \\ &\leq O(T(r, \wp(h))). \end{aligned} \tag{4.16}$$

Hence in view of (4.8) and the estimate $T(r, \wp(h)) = O\left(r^{2q}\right)$, we have $\rho(f) = \rho(\wp(h))$ and also $S(r, f) = S(r, \wp(h))$. So we have $T\left(r, e^{\alpha z}\right) = S(r, f)$. From the equation

$$-[\mathcal{L}_c(f)]^3 = f^3(z) - \left(e^{\frac{\alpha z+\beta}{3}}\right)^3 = \left(f(z) - e^{\frac{\alpha z+\beta}{3}}\right)\left(f(z) - \omega e^{\frac{\alpha z+\beta}{3}}\right)\left(f(z) - \omega^2 e^{\frac{\alpha z+\beta}{3}}\right),$$

we deduce that all the zeros of each of the following functions

$$\left(f(z) - e^{\frac{\alpha z+\beta}{3}}\right), \left(f(z) - \omega e^{\frac{\alpha z+\beta}{3}}\right) \text{ and } \left(f(z) - \omega^2 e^{\frac{\alpha z+\beta}{3}}\right)$$

are of multiplicities at least 3.

By Yamanoi's Second Fundamental Theorem (see [32]), we obtain

$$\begin{aligned}
2T(r,f) &\leq \bar{N}(r,f)+\bar{N}\left(r,\frac{1}{\left(f(z)-e^{\frac{\alpha z+\beta}{3}}\right)}\right)+\bar{N}\left(r,\frac{1}{\left(f(z)-\omega e^{\frac{\alpha z+\beta}{3}}\right)}\right)\\
&\quad+\bar{N}\left(r,\frac{1}{\left(f(z)-\omega^2 e^{\frac{\alpha z+\beta}{3}}\right)}\right)+S(r,f)\\
&\leq \frac{1}{3}N\left(r,\frac{1}{\left(f(z)-e^{\frac{\alpha z+\beta}{3}}\right)}\right)+\frac{1}{3}N\left(r,\frac{1}{\left(f(z)-\omega e^{\frac{\alpha z+\beta}{3}}\right)}\right)\\
&\quad+\frac{1}{3}N\left(r,\frac{1}{\left(f(z)-\omega^2 e^{\frac{\alpha z+\beta}{3}}\right)}\right)+N(r,f)+S(r,f)\\
&\leq T(r,f)+T\left(r,e^{\alpha z}\right)+N(r,f)+S(r,f)\\
&\leq T(r,f)++N(r,f)+S(r,f)
\end{aligned}$$

which implies that $T(r,f)=N(r,f)+S(r,f)$. It leads to $m(r,f)=S(r,f)=S(r,\wp(h))$. On the other hand, the form of the function f in (4.1) shows that

$$\frac{1}{2\wp(h(z))}=f(z)e^{-\frac{\alpha z+\beta}{3}}-\frac{\wp'(h(z))}{2\sqrt{3}\wp(h(z))}.$$

Therefore, by the lemma of the logarithmic derivative, it is easy to see that

$$\begin{aligned}
&m\left(r,\frac{1}{\wp(h(z))}\right)=m\left(r,\frac{1}{2\wp(h(z))}\right)+O(1) \qquad (4.17)\\
&\leq m(r,f)+m\left(r,e^{-\frac{\alpha z+\beta}{3}}\right)+m\left(r,\frac{h'(z)\wp'(h(z))}{\wp(h(z))}\right)+m\left(r,\frac{1}{h'(z)}\right)+O(1)\\
&\leq T\left(r,e^{-\frac{\alpha z+\beta}{3}}\right)+T\left(r,\frac{1}{h'(z)}\right)+S(r,\wp(h(z)))\\
&\leq T\left(r,e^{\alpha z}\right)+T(r,h'(z))+S(r,\wp(h(z)))\leq S(r,\wp(h(z))).
\end{aligned}$$

Combining equations (4.15) and (4.17) and observing that each pole of $\wp(z)$ is of multiplicity is exactly 2 (so that each pole $P(h)$ has multiplicity $2k$ for some integer $k\geq 1$), by applying Theorem 2.1 of Chiang and Feng [5], we obtain

$$\begin{aligned}
&T(r,\wp(h(z)))=T\left(r,\frac{1}{\wp(h(z))}\right)+O(1)\\
&=m\left(r,\frac{1}{\wp(h(z))}\right)+N\left(r,\frac{1}{\wp(h(z))}\right)+O(1)\\
&\leq \bar{N}\left(r,\frac{1}{\wp(h(z))}\right)+S(r,\wp(h(z)))\leq \bar{N}(r,\wp(h(z+c)))+S(r,\wp(h(z)))\\
&\leq \frac{1}{2}N(r,\wp(h(z+c)))+S(r,\wp(h(z)))\leq \frac{1}{2}T(r,\wp(h(z+c)))+S(r,\wp(h(z)))\\
&\leq \frac{1}{2}T(r,\wp(h(z)))+S(r,\wp(h(z)))+O\left(r^{\rho(\wp(h))-1+\epsilon}\right)
\end{aligned}$$

which yields that $T(r,\wp(h)) \leq S(r,\wp(h(z)))+O\left(r^{\rho(\wp(h))-1+\epsilon}\right)$. Therefore, we arrive at a contradiction. The proof of the theorem is complete. □

Proof of Theorem 2.2. For the details of proof of Theorem 2.2, we discuss here the case $n=1$ only because the cases $n \geq 2$ will follow from Theorem F of Han and Lü [16]. For $n=1$, equation (2.3) becomes

$$f(z)+f^{(k)}(z)=e^{\alpha z+\beta}. \tag{4.18}$$

The general solution of the differential equation (4.18) consist of two parts: one is complementary function $f_c(z)$ and the other is particular solution $f_p(z)$. The auxiliary equation here is $m^k+1=0$ which implies $m=\theta,\theta^2,\ldots,\theta^{k-1}$. It is easy to see that m can take value -1 also for the case when k is odd. Therefore, we have $f_c(z)=\sum_{j=1}^{k} a_j e^{\theta^j z}$, where a_j's are complex constants. Let us denote the differential operator D as $D \equiv d/dz$. Then equation (4.18) can be expressed as $\left(D^k+1\right)f(z)=e^{\alpha z+\beta}$. Therefore, we have

$$f_p(z)=\frac{1}{D^k+1}e^{\alpha z+\beta}.$$

If $\alpha \notin \{\theta,\theta^2,\ldots,\theta^{k-1}\}$, then a simple computations shows that the particular solution in this case is $f_p(z)=e^{\alpha z+\beta}/(\alpha^k+1)$. Hence the general solution is

$$f(z)=f_c(z)+f_p(z)=\sum_{j=1}^{k} a_j e^{\theta^j z}+\frac{ze^{\alpha z+\beta}}{\alpha^k+1}.$$

If $\alpha \in \{\theta,\theta^2,\ldots,\theta^{k-1}\}$, then we see that $\alpha^k=-1$. Therefore, we have

$$\begin{aligned}
f_p(z)&=\frac{1}{D^k+1}e^{\alpha z+\beta}=e^{\alpha z+\beta}\frac{1}{(D+\alpha)^k+1}(1)\\
&=e^{\alpha z+\beta}\frac{1}{D^k+\binom{k}{1}D^{k-1}\alpha+\binom{k}{2}D^{k-1}\alpha^2+\ldots+\binom{k}{k-1}D\alpha^{k-1}}(1)\\
&=e^{\alpha z+\beta}\frac{1}{\binom{k}{k-1}D\alpha^{k-1}}\left(1+\frac{1}{\binom{k}{k-1}\alpha^{k-1}}\left(D^{k-1}+\binom{k}{1}D^{k-2}+\ldots+1\right)\right)^{-1}(1)\\
&=e^{\alpha z+\beta}\frac{1}{\binom{k}{k-1}\alpha^{k-1}}\frac{1}{D}(1)=\frac{ze^{\alpha z+\beta}}{k\alpha^{k-1}}.
\end{aligned}$$

Hence, the general solution is

$$f(z)=f_c(z)+f_p(z)=\sum_{j=1}^{k} a_j e^{\theta^j z}+\frac{ze^{\alpha z+\beta}}{k\alpha^{k-1}}.$$

When in particular $\alpha=-1$, this case can be handled easily considering k as odd or even separately. □

Proof of Theorem 2.3. We split the whole proof into the follows two cases.

Case 1. Let the solution f be a transcendental entire function. Let us first consider the exponential case *i.e.,* $f(z) = de^{P(z)}$, where $P(z)$ is a polynomial in z. Then we have

$$d^2\left(e^{2P(z)-(\alpha z+\beta)} + e^{2P(z+c)-(\alpha z+\beta)}\right) = 1. \tag{4.19}$$

A simple computations shows that both the functions $2P(z) - (\alpha z + \beta)$ and $2P(z + c) - (\alpha z + \beta)$ must be constants, say, c_1 and c_2, respectively. Then an elementary calculation shows that

$$\alpha c = c_2 - c_1 = 2\left(P(z+c) - P(z)\right). \tag{4.20}$$

By the assumption, f is a finite order entire function and in view of (4.20), $\deg(P)$ must be equal to 1. Hence we can show that $P(z)$ takes the form $P(z) = (\alpha z + \beta)/2$. Thus it follows from (4.19) that $d^2 = 1/e^{\alpha c}$ with $d \neq \pm 1$ and α, c be such that $e^{\alpha c} \neq -1$.

Let $f(z)$ is not of the form $f(z) = de^{P(z)}$. We know from the result of Gross that any entire solution of $f^2(z) + g^2(z) = 1$ is of the form $f(z) = \sin(h(z))$ and $g(z) = \cos(h(z))$, where h is a an entire function.

The difference equation $f(z)^2 + f^2(z + c) = e^{\alpha z+\beta}$ can be written as

$$\left(\frac{f(z)}{e^{\frac{\alpha z+\beta}{2}}}\right)^2 + \left(\frac{f(z+c)}{e^{\frac{\alpha z+\beta}{2}}}\right)^2 = 1.$$

Therefore, by the result of Gross [8], it is easy to see that the general solution of $f(z)^2 + f^2(z + c) = e^{\alpha z+\beta}$ must be

$$f(z) = e^{\frac{\alpha z+\beta}{2}}\sin\left(h(z)\right) \quad \text{and} \quad f(z+c) = e^{\frac{\alpha z+\beta}{2}}\cos\left(h(z)\right)$$

for an entire function h. Therefore, we obtain $h(z + c) = h(z) + 2k\pi + \pi/2$ and $e^{\alpha c/2} = 1$, where k is an integer. Writing $h(z) = (4k+1)\pi z/2c + \mathcal{H}(z)$, it is easy to verify that $\mathcal{H}(z)$ is a c-periodic entire function. Therefore, the general non-constant entire solution can be written as

$$f(z) = e^{\frac{\alpha z+\beta}{2}}\sin\left(\frac{(4k+1)\pi z}{2c} + \mathcal{H}(z)\right).$$

In particular, if f is a finite order transcendental entire function, then by Pólya's theorem [25], the function $\mathcal{H}(z)$ must be constant, say, η. Hence, the general non-constant transcendental entire solution becomes

$$f(z) = e^{\frac{\alpha z+\beta}{2}}\sin\left(\frac{(4k+1)\pi z}{2c} + \eta\right).$$

Case 2. Let f be a meromorphic function.

The difference equation $f(z)^2 + f^2(z+c) = e^{\alpha z+\beta}$ can be written as

$$[f(z) + if(z+c)][f(z) - if(z+c)] = e^{\alpha z+\beta}. \tag{4.21}$$

From (4.21), it is easy to see that the functions $[f(z)+if(z+c)]$ and $[f(z)-if(z+c)]$ may have zeros and poles. Therefore, there exists a meromorphic function g and a complex number δ such that $[f(z)+if(z+c)]$ and $[f(z)-if(z+c)]$ can be expressed as

$$f(z) + if(z+c) = e^{\delta(\alpha z+\beta)} g(z) \tag{4.22}$$

and

$$f(z) - if(z+c) = e^{(1-\delta)(\alpha z+\beta)} \frac{1}{g(z)}. \tag{4.23}$$

Solving equations (4.22) and (4.23) for $f(z)$ and $f(z+c)$, we obtain

$$f(z) = \frac{1}{2}\left(e^{\delta(\alpha z+\beta)} g(z) + \frac{e^{(1-\delta)(\alpha z+\beta)}}{g(z)}\right) \tag{4.24}$$

and

$$f(z+c) = \frac{1}{2i}\left(e^{\delta(\alpha z+\beta)} g(z) - \frac{e^{(1-\delta)(\alpha z+\beta)}}{g(z)}\right). \tag{4.25}$$

Combining (4.24) and (4.25), it is easy to see that

$$\begin{aligned} &e^{\delta(\alpha z+\beta)} e^{\alpha\delta c} g(z+c) + \frac{e^{(1-\delta)(\alpha z+\beta)} e^{\alpha(1-\delta)c}}{g(z+c)} \\ &= -i\left(e^{\delta(\alpha z+\beta)} g(z) - \frac{e^{(1-\delta)(\alpha z+\beta)}}{g(z)}\right). \end{aligned} \tag{4.26}$$

Clearly, (4.26) shows that the functions $g(z)$ and $g(z+c)$ have the same set of zeros and poles with the same multiplicities, otherwise, comparing the zeros and poles of $g(z)$ and $g(z+c)$ from both sides of (4.26), we can arrive at a contradiction.

Therefore, there exists a polynomial $\mathcal{Q}(z)$ in z such that

$$\frac{g(z+c)}{g(z)} = e^{\mathcal{Q}(z)}. \tag{4.27}$$

If $e^{\mathcal{Q}(z)} \equiv 1$, then g becomes a c-periodic function. Now equating the coefficients in (4.26), we obtain,

$$ie^{\delta\alpha c} = 1 \text{ and } ie^{(1-\delta)\alpha c} = -1.$$

Therefore, we have $e^{\alpha c} = 1$ and $e^{\delta\alpha c} = -i$, which shows that $\delta = 1/4$ or $3/4$. Hence the possible forms of the function f is one of the following:

$$\begin{cases} f(z) = \dfrac{e^{\frac{1}{4}(\alpha z+\beta)}}{2}\left(g(z) + \dfrac{e^{\frac{1}{2}(\alpha z+\beta)}}{g(z)}\right) \\ f(z) = \dfrac{e^{\frac{1}{4}(\alpha z+\beta)}}{2}\left(e^{\frac{1}{2}(\alpha z+\beta)} g(z) + \dfrac{1}{g(z)}\right). \end{cases}$$

If $e^{\mathcal{Q}(z)} \neq 1$, then substituting $g(z+c) = e^{\mathcal{Q}(z)}g(z)$ in (4.26), we obtain that

$$g^2(z)e^{(2\delta-1)(\alpha z+\beta)} = -\frac{ie^{(1-\delta)\alpha c} + e^{\mathcal{Q}(z)}}{e^{\mathcal{Q}(z)}\left(ie^{\delta\alpha c} - 1\right)}. \tag{4.28}$$

Clearly, the function g in (4.28) cannot have any poles, hence g must be a transcendental entire function. But note that, all the zeros of $ie^{(1-\delta)\alpha c} + e^{\mathcal{Q}(z)}$ are the zeros of $g(z)$ are of multiplicities at least 2, which leads to a contradiction. This completes the proof. □

5. Future study

To continue the study, one can turn attention to the solutions of more general Fermat-type equations. For example, Ramanujan observed that $x = 9$, $y = 10$ and $z = -12$ is a solution of $x^n + y^n + z^n = 1$ for the case $n = 3$. Therefore, looking for the solutions of equation $x^n + y^n + z^n = 1$ for $n \geq 4$ will of great interests, and the study will become more effective if x, y and z be non-constant functions. Since the problem of finding solutions of (1.1) have been settled for the classes (i)-(iv) mentioned above, it is therefore natural to turn attention to the functional equation

$$f^n + g^n + h^n = 1, \tag{5.1}$$

where n is a positive integer and f, g and h are functions in any one of the above four function classes.

Finding non-constant entire as well as meromorphic solutions are effortless for $n = 1$. For example, for $n = 2$, one can verify that

$$(f, g, h) = (\sin(\phi)\cos(\psi), \sin(\phi)\sin(\psi), \cos(\phi))$$

is an immediate entire solution and

$$(f, g, h) = (i\sin(\phi)\tan(\phi), i\cos(\phi)\tan(\phi), \sec(\phi))$$

is a meromorphic solution of the equation (5.1), where ϕ and ψ are two entire functions. For $n \geq 3$, looking for non-constant entire as well as meromorphic solutions will be of utmost interest. For future course of work and to study Fermat-type functional equations, we refer the reader to go through the article of Gundersen [13] and references there in.

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