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NOTE ON A THEOREM OF RELATIVISTIC HYDRODYNAMICS

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A well known theorem of relativistic hydrodynamics states that the stream-lines of an isentropic perfect fluid are the future-pointing timelike (FPT) curves extremizing the integral $J = \int_{s_1}^{s_2} f ds$, where f is the so-called index function and s the proper time on the world line of the fluid particle. The integral being taken over all possible FPT curves with regular representations $x^i = x^i(s)$ joining the fixed end events E_1, E_2. The purpose of this note is to show that the stream-lines of an adiabatic perfect fluid can likewise be regarded as extremizing curves of the functional J provided the class of admissible curves consists of those FPT curves satisfying the side condition $u^i \partial_i S = 0$, u^i unit 4-velocity and S specific proper entropy of the fluid, with first end point fixed and second end point variable.

Key words: *Relativity:hydrodynamics*

1. *Introduction.* In relativistic hydrodynamics, the class of isentropic perfect fluids, i.e. those for which the specific proper entropy S takes a value S_0 independent of the space-time coordinates x^i ($i = 1, 2, 3, 4$) throughout the fluid, is of special interest since the stream-lines of such fluids are known to satisfy the variational principle (e.g. [1-3])

$$J = \int_{s_1}^{s_2} f \sqrt{g_{rs} \dot{x}^r \dot{x}^s}\, ds \quad \rightarrow \quad \text{extremum} \quad \left(\dot{x}^r = dx^r/ds\right) \tag{1}$$

for fixed end events.

The integral is taken over all possible future-pointing timelike (FPT) curves having regular representations $x^i = x^i(s)$, $s_1 \le s \le s_2$, in terms of proper time, i.e. curves $x^i(s)$ with a nonvanishing set of s-derivatives $\dot{x}^i$ equal to the unit 4-velocity u^i, $u^i u_i = 1$, and joining the fixed end events E_1, E_2. The positive scalar function f entering the integrand is the so-called index function of the fluid and is regarded as a known function of position in space-time; it is given by [2b,3]

$$f = 1 + c^{-2}\left(\varepsilon + \frac{p}{r}\right), \tag{2}$$

where p is the pressure, r the proper material density and ε the internal energy density; the proper energy density ρ of the fluid being set equal to

$$\rho = r\left(1 + \frac{\varepsilon}{c^2}\right) \tag{3}$$

An important property of the functional J is that its value is independent of the particular parameter chosen, a consequence of the homogeneity of the integrand in the variables $\dot{x}^i$. The corresponding Euler necessary condition for an extremum deduced from the first variation of the functional J assumes the form of the equation of motion of an isentropic perfect fluid

$$u^\ell \nabla_\ell u_i - \left(g_i^\ell - u^\ell u_i\right)\frac{\partial_\ell f}{f} = 0. \tag{4}$$

The differential system (4) expresses the fact that the stream-lines are geodesics of the metric $f^2 ds^2$, conformal to the metric ds^2 of the space-time[1].

A more general class of fluids consists of those for which the specific proper entropy S is constant only along each stream-line, constraint translated into

$$u^i \partial_i S = 0, \tag{5}$$

such fluids are said adiabatic. By virtue of the equation of continuity, condition (5) is equivalent to the equation, stating the conservation of proper material density [2b]

$$\nabla_i\left(ru^i\right) = 0. \tag{6}$$

The equation governing the motion of an adiabatic perfect fluid can be written in the form [2b,3]

$$u^\ell \nabla_\ell u_i - \left(g_i^\ell - u^\ell u_i\right)\left[\frac{\partial_l f}{f} - T\frac{\partial_\ell S}{c^2 f}\right] = 0, \tag{7}$$

where T is the proper temperature of the fluid.

The question which naturally arises is whether the functional J can be used to formulate an extremum principle such that along any FPT curve extremizing the integral J eq. (7) holds.

In this note we propose to show that if the class of admissible curves consists of those FPT curves satisfying the side condition (5), with the first end event fixed and the second end event variable, then the extremals of the functional J satisfy a differential equation which agrees with the differential system (7) provided we identify the Lagrange multiplier associated with the constraint (5), or more precisely its s-derivative, as the proper temperature T divided by c^2.

2. *Extremum principle for the stream-lines of an adiabatic perfect fluid.* Our problem is that of finding the necessary condition, known as the multiplier rule, which is imposed on the curve g of class C^1 with regular

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[1] *This property of the stream-lines in the case of a perfect fluid seems to have been first established by Eisenhart (Trans. Amer. Math. Soc. 26, 205, 1924) using a different approach.*

representation in terms of proper time

$$g : x^i = \bar{x}^i(s), \quad s_1 \le s \le s_2 \quad (i = 1,\dots,4) \tag{8}$$

in order that it furnishes an extremum to the functional

$$J = \int_{s_1}^{s_2} f \sqrt{g_{ij}\dot{x}^i\dot{x}^j}\, ds \quad \left(\dot{x}^i = dx^i/ds\right), \tag{9}$$

relative to neighboring admissible curves. The class of admissible curves consists of those FPT curves satisfying the side conditions

$$\phi_1(x,\dot{x}) = \sqrt{g_{ij}\dot{x}^i\dot{x}^j} = 1\,, \tag{10}$$

$$\phi_2(x,\dot{x}) = \dot{x}^i\,\partial_i S = 0\,. \tag{10'}$$

Curves satisfying differential side conditions of the form (10) are said differentially admissible.We further suppose that the matrix $\left\|\phi_{A\dot{x}^i}\right\|$, $A = 1, 2$ has rank 2 along g. A variational problem of this kind is designated in the calculus of variations as a parametric problem of Lagrange, a special case of the problem of Bolza whose theory in Riemannian space has been first discussed by Hestenes [4]. It should be noted that the equality constraint (10) not only specifies the parameter to be used but it also expresses the timelike character of the curves admitted to competition. A regular representation of g in terms of an arbitrary parameter θ, with $\dot{\theta}(s) > 0$, would yield an inequality constraint [5]. It remains to specify under which end conditions the functional J is to be extremized. In order that g be a "normal" extremizing curve, we have to treat an extremum problem with variable end point, i.e. the admissible curves neighboring g cannot have both their end points fixed respectively at the initial and final end points of g as it is the case for the variational principle satisfied by the stream-lines of an isentropic perfect fluid. The definition of normality will be given when discussing the so called multiplier rule. For variable-endpoint problems this term refers to both the Euler necessary condition and the transversality condition. If we consider our problem as a non-parametric problem in the space of coordinates (s, x^i) the above conditions follow from the corresponding conditions in non-parametric form [4,6a]. Adopting Morse's formulation of end conditions [6a, b], let us denote points near the initial and final end points of the extremal g: $x^i = \bar{x}^i(s)$, regarded as a curve in the space (s, x), by (s^A, x^{iA}), $A = 1, 2$. Without loss of generality we may assume that the class of terminally admissible curves neighboring g consists of those FPT curves whose end points (s^A, x^{iA}) are respectively given by

$$s^1 = 0\,,\ x^{i1} = \bar{x}^{i1}(s_1) = \bar{P}_1\,;\ s^2 = s^2(\sigma_1,\dots,\sigma_r),\ x^{i2} = x^{i2}(\sigma_1,\dots,\sigma_r)(1 \le r \le 4+1) \tag{11}$$

For values of (σ) near (0). It assumed that the functions appearing in (11) are of class C^2 and that for $(\sigma) = (0)$ they give the end point of g, i.e. $s^2(0) = s_2$, $x'^2(0) = \bar{x}^I(s_2) = \bar{P}_2$

We see that while (s^1, x'^1) has the prescribed value corresponding to the initial end value of $g(s_1 = 0, \bar{P}_1)$, the end condition leaves the value of s^2 undetermined in order to comply with the relativistic demand of the path dependence of proper time.

It follows from the non-parametric theory (e.g. [7-9]) that if g affords an extremum to the functional J there exists a constant λ_0 and two functions $\lambda_A(s)$, $A = 1, 2$ not all identically zero if $\lambda_0 = 0$; such that g satisfies the Euler-Lagrange (EL) equations

$$(a)\quad P_I = \frac{d}{ds} F_{\dot{x}^I} - F_{x^I} = 0, \quad (b)\quad \phi_1(x,\dot{x}) - 1 = \sqrt{g_{IJ}\dot{x}^I\dot{x}^J} - 1 = 0, \quad (c)\quad \phi_2(x,\dot{x}) = \dot{x}^I \partial_I S = 0, \tag{12}$$

where

$$F = \lambda_0 f \sqrt{g_{IJ}\dot{x}^I\dot{x}^J} + \lambda_1(s)(\phi_1 - 1) + \lambda_2(s)\phi_2 . \tag{13}$$

Moreover the following transversality relation holds

$$\left(F - \dot{x}^I F_{\dot{x}^I}\right) ds^2 + F_{\dot{x}^I} dx'^2 = 0 . \tag{14}$$

Taking into account the very useful identity

$$F + \lambda_1(s) = \dot{x}^I F_{\dot{x}^I} , \tag{15}$$

which follows from the relation $F(x, k\dot{x}, \lambda_1, \lambda_2) = k[F(x, \dot{x}, \lambda_1, \lambda_2) + \lambda_1]$, the transversality equation reduces to

$$-\lambda_1\, ds^2 + F_{\dot{x}^I} dx'^2 = 0 , \tag{14'}$$

where $(s, x, \dot{x})$ must be taken on g at the second end point of g. ds^2 and dx'^2 are the differentials of the functions appearing in (11) evaluated at $(\sigma) = (0)$, and (14') is regarded as an identity in these differentials. If there are no nontrivial multipliers $\lambda_0, \lambda_A(s)$ $(A = 1, 2)$ with $\lambda_0 = 0$ the admissible curve is called normal, otherwise abnormal. In the normal case we can always replace a nontrivial set of multipliers by $1, \lambda_A(s)/\lambda_0$ or realize the same end by simply setting $\lambda_0 = 1$. The multipliers $\lambda_0 = 1, \lambda_A(s)$ are then unique. A geometrical interpretation of normality has been given by Morse and Myers [7]. The importance of this property resides in the fact that if the admissible curve C is normal there are always neighboring curves belonging to the set of admissible curves so that the extremum problem is never trivial [8]. It is readily proved that under arbitrary transformations of space-time coordinates P_I in eq. (12) will transform as the covariant component of a 4-vector and the first member of the identity (14) will be an invariant provided the multipliers λ_0, λ_A belonging to g are invariants. Lagrange multipliers are often treated inadequately in the physical literature. In particular the function F with λ_0 arbitrarily chosen as unity is simply presented without a proof to exclude the abnormal case.

Making use of formula (15), the EL equation (12a) and the transversality condition (14) can be written explicitly as

$$\lambda_0 \, f u^\ell \nabla_\ell u_l - \lambda_0 \partial_\ell f\left(g_l^\ell - u_l u^\ell\right) + \lambda_1(s) u^\ell \nabla_\ell u_l + \dot{\lambda}_1(s) u_l + \dot{\lambda}_2(s) \partial_l S = 0 \quad (16)$$

and

$$-\lambda_1(s_2)\, ds^2 + \left[\lambda_0 f + \lambda_1(s) u_l + \lambda_2(s) \partial_l S\right]_2 dx^{l2} = 0 \quad (17)$$

where $u^l = \dot{x}^l$ is the unit 4-velocity and the subscript 2 indicates that $(s, x, \dot{x})$ in the functions appearing in the square bracket is to be taken on g at its second end point. Multiplying eq. (17) by u^l and noting that $u^\ell \nabla_\ell u_l = 0$, $u^l u_l$ being constant (=1), we obtain

$$\dot{\lambda}_1(s) = 0 \quad \text{or} \quad \lambda_1(s) = \text{const}. \quad (18)$$

Further useful informations on λ_1 and λ_2 will come from the transversality condition.

Since the curve $g : x^l = \overline{x}^l(s)$ joining the points

$$s^1(0) = 0 \quad x^{l1}(0) = \overline{x}^l(s_1) = \overline{P}_1\,; \quad s^2(0), x^{l2}(0) = \overline{x}^l(s_2) = \overline{P}_2 \quad (19)$$

in (s, x) space furnishes an extremum for the integral J on the class of admissible curves with end conditions (11), it will furnish an extremum compared to differentially admissible neighboring curves with end points $(0, \overline{P}_1)$ and $(s^2(\sigma), \overline{P}_2)$. The transversality condition with $dx^{l2} = 0$ then yields

$$\lambda_1(s_2) = 0. \quad (20)$$

Accordingly, as a consequence of the homogeneity of F and of the transversality condition, we see that if g is to extremize the functional J the multiplier λ_1 associated with g must be constant and equal to zero. Substitution of $\lambda_1 = 0$ in eqs (16) and (17) gives

$$\lambda_0 \, f u^\ell \nabla_\ell u_l - \lambda_0 \partial_\ell f\left(g_l^\ell - u_l u^\ell\right) + \dot{\lambda}_2(s) \partial_l S = 0, \quad (21)$$

$$\left[\lambda_0 f + \lambda_2(s) \partial_l S\right]_2 dx^{l2} = 0. \quad (22)$$

Comparison of the EL equation (21) with eq. (7) defining the streamlines of an adiabatic perfect fluid shows that λ_0 must be different from zero, i.e. the extremal g must be normal, if these two equations are to coincide. Let us show that this is indeed the case for our extremum problem with variable end point. Substitution of $\lambda_0 = 0$ in eqs (21) and (22) gives respectively

$$\dot{\lambda}_2(s) \partial_l S = 0 \quad (i = 1, \ldots, 4), \quad (23)$$

$$\left[\lambda_2(s) \partial_l S\right]_2 x_q^{l2} = 0, \quad (24)$$

where the subscript q attached to s^2 or x^{l2} means differentiation with respect to σ_q and evaluation at $\sigma = 0$.

These equations require that λ_2 be constant and equal to zero.

Accordingly if $\lambda_0 = 0$ then $\lambda_2 = \lambda_1 = 0$, but the multiplier rule assures that values for these multipliers exist that are not all zero; hence if there is an extremizing curve g it must be normal and we can set $\lambda_0 = 1$. This conclusion remains valid in the special case where just one of the space-time coordinates x^{i2} ($i=1, 2, 3, 4$), say x^{42}, is left undetermined while the space coordinates $x^{\alpha 2}$ ($\alpha = 1, 2, 3$) are set equal to the space coordinates $\bar{x}^{\alpha}(s_2)$ of the final end point $\bar{P}_2$ of g, provided we assume $\partial_4 S$ different from zero. If the admissible curves neighboring g have their final end events $(x^{12}(\sigma), ..., x^{42}(\sigma))$ fixed at $\bar{P}_2$, it turns out that λ_0 can have the value zero. Indeed, one easily verifies that the multiplier rule ensures the existence of an abnormal set of multipliers $\lambda_0 = 0$, $\lambda_1 = 0$ and $\lambda_2 = \text{const} \neq 0$. Returning to eq. (21) and setting $\lambda_0 = 1$ the EL equation now reads

$$u^{\ell} \nabla_{\ell} u_i - \frac{\partial_{\ell} f}{f}\left(g_i^{\ell} - u_i u^{\ell}\right) + \dot{\lambda}_2(s) \partial_i S = 0 . \tag{25}$$

This agrees with eq. (7) provided we identify $\dot{\lambda}_2(s)$ with T/c^2. The multiplier λ_2 or more exactly its s-derivative $\dot{\lambda}_2$ thus acquires a physical significance.

3. ***Conclusion***. Although the respective stream-lines of an adiabatic and isentropic perfect fluid extremize the same functional J, the timelike curves admitted to competition satisfy different side and end conditions so that we have to deal with different variational principles. For an adiabatic perfect fluid, the class of admissible curves consists of those timelike curves along which the specific proper entropy S is constant and whose second end points are undetermined. Translating these constraints into equations, we have seen that the derivation of the equation of motion from a variational principle requires the use of the transversality condition. This condition furnishes further informations on the multipliers not obtainable from the Euler equations. For instance, without the use of this condition, one cannot prove that the multiplier associated with the constraint specifying the timelike nature of the 4-velocity, $\sqrt{u^i u_i} = 1$, is a constant equal to zero; the only information furnished by the Euler equation is the constancy of the corresponding multiplier.

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ОБ ОДНОЙ ТЕОРЕМЕ РЕЛЯТИВИСТСКОЙ ГИДРОДИНАМИКИ

Р.А.КРИКОРЯН

Хорошо известная теорема релятивистской гидродинамики гласит, что линиями тока изоэнтропической идеальной жидкости являются направленные к будущему временоподобные (futurepointing timelike, FPT) кривые, сообщающие экстремум интегралу $J = \int_{s_1}^{s_2} f ds$, где f - так называемая индекс-функция, а s - соответствующее время на мировой линии частицы. Интеграл берется по всевозможным FPT кривым с регулярными представлениями $x^l = x^l(s)$, соединяющими фиксированные конечные события E_1, E_2. Цель данной заметки показать, что линии тока адиабатической идеальной жидкости могут подобным образом рассматриваться как кривые, сообщающие экстремум функционалу J при условии, что класс допустимых кривых состоит из FPT кривых, которые удовлетворяют условиям $u^l \partial_l S = 0$, где u^l - единичная 4-скорость, а S - соответствующая удельная энтропия жидкости, когда первая конечная точка фиксирована, а вторая - переменна.

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