АСТРОФИЗИКА

ТОМ 46 НОЯБРЬ, 2003 ВЫПУСК 4

УДК: 524.354.4

ON THE TETRAD FORMULATION OF THE EQUATIONS OF MOTION IN GENERAL RELATIVITY

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Received 20 May 2003
Accepted 25 June 2003

It is shown that the equation of motion $Du_j / Ds = (e / mc^2) F_{ji} u^i$, natural generalization to curved space-time of the Heaviside-Lorentz law of ponderomotive force, is equivalent to the metric independent and covariant Van Dantzig's equations of motion $dx^j \partial_{[j} p_{i]} = 0$ or $L_v p_i = 0$ where p_i is the conjugate momentum 4-vector and v a vector determined by the condition $p_i v^i = 1$, only with respect to holonomic coordinates. With respect to an anholonomic system, the Heaviside-Lorentz equation is a particular case of VD equations valid for a privileged class of anholonomic frames, those consisting of orthogonal unit vectors.

Key words: *Relativity:equations of motions*

1. *Introduction.* Kottler [1], Cartan [2] and later Van Dantzig [3], quite independently of one another, concluded on the natural invariance of Maxwell's equations under arbitrary transformations of the space-time coordinates. Adopting the point of view that the fundamental laws of physics, not only of electromagnetism, must be formulated in a form independent of the metric of space-time, Van Dantzig was the first to develop a relativistic theory of electromagnetism, thermodynamics and thermo-hydrodynamics independent of metrical geometry. In his article entitled "Electromagnetism independent of metrical geometry III. Mass and motion" [3b] he showed that the equations of dynamics can also be written in a form which is covariant under all holonomic coordinate transformations and independent of any metric or linear connection, and presented a beautiful interpretation of these equations in terms of the Lie derivative. Van Dantzig applied his new results to the motion of a charged test particle of mass m, charge e, and 4-velocity u^i moving in a given gravitational field (g_{ij}) and electromagnetic field (F_{ij}), and recovered the usual general relativistic equations of motion, natural generalization to curved space-time of the Heaviside-Lorentz (HL) law of ponderomotive force,

$$\frac{Du_j}{Ds} = \frac{e}{mc^2} F_{ji} u^i , \qquad (1)$$

where D/Ds indicates the absolute derivative with respect to proper time s along the world line of the particle. The question which naturally arises is whether or not we would reach the same conclusion if, instead of an holonomic coordinate system $\{x^i\}$ $(i = 1, 2, 3, 4)$ with basis vectors $\{e_i\}$ tangent to the parameter curves $x^k = \text{const}$ $(k \neq i)$, we use an orthogonal frame of basis vectors $\{e_{\bar{a}}\}(\bar{a} = 1, 2, 3, 4)$, one of the vectors (say $e_{\bar{4}}$) being timelike and the other three $\{e_{\bar{\alpha}}\}(\bar{\alpha} = 1, 2, 3)$ spacelike, which are not the tangents to the parameter curves of any system of allowable coordinates. Such a system of orthogonal basis vectors, called by Schouten [4] an anholonomic coordinate system, constitutes, as pointed out by Synge [5] and Pirani [6], a much more natural and convenient device for ascribing a physical significance to the components of a tensor than the natural basis vectors $\{e_i\}$. In physical terms, $e_{\bar{4}}$ is interpreted as the 4-velocity of an observer at a given event and the triad $\{e_{\bar{\alpha}}\}$ forms the reference frame used by the observer at the event in question. Physical quantities measured by the observer are then just the tetrad components of the corresponding tensor field.

As emphasized by Schouten, "if any mathematical expression with respect to holonomic coordinates is transformed with respect to an anholonomic system, correction terms appear, all containing the object of anholonomity." We shall see that, with respect to an "anholonomic coordinate system", the HL equation of motion is a particular case of Van Dantzig's general theory.

In sec.II, we briefly recall Van Dantzig's derivation of his new form of the equations of dynamics. In sec.III, we give the mathematical prerequisites for the tetrad formulation of the equations of dynamics. We show that the anholonomic VD and HL equations of motion do agree with each other only and only if the anholonomic frame consists of mutually orthogonal unit vectors.

2. *Van Dantzig's new form of dynamical equations.* For purposes of reference, we briefly recall Van Dantzig's formulation of the equations of motion of a material particle. Let $d\Lambda$ be the action-differential assumed to be positive homogeneous of degree unity in the differentials dx^i $(i = 1, 2, 3, 4)$. By variation of $\int d\Lambda = \int p_i dx^i$, where

$$p_i = \frac{\partial\, d\Lambda}{\partial\, dx^i}, \tag{2}$$

one obtains the equation of motion

$$dp_i - \frac{\partial\, d\Lambda}{\partial x^i} = 0. \tag{3}$$

Following Euler's condition, the p_i are homogeneous of degree zero in the dx^i, hence

$$\frac{\partial\, d\Lambda}{\partial x^i} = (\partial_i\, p_j)dx^j, \tag{4}$$

and eq. (3) is therefore equivalent to

$$2\, dx^j\, \partial_{[j}\, p_{i]} = 0. \tag{5}$$

Now if we put

$$v^i = \frac{dx^i}{d\Lambda}, \tag{6}$$

we have

$$p_i v^i = 1, \tag{7}$$

from which it results that

$$(\partial_j p_i)v^i + p_i \partial_j v^i = 0. \tag{8}$$

Thus the Lie derivative of p_i with respect to v^i is given by

$$L_v p_i \equiv v^j \partial_j p_i + p_j \partial_i v^j = 2 v^j \partial_{[j} p_{i]}. \tag{9}$$

From (9) we see that the equation of motion (5) can be written in the form

$$L_v p_i = 0, \tag{10}$$

which expresses the fact that the conjugate momentum 4-vector is constant, in the sense of the Lie derivative, during the motion under the condition that v^i be defined by (7).

Van Dantzig applied the above general results to the relativistic motion of a mass point particle in an external electromagnetic field, neglecting the emission of radiation. The relativistic action-differential is

$$d\Lambda = mcds + \frac{e}{c}\varphi_i\, dx^i, \tag{11}$$

where

$$ds = \sqrt{g_{ij} dx^i dx^j} \tag{12}$$

is the Riemannian line-element of general relativity and φ_i the 4-potential. From eq (2) it follows that

$$p_i = mcu_i + \frac{e}{c}\varphi_i, \quad \left(u^i = \frac{dx^i}{ds}\right). \tag{13}$$

The Riemannian connection being symmetrical, eq (5) can be written with covariant derivatives. Hence

$$u^i \nabla_j p_i = u^i \nabla_i p_j. \tag{14}$$

Substitution of (13) in (14) yields

$$mc\frac{D}{Ds}u_j + \frac{e}{c}u^i \nabla_i \varphi_j = mcu^i \nabla_j u_i + \frac{e}{c}u^i \nabla_j \varphi_i. \tag{15}$$

Because $u_i u^i = 1$, the term in $u^i \nabla_j u_i$ vanishes and we recover the usual relativistic equation of motion

$$\frac{D}{Ds}u_j = \frac{e}{mc^2}u^i(\nabla_j\varphi_i - \nabla_i\varphi_j) = \frac{e}{mc^2}u^i F_{ji}, \tag{16}$$

with

$$F_{ji} \equiv \nabla_j\varphi_i - \nabla_i\varphi_j = \partial_j\varphi_i - \partial_i\varphi_j. \tag{17}$$

3. *The anholonomic equations of motion.* Let us introduce a field of anholonomic frames consisting of orthogonal vectors $e^i_{\bar{a}}$ $(\bar{a}, i = 1, 2, 3, 4)$, with $e^i_{\bar{4}}$ timelike and $e^i_{\bar{\alpha}}$ $(\alpha = 1, 2, 3)$ spacelike, and the reciprocal system $e^{\bar{a}}_i$, we have

$$g_{\bar{a}\bar{b}} = g_{ij} e^i_{\bar{a}} e^j_{\bar{b}}, \quad e^i_{\bar{a}} e^{\bar{a}}_j = \delta^i_j, \quad e^i_{\bar{a}} e^{\bar{b}}_i = \delta^{\bar{b}}_{\bar{a}}. \tag{18}$$

The Lorentz indices are raised and lowered by $g^{\bar{a}\bar{b}}$ or $g_{\bar{a}\bar{b}}$ according to $e^{\bar{a}i} = g^{\bar{a}\bar{b}} e^i_{\bar{b}}$ (resp. $e^i_{\bar{a}} = g_{\bar{a}\bar{b}} e^{i\bar{b}}$).
The anholonomic components of a tensor, for example, T^i_j are

$$T^{\bar{a}}{}_{\bar{b}} = e^{\bar{a}}_i e^j_{\bar{b}} T^i_j \tag{19}$$

and from (18) it follows that

$$T^i_j = e^i_{\bar{a}} e^{\bar{b}}_j T^{\bar{a}}{}_{\bar{b}}. \tag{20}$$

Equations with respect to a tetrad frame nearly always contain correction terms with the object of anholonomity [4]

$$\Omega_{\bar{b}\bar{a}}{}^{\bar{c}} \equiv e^j_{\bar{b}} e^i_{\bar{a}} \partial_{[j} e^{\bar{c}}_{i]}, \tag{21}$$

for instance

$$e^j_{\bar{b}} e^i_{\bar{a}} \partial_{[j} v_{i]} = \partial_{\bar{b}} v_{\bar{a}} - \partial_{\bar{a}} v_{\bar{b}} + \Omega_{\bar{b}\bar{a}}{}^{\bar{c}} v_{\bar{c}}. \tag{22}$$

The covariant derivative of a vector with respect to an anholonomic coordinate system is defined by

$$\begin{aligned} \nabla_{\bar{b}} v_{\bar{a}} &\equiv e^j_{\bar{b}} e^i_{\bar{a}} \nabla_j v_i = \partial_{\bar{b}} v_{\bar{a}} - \Gamma^{\bar{c}}_{\bar{b}\bar{a}} v_{\bar{c}}, \\ \nabla_{\bar{b}} v^{\bar{a}} &\equiv e^j_{\bar{b}} e^{\bar{a}}_i \nabla_j v^i = \partial_{\bar{b}} v^{\bar{a}} + \Gamma^{\bar{a}}_{\bar{b}\bar{c}} v^{\bar{c}}, \end{aligned} \tag{23}$$

where

$$\partial_{\bar{a}} = e^i_{\bar{a}} \partial_i \tag{24}$$

is the Pfaffian derivative, and

$$\Gamma^{\bar{c}}_{\bar{b}\bar{a}} \equiv e^j_{\bar{b}} e^{\bar{c}}_i \nabla_j e^i_{\bar{a}}. \tag{25}$$

The transformation law of linear connections, associated with a change of natural basis vectors, remaining valid for an anholonomic system, we have

$$\Gamma^{\bar{c}}_{\bar{b}\bar{a}} = e^j_{\bar{b}} e^k_{\bar{a}} \left[e^{\bar{c}}_i \Gamma^i_{jk} - \partial_k e^{\bar{c}}_k \right]. \tag{26}$$

From this it follows that

$$\Omega_{\bar{b}\bar{a}}{}^{\bar{c}} = \Gamma^{\bar{c}}_{\bar{a}\bar{b}} - \Gamma^{\bar{c}}_{\bar{b}\bar{a}}. \tag{27}$$

Remark: Note that with Schouten's notation $\Gamma^{\bar{c}}_{\bar{a}\bar{b}} - \Gamma^{\bar{c}}_{\bar{b}\bar{a}} = 2\Omega_{\bar{b}\bar{a}}{}^{\bar{c}}$.
With reference to eq (23), one finds that the tetrad components of the LH equation of motion (1) are

$$\frac{Du_{\bar{b}}}{Ds} = \frac{du_{\bar{b}}}{ds} - \Gamma^{\bar{c}}_{\bar{a}\bar{b}} u_{\bar{c}} u^{\bar{a}} = \frac{e}{mc^2} F_{\bar{b}\bar{a}} u^{\bar{a}}. \tag{28}$$

Likewise, with reference to (22), the tetrad components of VD equation of motion (5) are

$$dx^{\bar{a}}\left(\partial_{\bar{b}}\, p_{\bar{a}} - \partial_{\bar{a}}\, p_{\bar{b}}\right) + \Omega_{\bar{b}\bar{a}}{}^{\bar{c}}\, p_{\bar{c}} dx^{\bar{a}} = 0. \tag{29}$$

Substitution of $u^{\bar{a}} = \dfrac{dx^{\bar{a}}}{ds}$ and $p_{\bar{a}} = mcu_{\bar{a}} + \dfrac{e}{c}\varphi_{\bar{a}}$ in eq (29) gives

$$\frac{du_{\bar{b}}}{ds} = \frac{e}{mc^2} u^{\bar{a}}\left[\partial_{\bar{b}}\varphi_{\bar{a}} - \partial_{\bar{a}}\varphi_{\bar{b}}\right] + \Omega_{\bar{b}\bar{a}}{}^{\bar{c}}\varphi_{\bar{c}} + \Omega_{\bar{b}\bar{a}}{}^{\bar{c}}\, u^{\bar{a}} u_{\bar{c}}\,. \tag{30}$$

From eq (30) it follows that the simple expression $F_{\bar{b}\bar{a}} = \partial_{\bar{b}}\varphi_{\bar{a}} - \partial_{\bar{a}}\varphi_{\bar{b}}$ is not acceptable as a definition of $F_{\bar{b}\bar{a}}$ when employing a tetrad frame and must be replaced by

$$F_{\bar{b}\bar{a}} = \partial_{\bar{b}}\varphi_{\bar{a}} - \partial_{\bar{a}}\varphi_{\bar{b}} + \Omega_{\bar{b}\bar{a}}{}^{\bar{c}}\varphi_{\bar{c}}\,, \tag{31}$$

in agreement with Corum's remark [7]. Eq (30) may now be written

$$\frac{du_{\bar{b}}}{ds} = \frac{e}{mc^2} u^{\bar{a}} F_{\bar{b}\bar{a}} + \Omega_{\bar{b}\bar{a}}{}^{\bar{c}}\, u^{\bar{a}} u_{\bar{c}}\,. \tag{32}$$

The equation of motion (29), as in the holonomic case, can be expressed in terms of the Lie derivative. With respect to an anholonomic coordinate system, one can prove that the Lie derivative $L_v w_{\bar{a}}$ of a covariant vector $w_{\bar{a}}$ is given by [4]

$$L_v w_{\bar{a}} = v^{\bar{b}}\partial_{\bar{b}} w_{\bar{a}} - v^{\bar{b}}\partial_{\bar{a}} w_{\bar{b}} + \Omega_{\bar{b}\bar{a}}{}^{\bar{c}} v^{\bar{b}} w_{\bar{c}} + \partial_{\bar{a}}\left(v^{\bar{b}} w_{\bar{b}}\right). \tag{33}$$

Putting $v^{\bar{a}} = dx^{\bar{a}}/d\Lambda$ so that $p_{\hat{a}} v^{\hat{a}} = 1$, we see that eq (29) is equivalent to

$$L_v p_{\bar{a}} \equiv v^{\bar{b}}\,\partial_{\bar{b}}\, p_{\bar{a}} - v^{\bar{b}}\,\partial_{\bar{a}}\, p_{\bar{b}} + \Omega_{\bar{b}\bar{a}}{}^{\bar{c}}\, v^{\bar{b}} p_{\bar{c}} = 0. \tag{34}$$

From eq (34) it is clear that, when a transformation from a system of natural basis vectors $\{e_i\}$ to a tetrad frame $\{e_{\bar{a}}\}$ is performed, the vanishing of $v^{\bar{b}}\left(\partial_{\bar{b}}\, p_{\bar{a}} - \partial_{\bar{a}}\, p_{\bar{b}}\right)$ is not sufficient to ensure the constancy of $p_{\bar{a}}$ in the sense of the Lie derivative.

The familiar language of physics may be introduced by defining the physical electric and magnetic fields $E_{\bar{\alpha}}$ and $H_{\bar{\alpha}}$, measured by the observer with 4-velocity $e_{\bar{4}}$, by the invariants [5]

$$E_{\bar{\alpha}} = F_{\bar{\alpha}\bar{4}}, \quad H_{\bar{\alpha}} = \frac{1}{2}\varepsilon_{\bar{\alpha}\bar{\beta}\bar{\gamma}} F_{\bar{\beta}\bar{\gamma}}, \quad F_{\bar{a}\bar{b}} = e^i_{\bar{a}} e^j_{\bar{b}} F_{ij}\,. \tag{35}$$

The first three components of eqs (28) and (30) may thus be exhibited in the form

$$\frac{du_{\bar{\alpha}}}{ds} = \frac{e}{mc^2}\left[E_{\bar{\alpha}} u^{\bar{4}} + \varepsilon_{\bar{\alpha}\bar{\beta}\bar{\gamma}}\, u_{\bar{\beta}} H_{\bar{\gamma}}\right] + \Gamma^{\bar{c}}_{\bar{a}\bar{\alpha}}\, u_{\bar{c}} u^{\bar{a}}\,, \tag{36}$$

$$\frac{du_{\bar{\alpha}}}{ds} = \frac{e}{mc^2}\left[E_{\bar{\alpha}} u^{\bar{4}} + \varepsilon_{\bar{\alpha}\bar{\beta}\bar{\gamma}}\, u_{\bar{\beta}} H_{\bar{\gamma}}\right] + \Omega_{\bar{\alpha}\bar{a}}{}^{\bar{c}}\, u_{\bar{c}} u^{\bar{a}}\,. \tag{37}$$

Eqs (36) and (37) are seen to have the same form as in special relativity with an added term representing the influence of anholonomity on the description of the motion of the charged particle relative to the reference frame $\{e_{\bar{a}}\}$. The component $u^{\bar{4}} = dx^{\bar{4}}/ds$ may be interpreted in physical terms as the ratio of the elapsed time registered by the observer's clock between two adjacent events x^l and $x^l + dx^l$ in the history of the particle to the corresponding time ds registered by the clock carried by the particle.

To pursue further the comparision of eqs (28) and (32) we have to substitute in eq (28) the relation connecting $\Gamma^{\bar{c}}_{\bar{a}\bar{b}}$ and $\Omega_{\bar{b}\bar{a}}{}^{\bar{c}}$. According to eq (27), in a Riemannian space we have [4,8]

$$\Gamma^{\bar{c}}_{\bar{a}\bar{b}} = \begin{Bmatrix} \bar{c} \\ \bar{a}\ \bar{b} \end{Bmatrix} + \frac{1}{2}\left[-\Omega_{\bar{a}\bar{b}}{}^{\bar{c}} + g_{\bar{a}\bar{l}}\, g^{\bar{c}\bar{m}}\, \Omega_{\bar{b}\bar{m}}{}^{\bar{l}} + g_{\bar{b}\bar{l}}\, g^{\bar{c}\bar{m}}\, \Omega_{\bar{a}\bar{m}}{}^{\bar{l}}\right], \tag{38}$$

where $\begin{Bmatrix} \bar{c} \\ \bar{a}\ \bar{b} \end{Bmatrix}$ are the Christoffel symbols of the metric $g_{\bar{a}\bar{b}}$

$$\begin{Bmatrix} \bar{c} \\ \bar{a}\ \bar{b} \end{Bmatrix} = \frac{1}{2} g^{\bar{c}\bar{d}} \left[\partial_{\bar{a}}\, g_{\bar{d}\bar{b}} + \partial_{\bar{b}}\, g_{\bar{d}\bar{a}} - \partial_{\bar{d}}\, g_{\bar{a}\bar{b}}\right]. \tag{39}$$

When the tetrad $\{e^l_{\bar{a}}\}$ is orthonormal

$$g_{\bar{a}\bar{b}} = \eta_{\bar{a}\bar{b}} = \mathrm{diag}(-1, -1, -1, 1), \tag{40}$$

the Christoffel symbols in eq (38) vanish.

Substituting (38) in (30) and using (39) we get

$$\frac{du_{\bar{b}}}{ds} = \frac{e}{mc^2} u^{\bar{a}} F_{\bar{b}\bar{a}} + \frac{1}{2} g^{\bar{c}\bar{d}} u_{\bar{c}} u^{\bar{a}}\, \partial_{\bar{b}}\, g_{\bar{d}\bar{a}} + \Omega_{\bar{b}\bar{a}}{}^{\bar{c}}\, u_{\bar{c}} u^{\bar{a}}\ . \tag{41}$$

This agrees with the anholonomic VD equation of motion (32) only if the tetrad frame $\{e_{\bar{a}}\}$ consists of orthonormal vectors. On account of this, it is not quite correct to say that the formulation of dynamics in the ordinary general relativity theory is equivalent to D.Van Dantzig's naturally covariant and metric independent approach.

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О ТЕТРАДНОЙ ФОРМУЛИРОВКЕ УРАВНЕНИЯ ДВИЖЕНИЯ В ОБЩЕЙ ТЕОРИИ ОТНОСИТЕЛЬНОСТИ

Р.А.КРИКОРЯН

Показано, что уравнение движения $Du_j/Ds = (e/mc^2)F_{jl}u^l$, которое является естественным обобщением закона Хевисайда-Лоренца о пондеромоторной силе на случай искривленного пространства-времени, эквивалентно независимому от метрики, ковариантному уравнению движения Ван Данцинга $dx^J \partial_{[J}\, p_{I]} = 0$, или $L_\nu p_I = 0$ (где p_I - сопряженный 4-вектор импульса и ν - вектор, определяемый условием $p_I \nu^I = 1$) лишь относительно голономных координат. При неголономных системах уравнение Хевисайда-Лоренца является частным случаем уравнения Ван Данцинга лишь для избранного класса таких систем, образуемых взаимно ортогональными единичными векторами.

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