АСТРОФИЗИКА

ТОМ 45 НОЯБРЬ, 2002 ВЫПУСК 4

УДК: 524.3-6

MULTILEVEL LINE TRANSFER WITH THE IMPLICIT INTEGRAL METHOD

L.CRIVELLARI[1,2], O.CARDONA[3*], E.SIMONNEAU[4]

Received 10 July 2002
Accepted 10 September 2002

Once assumed the need of an iterative procedure in order to solve the problem of the formation of spectral lines in the case of a model atom with many energy levels, the sequel is to seek for the most effective form of such an iterative scheme. It is an almost universal practice within all the iterative methods for the solution of those radiative transfer problems, in which the transfer equations are coupled to the state of the matter, to take as the input of each step of iterations the values of the opacity coefficients obtained as a result of the previous one. This is, for instance, the procedure used to correct the temperature in the computation of stellar atmosphere models, or that to build the Λ -operator (either the exact or the approximated one) within the Accelerated Lambda Iteration methods. Yet, if we assume, in order to solve the multilevel line transfer problem, that at each step of iterations the line opacities are known, we can express via the statistical equilibrium equations the populations of the energy levels - and consequently the source functions of the relevant spectral lines - as a linear function of the full set of the corresponding mean intensities of the radiation field. Once obtained such linear forms for the source functions, we are able to solve without the need of any further approximation the radiative transfer equations for all the lines, now linearly coupled through the above linear forms of the statistical equilibrium equations. This is achieved by means of the Implicit Integral Method that we already presented in a series of previous papers.

Key words: *line: formation - radiative transfer*

1. *The statement of the problem.* The problem of spectral line formation in the case of a multilevel model atom implies the solution of the radiative transfer (RT) equations relevant to the specific intensities for all the spectral lines. These lines correspond to the different transitions that can originate between the energy levels of the atom considered.

In the case of a line brought about by the transition between two discrete levels j and k (with $E_k > E_j$), the absorption and emission coefficients, $\chi_{jk}(\nu)$ and $\eta_{jk}(\nu)$ respectively, are given by the well known Füchtbauer-Ladenburg relations

$$\chi_{jk}(\nu) = \frac{h\nu_{jk}}{4\pi}\left(N_j B_{jk} - N_k B_{kj}\right)\varphi_{jk}(\nu) \qquad (1)$$

and

$$\eta_{jk}(\nu) = \frac{h\nu_{jk}}{4\pi} N_k A_{kj}\, \varphi_{jk}(\nu), \qquad (2)$$

where the quantities B_{jk}, B_{kj} and A_{kj} are the Einstein coefficients for the

* *On sabbatical leave in institut d'Astrophysique CNRS from Instituto de Astrofisica, Optica y Electrónica, Puebla, Mexico.*

transition considered. We denote with $\varphi_{jk}(\nu)$ the profile of each spectral line. In this paper we will assume that the $\varphi_{jk}(\nu)$'s are known properties of the material, and are independent of the radiation field; namely, we will consider the case of complete redistribution. On the contrary, the level populations N_j and N_k are the unknowns relevant to the state of the matter, as the specific intensity $I_{jk}(\nu)$ is for the radiation field. Under the conditions typical of a stellar atmosphere, the processes that populate and depopulate the atomic levels are inelastic electron collisions, whose transition probabilities will be denoted by C_{jk} and C_{kj} respectively, and radiative transitions, whose probabilities are given by the Einstein coefficients B_{jk}, B_{kj} and A_{kj} together with the values of the mean intensity of the corresponding spectral lines. Consequently, the level populations N_j and N_k are strongly dependent on the intensity of the radiation field.

We will denote with $J_{jk}(\nu)$ the specific mean intensity of the radiation field in the spectral line originated by the transition between the levels j and k, and with $\bar{J}_{jk}$ the profile integrated mean intensity (i.e. the average of $J_{jk}(\nu)$ over the frequencies, weighted with the profile $\varphi_{jk}(\nu)$. Under steady state conditions, the statistical equilibrium (SE) equations that yield the values of the level populations read (for any level j):

$$\sum_{i<j} C_{ij} N_i + \sum_{k>j} (C_{kj} + A_{kj}) N_k - \left[\sum_{i<j} (C_{ji} + A_{ji}) + \sum_{k>j} C_{jk} \right] N_j =$$
$$= -\sum_{i<j} (N_i B_{ij} - N_j B_{ji}) \bar{J}_{ij} + \sum_{k>j} (N_j B_{jk} - N_k B_{kj}) \bar{J}_{jk} \,. \qquad (3)$$

We have cast eq. (3) in a form slightly different from the customary one: the terms induced by the radiation field (both absorptions and stimulated emission) have been groupped in the right-hand-side, with the aim of showing explicitly that the coefficients of the terms $\bar{J}_{ij}$ and $\bar{J}_{jk}$ are (but for the constant proportionality factor $h\nu/4\pi$) the corresponding absorption coefficients, as given by eq. (1).

Once we have formulated the SE equations (eq. [3]) that yield the populations of the energy levels, together with eqs. (1) and (2) that express the absorption and emission coefficients in terms of the relevant populations, we have to solve simultaneously the SE and the RT equations. In the former the values of the terms $\bar{J}_{ij}$ and $\bar{J}_{jk}$ are determined by the solution of the corresponding RT equations, whose coefficients $\chi_{jk}(\nu)$ and $\eta_{jk}(\nu)$ shall depend on the solution itself of the system of the SE equations. That is to say, we face a problem in which the different "protagonist" variables (i.e., the populations of the different energy levels and the mean intensities of the different spectral lines) are strongly interlocked. The solution of the problem will be achieved by means of an iterative procedure.

Two basic schemes can be envisaged in order to obtain the joint solution

of the two systems of equations, statistical equilibrium and radiative transfer, by mean of an iterative procedure. The first one is based on the linearization around the values obtained from the previous step of iterations of all the non-linear terms that enter into the equations of the two systems. These terms consist, in general, of the products of the absorption coefficient, i.e. the level population, times the specific intensity in the RT equations, and the frequency integrated mean intensity in the SE equations. These values, around which the non-linear terms are linearized, are progressively corrected at each step of the iterative procedure.

But, due to the fact that the RT equations couple all the points of the medium and therefore the SE equations must be solved simultaneously at all these points, the dimensions of the problem result very large. Hence the risk of numerical instabilities is very high. Consequently the method is not robust enough, especially when the number of energy levels and that of depth points required for a proper solution of the RT equations are very large.

The second scheme consists of a sequential procedure. Here each step of the iterative process is splitted into two parts. Within each of them, one of the two system of equations is solved by taking as data the solution, or part of the solution, of the other one. According to the amount of information that comes from the previous solution of one of the two systems (and is used successively as a data in the solution of the second one) the convergence will be more or less rapid. The global coupling of the two solutions is eventually obtained at the end of the iterative procedure, provided that the convergence to the correct solution have been achieved.

We are going to present in the next sec.2 and 3 another iterative procedure that, while retaining certain properties common to that above described, is however not sequential. Both systems of equations, SE and RT, are solved jointly, but the original non-linear coupling between them has been reduced to a linear one, so that the simulateneous solution becomes possible. The linear relations that link the protagonist functions of both the systems of equations are progressively improved by correcting step by step the corresponding coefficients.

At this point, we will briefly review the general properties of the sequential iterative methods nowadays in use, for the sake of a future comparison with our own, that may show some similarities with the former.

The simplest iterative scheme, in which each single step of iterations can be splitted into two parts, is the so-called Λ-iteration. In the first part of each step, the frequency integrated mean intensities, $\bar{J}_{jk}$, are computed via the RT equations by using as data the corresponding absorption and emission coefficients, computed from the solution of the SE equations achieved in the second part of the previous iteration. Then, in the second part of the current step, the new values of the populations are determined, via the SE

equations, by using as data the values of the mean intensities $\bar{J}_{jk}$, just computed. But it is well known that the rate of convergence of this straightforward scheme is exceedingly slow; in practice, it does not converge from the numerical standpoint.

It is possible to get rid of this severe drawback, at least to a certain degree, by employing Accelerated Lambda Iteration (ALI) methods (see, e.g., Rybicki and Hummer, [1], for its application to the problem here under study). The structure of such algorithms is similar to that considered above. Again, the values of the level populations required to compute the absorption and emission coefficients of the RT equations to be solved are those obtained from the previous solution of the SE equations. But it can be realized at the moment of solving the RT equations, that the mean intensities $\bar{J}_{jk}$ can be expressed as the sum of a local contribution, $\bar{J}_{jk}^{loc} = \lambda_{jk} S_{jk}$, directly proportional to the source function S_{jk} through the coefficient λ_{jk} that is the corresponding diagonal element of the Λ-operator, and a non-local residual contribution, $\bar{J}_{jk}^{res} = \bar{J}_{jk} - \lambda_{jk} S_{jk}$.

Then, in the second part of each step of the iterative process, the explicit values of the diagonal elements λ_{jk} and the residual mean intensities $\bar{J}_{jk}^{res}$ are employed as a data in the SE equations, instead of those of the whole mean intensities $\bar{J}_{jk}$, like in the ordinary Λ-iteration. In such a way it becomes possible to eliminate the mean intensities in the SE equations, and consequently to decouple the two systems and solve them sequentially. As from the first to the second part of each step of iterations a lesser amount of information than in the case of the ordinary Λ-iteration is carried on, the rate of convergence resulted here is much higher.

On the contrary, our own iterative method, while as straightforward as the previous ones, differs in that the two systems of equations, SE and RT, are not decoupled; they are solved simultaneously. But the strong non-linear character of the original coupling is made weaker: it is reduced in a natural way to a linear one. As this is achieved by carrying on, from one step of iterations to the next one, an amount of information lesser than that carried on by the above methods, it results in that the rate of convergence is even more high.

2. *The linear form of the SE equations.* Like in all the other iterative methods above considered to solve the RT equations coupled with the radiation dependent state of matter, we will assume that the absorption coefficients of all the spectral lines be known at the beginning of each iteration. That is, at each iterative step we shall take as known the values of the level populations only at the stage of computing the absorption coefficients, while in the run of the iteration itself we will determine consistently - via the joint solution of the SE and RT systems of equations - both the emission coefficients (and by the way. the new values of the level populations)

and the corresponding mean intensities. These new values of the populations will be employed to initialize the next run. In order to bootstrap the procedure, we can use the absorption coefficients computed from the equilibrium values N_j^* of the level populations, or even better the values of the populations resulting from the solution of a series of Two-Levels poroblems, each one corresponding to the transition between the fundamental level and each one of the excited levels independently.

However, the fact that we considered the old values of the absorption coefficients when solving (at each step of iterations) the RT equations, allows us to consider them also for the solution of the system of the SE equations, that is to use the old values of the absorption coefficients in the right-hand side of eq. (3), too. Thanks to this choice, the non-linear terms of eq. (3) become simpler terms proportional to the mean intensities $\bar{J}_{jk}$, without the need of any further simplifying hypothesis.

Within this scheme, from the system of the SE equations (eq. [3]), just by means of the inversion of a matrix that includes only the inelastic electron collision and spontaneous emission terms, it is easy to derive - for each level j - the coefficients Γ_{jk} of the linear form

$$N_j = N_j^c + \sum_{k \neq j} \Gamma_{jk} \bar{J}_{jk} , \tag{4}$$

where N_j^c denotes the coronal population of level j, namely the population that would correspond to the case in which spontaneous emissions do occur, but not absorptions and stimulated emissions. The coefficients Γ_{jk} shall vary from one run of iterations to the next one, as do the "known" populations used to compute the absorption coefficients.

Once we have obtained the linear form given by eq. (4) for the SE equations, because the absorption coefficients for all the transitions $j \leftrightarrow k$ are known, it is straigthforward to derive from the expression for the emission coefficient, as given by eq. (2), the coefficients $S^c(L)$ and $\sigma(L, L')$ of the system of linear equations

$$S(L) = S^c(L) + \sum_{L'} \sigma(L, L') J(L') \tag{5}$$

for the source functions S(L), where the index L denotes each one of the lines (whose number is NLT) that originate from the transitions between the atoms levels (whose number is NEL). Hereinafter $J(L)$ stands for $\bar{J}_{jk}$.

That is, starting from an hypothesis quite standard in the study of stellar atmospheres, we found out the way of expressing the source functions of each one of the NLT spectral lines formed as a linear function of all the integrated mean intensities J of the radiation field in these lines. In such a way, at least within an iterative scheme, we are back to a situation akin to that of the two-level-atom. That suggested us to solve the system of eqs. (5) together with the coresponding RT equations, in which the absorption

coefficients are assumed to be known, too, by employing the Implicit Integral Method [2] (hereafter IIM) shortly described in the next section.

3. *The IIM for the Multi-Level-Atom.* 3.1. *Bases of the method.* It happens in many classical stellar atmosphere problems that one has to solve a large number of individual RT equations in order to account for the evolution of the corresponding specific intensities, one for each direction and frequency. In most of these problems the source function of each individual RT equation includes an integral that couples all the specific intensities, i.e. a scattering-like integral. Consequently all the RT equations are coupled through these integrals, so that all of them must be solved simultaneously, which makes the problem extraordinarily complex. For some of these instances, the scattering-like integral is the same in all the RT equations. This is the case of isotropic coherent scattering in monochromatic problems, or the case of the specific mean intensity integrated over the absorption profile in the problem of two-level spectral line formation under conditions of complete redistribution. With the specific aim of solving this kind of scalar coupling problems (in which the same integral couples all the RT equations), the IIM was developed for systems formed by both plane-parallel layers (Simonneau and Crivellari, [2]) and spherical layers (Gros et al., [3]), systems that are typical of stellar atmospheres.

In this Section we generalize the IIM in order to solve directly (i.e. without iterations) the spectral line formation problem for a multilevel atom model with NEL energy levels, once the corresponding source functions (for all the spectral lines) have been linearized according to the form given by eq. (5) of the previous Section.

Like most of the numerical methods employed for the solution of the RT problem, the IIM requires a discrete optical depth scale. Such a grid of points $\tau(D)$, with $D=1, 2, \ldots, ND$, delimites a series of layers $(D, D+1)$, in everyone of which the relevant physical variables are assumed to vary in a prescribed manner. Here, in particular, we assume a parabolic behaviour for each specific source function, which enables us to warrant their continuity (as well as that of their first derivatives) at the points $\tau(D)$ that separate the layers. It is at these points that we evaluate the numerical values of all the variables of the problem.

At each depth point $\tau(D)$ we will denote by $I_D^{\pm}(\nu, \mu, L)$ the upgoing (+) and downgoing (-) intensities. As customary, ν is the frequency, μ is the cosine of the angle between the direction of the intensity and the radial direction, L specifies each one of the NLT spectral lines.

The natural boundary conditions of the problem are given by the specific downgoing incident intensities $I_1^-(\nu, \mu; L)$, striking onto the upper boundary surface of the system $\tau(D=1)$, and the specific upgoing incident intensities $I_{ND}^+(\nu, \mu; L)$, striking onto the lower boundary surface $\tau(ND)$. Actually, we

use the name of boundary conditions because we solve the problem for the mean intensities or, alternatively, the corresponding source functions. Indeed, the above incident intensities are the initial conditions for the evolution of the relevant specific intensities.

In order to solve the RT problem, we have stratified the stellar atmosphere into a series of layers $(D,\ D+1)$; the IIM shall tackle each of them one by one, as if they were independent systems. Therefore, we shall need the boundary conditions relevant to each of them. By analogy with the case of the global system, we will consider here as initial (boundary) conditions the "implicit knowledge" of the corresponding downgoing incident intensities $I_D^-(\nu, \mu; L)$ and upgoing incident intensities $I_{D+1}^+(\nu, \mu; L)$.

The essence of the elimination procedure at the basis of the IIM shows up in the treatment of the upper boundary conditions. During the progressive study of each individual layer, from the top to the bottom, we transport from the previous ones the form of the upper boundary conditions. The downgoing incident intensities are explicitly known at the surface, where their value is usually zero. For the succesive layer $(D,\ D+1)$ the values of $I_D^-(\nu, \mu; L)$ are not explicitly known, but the process of eliminating the previous slabs enables us to compute the coefficients of certain linear relations that link the downgoing incident intensities with the other intervening variables evaluated at the same point $\tau(D)$, namely the source functions, their first derivatives and the upgoing intensities. These linear relations will constitute the upper boundary conditions for the study (and elimination) of the layer $(D,\ D+1)$.

The lower boundary condition shall be expressed through the formal use of the so far unknown upgoing incident intensities $I_{D+1}^+(\nu, \mu; L)$.

In this way, the IIM works by eliminating layers of the system and replacing them by means of a suitable boundary condition at the upper surface of the first of the remaining layers. This constitutes a typical forward-elimination process.

Yet, the study of the layer $(D,\ D+1)$ also yields the coefficients of another set of linear relations that express the source function $S_D(L)$ of each line L in terms of all the so far unknown source functions $S_{D+1}(L')$ and their derivatives $S'_{D+1}(L')$, as well as the full set (frequencies, directions and lines) of the unknown upgoing incident intensities $I_{D+1}^+(\nu, \mu; L')$. These relations will be enough to compute later the numerical values of the corresponding variables, within a back-substitution process.

Now, the coefficients of the implicit upper boundary condition for each individual layer, that link the downgoing incident intensities $I_D^-(\nu, \mu; L)$ with the intensities $I_D^+(\nu, \mu; L)$, the source functions and their derivatives, can be overridden at each succesive step of elimination. On the contrary, those that link the source functions $S_D(L)$ with all the variables at $\tau(D+1)$, necessary to compute the values of the former in the back-process, must be stored in memory.

3.2. *Dynamics of the Implicit Integral Method.* Let us undergo now the study of the generic layer $(D, D+1)$. We assume that, from the previous study of the foregoing layers, we have determined the coefficients of the linear relations (one for each specific downgoing incident intensity, i.e. one for each triple set of values of the indexes N, M and L that respectively denote the frequency, the direction, and the spectral line)

$$I_D^-(N, M, L) = \mathcal{LR}.\left[\{S_D(L')\}, \{S'_D(L')\}, \{I_D^+(N', M', L')\}\right], \tag{6}$$

which constitute the implicit upper boundary conditions for the study of the layer $(D, D+1)$. Hereinafter the symbol $\mathcal{LR}.$ will denote a linear relation. We employ the notation $\{...\}$ to mean a set of functions, numbered according to their relevant indexes.

The above linear relations are the initial conditions required to study the propagation of the downgoing intensities falling onto the upper boundary surface at $\tau(D)$. For the first layer, the coefficients of eq. (6) must be all equal to zero, in order to fulfill the initial conditions at the surface. The other set of boundary conditions consists of the so far unknown upgoing intensities $I_{D+1}^+(N, M, L)$.

Besides, as already remarked above, from the previous study of the foregoing layer $(D-1, D)$ we have for all the lines L the coefficients of the linear relation

$$S_{D-1}(L) = \mathcal{LR}.\left[\{S_D(L')\}, \{S'_D(L')\}, \{I_D^+(N, M, L')\}\right], \tag{7}$$

Now, by making use of the piecewise parabolic approximation assumed for the behaviour of the source functions, the integral form of the RT equation applied to $I^+(N, M, L, \tau)$ between $\tau(D+1)$ and $\tau(D)$ yields straightforwardly the coefficients of the linear form

$$I_D^+(N, M, L) = \mathcal{LR}.\left[S_D(L), S_{D+1}(L), S'_{D+1}(L), I_{D+1}^+(N, M, L)\right]. \tag{8}$$

The set of eqs. (8) represents the RT equations for the upgoing intensities under the above hypothesis of a parabolic behaviour of $S(L, \tau)$ within the layer $(D, D+1)$. The same hypothesis, i.e. piecewise parabolic behaviour of the source function, enables us to eliminate the set of values of $S'_D(L')$ in eq. (7), and to rewrite the linear form in terms only of the source functions $S_D(L')$ and $S_{D+1}(L')$, and the derivatives $S'_{D+1}(L')$.

Thanks to this elimination, and the substitution of the set of values $\{I_D^+(N, M, L')\}$ by the corresponding implicit values given by eq. (8), we can rewrite the linear expression for $I_D^-(N, M, L)$, given by eq. (6), in the form

$$I_D^-(N, M, L) = \mathcal{LR}.\left[\{S_D(L')\}, \{S_{D+1}(L')\}, \{S'_{D+1}(L')\}, \{I_{D+1}^+(N', M', L')\}\right], \tag{9}$$

whose coefficients are computed straightforwardly. Equations (8) and (9) yield respectively the values of $I_D^+(N, M, L)$ and $I_D^-(N, M, L)$ at depth $\tau(D)$ as linear functions of the same protagonist variables of the problem. Due to

that, by integrating over frequencies and directions (more precisely by summing over the discrete indexes N and M), we can now obtain the coefficients of the linear expression

$$J_D(L) = \mathcal{LR}.\left[\{S_D(L')\}, \{S_{D+1}(L')\}, \{S'_{D+1}(L')\}, \{I^+_{D+1}(N, M, L')\}\right], \qquad (10)$$

that gives the mean intensity $J_D(L)$ at depth $\tau(D)$.

At this point, by making use of eq. (5), that yields each source function $S_D(L)$ in terms of all the mean intensities $J_D(L')$, we are able, via the division by a matrix whose dimension is equal to the number NLT of lines considered, to express the individual source functions in terms of the protagonist variables, in the form

$$S_D(L) = \mathcal{LR}.\left[\{S_{D+1}(L')\}, \{S'_{D+1}(L')\}, \{I^+_{D+1}(N', M', L')\}\right]. \qquad (11)$$

This is the fundamental relation, whose coefficients, straightforwardly computed, must be stored in memory in order to compute the explict values of the source functions $S_D(L)$ in the successive process of back-substitution. The relation given by eq. (11) is akin to that given by eq. (7), that we assumed known from the previous study of the foregoing layer $(D-1, D)$.

In the same way as we derived eq. (8) for the upgoing intensities, we can now write the RT equations for the downgoing intensities in the form

$$I^-_{D+1}(N, M, L) = \mathcal{LR}.\left[S_D(L), S_{D+1}(L), S'_{D+1}(L), I^-_D(N, M, L)\right]. \qquad (12)$$

Again, we have made use of the piecewise parabolic approximation for the source functions $S(L, \tau)$, in order to solve the RT equations for the downgoing intensities $I^-(N, M, L, \tau)$ between $\tau(D)$ and $\tau(D+1)$.

By replacing into eq. (12) the expression given by eq. (9) for the downgoing intensities $I^-_D(N, M, L)$, and that given by eq. (11) for the source functions $S_D(L)$ at $\tau(D)$, we can easily derive the coefficients of the linear expressions

$$I^-_{D+1}(N, M, L) = \mathcal{LR}.\left[\{S_{D+1}(L')\}, \{S'_{D+1}(L')\}, \{I^+_{D+1}(N', M', L')\}\right], \qquad (13)$$

which are akin to those given by eq. (6). Equation (13) shall be used to provide the upper initial conditions necessary to perform the elimination process in the next layer $(D+1, D+2)$. Then the process is iterated till the bottom.

Along the forward-elimination, we have computed and stored at each depth point $\tau(D)$ $(D=1, 2, \ldots, D\text{-}1)$, the coefficients of the linear relations between the source functions $S_D(L)$ given by eq. (11).

When we reach the bottom, the downgoing intensities take the form

$$I^-_{ND}(N, M, L) = \mathcal{LR}.\left[\{S_{ND}(L')\}, \{S'_{ND}(L')\}, \{I^+_{ND}(N', M', L')\}\right]. \qquad (14)$$

At $\tau(ND)$, the upgoing intensities $I^+_{ND}(N', M', L')$ and the asymptotical behaviour of the source function, i.e. $S'(L', ND)$, constitute the known initial conditions for the corresponding RT equations. Then, we are able to

compute easily the coefficients of the linear relation

$$I^{-}_{ND}(N, M, L) = \mathcal{LR}.[\{S_{ND}(L')\}] \tag{15}$$

and, because the values of $I^{+}_{ND}(N, M, L)$ are known, it becomes possible to express $J_{ND}(L)$ in terms of all the source functions at $\tau(ND)$:

$$J_{ND}(L) = \mathcal{LR}.[\{S_{ND}(L')\}]. \tag{16}$$

If we take into account the linear dependence of each source function $S_{ND}(L)$ on the set of mean intensities $J_{ND}(L')$, as given by eq. (5), the explicit values of the former quantities are easily computed.

Once we have the explicit values of the protagonist variables $S_{ND}(L)$, $S'_{ND}(L)$ and $I^{+}_{ND}(N, M, L)$, by means of the fundamental relation given by eq. (11) - whose coefficients were stored in memory during the forward-elimination - it is trivial to compute the explicit values of the individual source functions $S_{ND-1}(L)$ at depth $\tau(ND-1)$. The piecewise parabolic behaviour of $S(L, \tau)$ between $\tau(ND-1)$ and $\tau(ND)$ allows us to compute the explicit values of $S'_{ND-1}(L)$; then the RT equations, in the form given by eq. (8), yield those of the upgoing intensities $I^{+}_{ND-1}(N, M, L)$.

Thus, the back-substitution procedure can be carried on up to the surface. In such a way, all the values of the source functions $S_D(L)$ and those of their derivatives $S'_D(L)$ can be obtained, together with those of the upgoing intensities $I^{+}_{D}(N, M, L)$. In particular, the upgonig intensities computed at the last step of the back-substitution shall constitute the emergent specific intensity.

4. *Results for a paradigm case and conclusions.* We have shortly described in sec.2 how it is possible, for an NEL level atom, to express the source functions of the NLT spectral lines as linear functions of the NLT integrated mean intensities. Succesively, in sec.3, we showed the way of solving the RT equations for these lines, without any approximation other than those intrinsic to the numerical algorithm: approximate quadrature of the integrals for the average of the specific intensities by means of the method of discrete ordinates, and analytical integration of the RT equations, in which the corresponding source functions are approximated by arcs of parabola between pairs of successive depth points. Now we have only to present an example, and to draw a few conclusions about the best way of applying the ideas that we have introduced.

We applied our method to the case of an hydrogen atom with 5 energy levels, which implies 10 spectral lines. Given the Einstein coefficients of the corresponding transitions, we assume that all the collisional cross sections had the value $C_{ij} = 10^{-6} A_{ij}$. That is, each line shall behave like in the case of a Two-Level-Atom with a parameter ε equal to 10^{-6}.

In order to compute in the first run of iterations the values of the coefficients that, according to eq. (5), link the individual source functions with the set of the different mean intensities, we start with the values of

the absorption coefficients, i.e. with the values of the populations that correspond to the case of equilibrium. (We take a temperature of 10000 K). The evolution of the results, iteration by iteration, is presented in Fig.1, which is in our opinion self-explanatory.

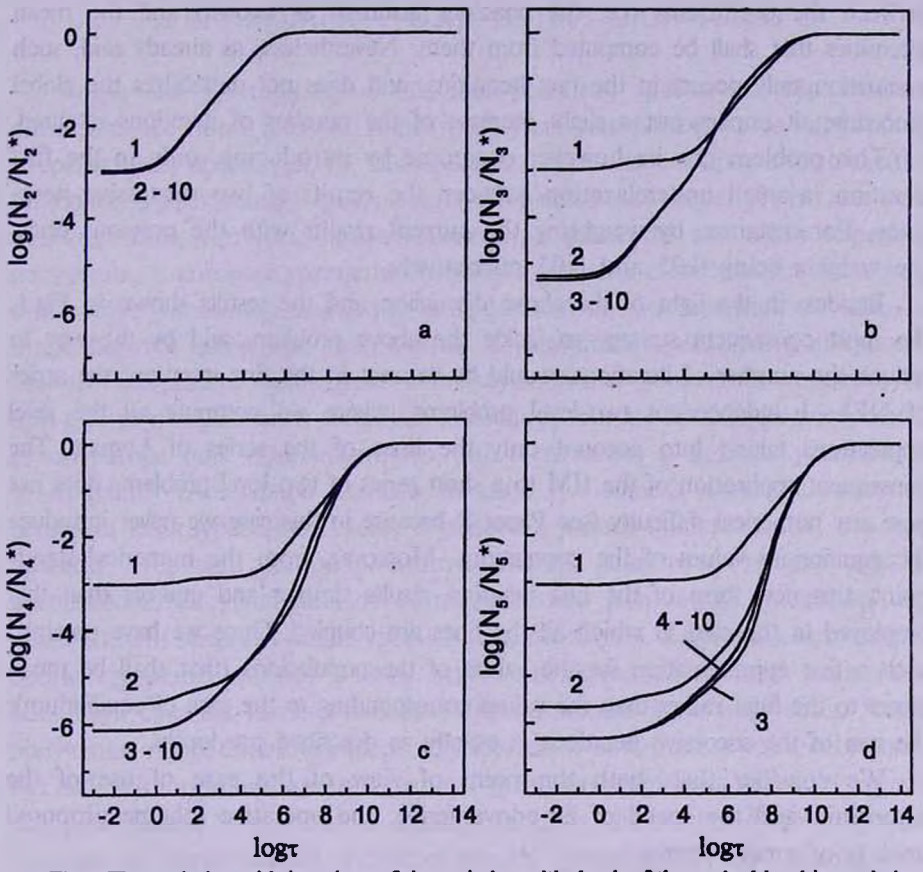

Fig.1. The evolution with iterations of the variation with depth of the excited levels' populations for the case of an hydrogen atom in an isothermal atmosphere at 10000 K. For all the transitions we have set $C_{ij} = 10^{-4} A_{ij}$. The iterations are labelled with the naumbers from 1 through 10. The optical depth τ in abscissa corresponds to the Lyman mean line opacity.

However, we have to remark that in some cases, usually when steep gradients occur in the source functions, a slight problem may arise in the first iteration: unphysical solutions (i.e. very small and sometimes negative values of the populations) may show up, but they disappear in the successive runs of iterations. This fact, possibly due to round-off errors, is however not critical, because it does not destabilize the procedure, but only delays it by a certain number of iterations (typically of the order of the number of levels considered).

Such unphysical solutions are brought about by the fact that, at the beginning, in order to compute the coefficients of the linear form that links the source

functions with the mean intensities, we use the values of the opacities that correspond to the equilibrium populations of the levels. These equilibrium values are very far from those consistent with the correct solution for the mean intensities and the corresponding source functions. Therefore, there is a certain "unbalance" between the coefficients (i.e. the opacities assumed as known) and the mean intensities that shall be computed from them. Nevertheless, as already said, such a situation only occurs in the first iterations, and does not destabilzes the global procedure; it implies just a slight increase of the number of iterations required.

This problem can be however overcome by introducing, only in the first iteration, a small underelaxation between the results of two successive iterations. For instance, by weighting the current results with the previous ones, the weights being 0.95 and 0.05 respectively.

Besides, in the light of the above discussion and the results shown in Fig.1, the most consequent strategy to tackle the above problem, and by the way to reduce the number of iterations, would be to treat in the first iteration, the series of NEL - 1 independent two-level problems, where we compute all the level populations taking into account only the lines of the series of Lyman. The consequent application of the IIM to a short series of two-level problems does not pose any numerical difficulty (see Paper I) because in this case we never introduce the equilibrium values of the populations. Moreover, from the numerical standpoint, this new form of the first iteration results simpler and quicker than that employed in the case in which all the lines are coupled. Once we have obtained such a first approximation for the values of the populations (that shall be much closer to the final values than the values corresponding to the case of equilibrium), the run of the succesive iterations is exactly as described previously.

We consider that, both the point of view of the ease of use of the algorithm, and the speed of its convergence, the operative scheme proposed here is of great interest.

Acknowledgements. Part of this work was supported by Italian MURST (COFIN 2000).

[1] Osservatorio Astronomico di Trieste,
Italy, e-mail: luc@ll.iac.es
[2] Instituto de Astrofisica de Canarias, Spain
[3] Instituto Nacional de Astrofizica, Optica y Electronica, Pueblo, Mexico
[4] Institut D'Astrophysique CNRS, France

ПРИМЕНЕНИЕ МЕТОДА НЕЯВНОГО ИНТЕГРАЛА ПРИ МНОГОУРОВЕННОМ ПЕРЕНОСЕ ИЗЛУЧЕНИЯ В ЛИНИИ

Л.КРИВЕЛАРИ[1,2], О.КАРДОНА[3], Э.СИМОНЮ[4]

Когда мы предполагаем, что для решения задачи об образовании спектральных линий в случае многоуровенных модельных атомов необходима итерационная процедура, то, как следствие, возникает вопрос о нахождении наиболее эффективных форм таких итерационных схем. Почти во всех итерационных методах, применяемых при решении тех задач переноса излучения, в которых уравнения переноса связаны с состоянием вещества, в качестве исходных значений коэффициентов поглощения на каждом этапе берутся значения, полученные в результате предыдущей итерации. Такова, например, процедура, используемая для коррекции температуры при построении моделей звездных атмосфер, или же процедура построения Λ-оператора (как приближенного, так и точного) в методах ускоренной Λ-итерации (Accelarated Lambda Iteration - methods). Однако, если при решении многоуровенных задач переноса в линии предположить, что при каждой итерации непрозрачности линий известны, то мы можем воспользоваться уравнениями статистического равновесия и представить населенности энергетических уровней (следовательно, и функции источников соответствующих линий) в виде линейных функций от всей последовательности средних интенсивностей, определяющих поле излучения. После получения таких линейных форм для функций источников мы в состоянии без дальнейших приближений решить уравнения переноса для всех линий, которые теперь посредством вышеуказанных линейных форм линейно связаны с уравнениями статистического равновесия. Это достигается посредством метода "неявного интеграла" (Implicit Integral Method), который был описан в серии предыдущих работ.

REFERENCES

1. *G.B.Rybicki*, *D.G.Hummer*, Astron. Astrophys., **245**, 171, 1991.
2. *E.Simonneau*, *L.Crivellari*, Astrophys. J., **409**, 830, 1993.
3. *M.Gros*, *L.Crivellari*, *E.Simonneau*, Astrophys. J., **489**, 331, 1997.