АСТРОФИЗИКА

ТОМ 44 НОЯБРЬ, 2001 ВЫПУСК 4

УДК: 524.354.4

ON PULSAR ELECTRODYNAMICS IN ROTATING FRAMES

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Received 13 April 2001

In this paper we have considered a rotating, perfectly conducting sphere and have calculated the electric and magnetic field distributions measured by the rotating observer using the anholonomic approach. The calculations have been done for the following two cases: (1) rotating charged spherical shell and (2) uniformely magnetized sphere. We have shown that in the limiting situation $(\omega a/c)^2 \ll 1$ and $\gamma \approx 1$, the magnetic field distribution is the same for both observers, inertial and noninertial. The expressions obtained for the electric field components in the rotating frame have been compared with the corresponding expressions in the inertial frame, where the observer is at rest. Some of results are in agreement with Post's approach to noninertial electrodynamics.

1. *Introduction*. The formulation of electrodynamics in noninertial frames has been examinated by various authors adopting different approaches. An inspection of literature shows that the most widely used approaches are the approach of anholonomic frames and that which is based on the Kotter-Cartan-Van Dautzing (KCD) formalism of electrodynamics, extensively used by Post and collaborators. However neither of this approaches being free from objections and further theoretical investigations should be pursued for a better understanding of electrodynamics in noninertial frames.

Using the anholonomic approach Corum [1] has investigated the electromagnetic fields produced by rotating charge distributions and solved a number of paradoxes. However the expressions of the electric and magnetic field components computed for a rotating charged sphere with a uniform charge distribution when observed from the noninertial frame cannot be considered as satisfactory because, as we shall see, the field Frenet-Serret (FS) frames used in calculations need to be corrected. Let us also note that in referee [2], the electric and magnetic field distribution produced by a rotating magnetized sphere have been calculated using the "Post" approach.

In this paper using the anholonomic approach we reconsidered the above mentioned problem of Corum and also treat the case of rotating magnetized sphere. The field of FS frames used for calculating the electromagnetic fields components is obtained by explicitly solving the FS equations. Our solutions differ from FS frames used by Corum, because his frames are a simple extension of the tetrad components in cylindrical coordinates to the case of spherical ones. Finally the expressions of the electromagnetic field components obtained for the rotating magnetized sphere are compared to those derived by using the "Post" approach [2,3].

2. *The field of FS frames for relativistic rotation.* To calculate the physical quantities measured in the rotating frame being the tetrad components of the corresponding tensor field, we have to determine the components of the FS tetrad. At each event on the worldline of the rotating observer let us associate the FS tetrad $\mu_{(a)}$ $(a = 1, 2, 3, 4)$ consisting of his 4-velocity $\mu_{(4)}$ and the orthonormal triad $\{\mu_{(\alpha)}\}$ is the reference frame used by the observer at the event in question.

The FS tetrad is determined by equations [4]

$$\begin{aligned} \frac{D\mu^i_{(4)}}{Ds} &= b\mu^i_{(1)} \\ \frac{D\mu^i_{(1)}}{Ds} &= C\mu^i_{(2)} + b\mu^i_{(4)} \\ \frac{D\mu^i_{(2)}}{Ds} &= d\,\mu^i_{(3)} + C\mu^i_{(1)}, \end{aligned} \tag{1}$$

where $\frac{D}{Ds}$ indicates the absolute derivative with respect to observers proper time. The solutions of these equations for spherical coordinates are

$$\begin{aligned} \mu_{(1)} &= \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}, \\ \mu_{(2)} &= \frac{\gamma}{r\sin\theta}\frac{\partial}{\partial\varphi} + \frac{\gamma\omega r\sin\theta}{c}\frac{\partial}{c\partial t}, \\ \mu_{(3)} &= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}, \\ \mu_{(4)} &= \frac{\gamma\omega}{c}\frac{\partial}{\partial\varphi} + \gamma\frac{\partial}{c\partial t}, \end{aligned} \tag{2}$$

where

$$\gamma = \left(1 - \frac{r^2\omega^2\sin^2\theta}{c^2}\right)^{-1/2}$$

and

$$b = -r\frac{\gamma^2\omega^2}{c^2}\sin\theta,\quad C = \frac{\omega\gamma^2}{c},\quad d = 0.$$

Let us note that Corum in his investigations on rotational electrodynamics used the following FS tetrad:

$$\begin{aligned} e_{(1)} &= \frac{\partial}{\partial r}, & e_{(2)} &= \frac{\partial}{\partial\theta}, \\ e_{(3)} &= \gamma\frac{\partial}{\partial\varphi} + \frac{\gamma\omega r^2\sin^2\theta}{c}\frac{\partial}{c\partial t}, & e_{(4)} &= \frac{\gamma\omega}{c}\frac{\partial}{\partial\varphi} + \gamma\frac{\partial}{c\partial t}. \end{aligned} \tag{3}$$

Taking into account the reliability $e_{(2)} \leftrightarrow e_{(3)}$ and comparing (2) and (3) we clearly see that they differ from each other. It would not matter of one could obtain $\mu_{(\alpha)}$ from $e_{(\alpha)}$ by a rotation since both tetrads are orthogonal to the observer's 4-velocity. It is easy to show that the matrix $M_{(\alpha\beta)}$

$$M_{(\alpha\beta)} = \begin{pmatrix} \sin\theta, & 0, & \cos\theta \\ r\cos\theta, & 0, & -r\sin\theta \\ 0, & r\sin\theta, & 0 \end{pmatrix} \tag{4}$$

such that $e_{(\alpha)} = M_{(\alpha\beta)}\mu_{(\beta)}$ is not orthogonal, i.e. $M_{(\alpha\beta)} M^{-1}_{(\alpha\beta)} \neq I$. From this result we infer that these tetrads are different. This is not surprising since Corum obtained the expression for the tetrad component $e_{(3)}$ by replacing the cylindrical coordinate r by $r\sin\theta$ in the solution

$$e_{(3)} = \gamma\frac{\partial}{\partial\varphi} + \frac{\gamma r^2\omega}{c}\frac{\partial}{c\,\partial t}$$

of FS equations for cylindrical coordinates.

For this reason, Corum's computations of the electromagnetic fields produced by a rotating charged sphere are not free of objections. Therefore in the next section we shall reconsider this problem.

3. *Rotating charged spherical shell.* In the inertial frame of the observer, if the charge density on the spherical shell of radius a is taken as uniform, say

$$\rho_0 = \frac{Q}{4\pi a}\delta(r' - a), \tag{5}$$

where Q is the total charge, the components of the 4-current density J^i $(i = 1, 2, 3, 4)$ are specified

$$J^i = \left(0, 0, \frac{\omega\rho_0}{c}, \rho_0\right). \tag{6}$$

The solutions of Maxwell's equations for the 4-potential are

$$A_4 = A_{(4)}(\lambda) = \begin{cases} -Q/a, & r \le a, \\ -Q/r, & r \ge a, \end{cases} \tag{7}$$

$$A_3 = A_{(3)}(\lambda) r\sin\theta = \begin{cases} \dfrac{Q\omega r^2}{3ca}, & r \le a \\ \dfrac{Q\omega a^2}{3cr}\sin^2\theta, & r > a \end{cases} \tag{8}$$

where $A_{(a)}(\lambda) = \lambda^i_{(a)} A_i$ are the physical components and $\lambda_{(a)}$ is the unit vector in the direction of the parametric-line $x^{(a)}$ $\left(x^1 = r, x^2 = \theta,\ x^3 = \varphi, x^4 = ct\right)$. Taking (7), (8), from the definition of the electromagnetic field tensor F_{ij} in terms of A_i

$$F_{ij} = \nabla_i A_j - \nabla_j A_i \tag{9}$$

we obtain the following expressions for the nonvanishing components of F_{ij}:

$$F_{14} = \begin{cases} 0, & r \le a \\ \dfrac{Q}{r^2}, & r \ge a \end{cases} \tag{10}$$

$$F_{13}=\begin{cases}\dfrac{2Q\omega r}{3ca}\sin^2\theta, & r\le a\\ -\dfrac{Q\omega a^2}{3cr^2}\sin^2\theta, & r\ge a\end{cases}\tag{11}$$

$$F_{23}=\begin{cases}\dfrac{2Q\omega r^2}{3ca}\sin\theta\cos\theta, & r\le a\\ \dfrac{2Q\omega a^2}{3cr}\sin\theta\cos\theta, & r\ge a.\end{cases}\tag{12}$$

The components of the vector-potential $A_{(a)}$ and electromagnetic tensor $F_{(ab)}$ in the rotating frame $\{\mu_{(\alpha)}\}$ are related to the inertial components A_i, F_{ij} by the transformation

$$A_{(a)}(\mu)=\mu^i_{(a)}A_i\tag{13}$$

$$F_{(ab)}(\mu)=\mu^i_{(a)}\mu^j_{(b)}F_{ij}.\tag{14}$$

We list the nonzero results:

$$A_{(2)}(\mu)=\begin{cases}-\dfrac{2Q\gamma\omega r}{3ca}\sin\theta, & r\le a\\ -\dfrac{Q\gamma\omega}{c}\left(1-\dfrac{a^2}{3r^2}\right)\sin\theta, & r\ge a\end{cases}\tag{15}$$

$$A_{(4)}(\mu)=\begin{cases}-\dfrac{Q\gamma}{a}\left(1-\dfrac{\omega^2r^2\sin^2\theta}{3c^2}\right), & r\le a\\ -\dfrac{Q\gamma}{r}\left(1-\dfrac{\omega^2a^2\sin^2\theta}{3c^2}\right), & r\ge a,\end{cases}\tag{16}$$

and

$$F_{14}(\mu)=\begin{cases}\dfrac{2Q\gamma\omega^2r}{3ac^2}\sin\theta, & r\le a\\ \dfrac{Q\gamma\sin\theta}{r^2}\left(1+\dfrac{2}{3}\dfrac{a^2\omega^2}{c^2}P_2(\cos\theta)\right), & r\ge a\end{cases}\tag{17}$$

$$F_{(34)}(\mu)=\begin{cases}0, & r\le a\\ \dfrac{Q\gamma}{r^2}\left(1-\dfrac{\omega^2a^2\sin^2\theta}{c^2}\right)\cos\theta, & r\ge a\end{cases}\tag{18}$$

$$F_{(12)}(\mu)=\begin{cases}\dfrac{2Q\gamma\omega}{3ca}, & r\le a\\ \dfrac{Q\gamma\omega}{cr}\left(\sin^2\theta+\dfrac{2}{3}\dfrac{a^2}{r^2}P_2(\cos\theta)\right), & r\ge a\end{cases}\tag{19}$$

$$F_{(23)}(\mu)=\begin{cases}0, & r\le a\\ -\dfrac{Q\gamma\omega}{r}\left(1-\dfrac{a^2}{r^2}\right)\sin\theta\cos\theta, & r\ge a.\end{cases}\tag{20}$$

Comparing our results with those of Corum (with the reliability $e_{(2)}\leftrightarrow e_{(3)}$) we see that the expressions for $A_{(a)}(\mu)$ and $F_{(ab)}(\mu)$ are different [1]. This disagreement is not surprising since, as explained in section 2, our FS tetrad $\mu_{(\alpha)}$ does not coincide with the tetrad $e_{(\alpha)}$, used by Corum. It is important to recall that $\mu_{(\alpha)}$ cannot be obtained from $e_{(\alpha)}$ by a rotation having as axis the 4-velocity of the noninertial observer.

4. *Uniformly magnetized rotating sphere.* The uniformly magnetized, perfect conducting sphere with an angular velocity $\bar{\omega}$ parallel to $\bar{B}$ is considered in the astrophysical literature as the simplest model of pulsars [5,6]. Let us suppose that in the inertial frame of the observer we have a uniformly magnetized, slowly rotating $\left((\omega a/c)^2 << 1\right)$, perfect conducting sphere. The magnetic field lines being parallel to the rotation inside the sphere, while outside we have a dipolar distribution given by

$$\begin{aligned} B_r &= \frac{2m}{r^3}\cos\theta, \quad r \geq a, \\ B_\theta &= \frac{m}{r^3}\sin\theta, \quad r \geq a, \end{aligned} \tag{21}$$

where m is the magnetic moment of the sphere. As have been shown in [6], the unipolar induction of the rotating sphere surrounded by vacuum, will generate a stationary electric field outside, the distribution of which is:

$$\begin{aligned} E_r &= -\frac{B_0\,\omega\,a^5}{2cr^4}\left(3\cos^2\theta - 1\right), \\ E_\theta &= -\frac{B_0\,\omega\,a^5}{cr^4}\sin\theta\cos\theta. \end{aligned} \tag{22}$$

On the surface of the sphere the electric field components have the following expressions:

$$\begin{aligned} E_r &= \frac{B_0\,\omega\,a}{2c}\sin^2\theta, \\ E_\theta &= -\frac{B_0\,\omega\,a}{c}\sin\theta\cos\theta, \end{aligned} \tag{23}$$

where $B_0 = 2m/a^3$ is the value of the magnetic field at the pole. Since the normal component of the electric field is discontinuous for $r=a$ we must have a charge distribution σ on the surface of the sphere given by

$$\sigma = -\frac{2m\omega}{ac}\cos^2\theta. \tag{24}$$

Let us now find the distributions of the electric and magnetic fields as measured by an observer rotating with the sphere. As we have seen in section 3, the noninertial field components are connected with the inertial ones by the transformation law (13,14). Assuming $(\omega a/c)^2 << 1$, we obtain the following nonzero results for magnetic field components:

$$\begin{aligned} F_{(23)}(\mu) &= B_{(1)}(\mu) = \frac{3m\gamma}{r^3}\sin\theta\cos\theta, \\ F_{(12)}(\mu) &= B_{(3)}(\mu) = \frac{2m\gamma}{r^3}P_2(\cos\theta), \end{aligned} \qquad r > a \tag{25}$$

and

$$F_{(23)}(\mu) = B_{(1)}(\mu) = \frac{3m}{a^3}\sin\theta\cos\theta, \qquad r=a \tag{26}$$
$$F_{(12)}(\mu) = B_{(3)}(\mu) = \frac{2m}{a^3}P_2(\cos\theta),$$

and for the electric field components:

$$F_{(14)} = E_{(1)}(\mu) = \frac{\gamma m\omega}{cr^2}\left[3\cos^2\theta - 1 - \frac{a^2}{r^2}\left(4\cos^2\theta - 1\right)\right]\sin\theta, \qquad r>a \tag{27}$$
$$F_{(34)} = E_{(3)}(\mu) = -\frac{\gamma m\omega}{cr^2}\left[3\sin^2\theta - \frac{2a^2}{r^2}\left(\sin^2\theta - \frac{3\cos^2\theta - 1}{2}\right)\right]\cos\theta.$$

Finally, the electric field components on the surface of the sphere $E_{(1)}(\mu)$ and $E_{(3)}(\mu)$ are equal to zero. It is worth to mention that this last result has been used in [6] to obtain the expressions of the electric fields (23) on the surface of the sphere as measured by an inertial observer.

The above-derived expressions of the electric field components for the noninertial observer are valid only under laboratory conditions, when no charge can escape from the metallic sphere. Applying these solutions to the case of pulsar, Goldreich and Julian in [6] concluded that "rotating magnetic neutron star can not be surrounded by a vacuum", since the electric force along the direction of the magnetic field exceeds the gravitational force in the same direction. As a consequence of the escaped charges from the star surface, the closed magnetic field lines may be regarded as equipotentials. Taking into account that the interior of a neutron star is a perfect conductor they obtained for the electric field components in the inertial frame the following expressions [5]:

$$E_r = \frac{\omega B_0 a}{2c}\left(\frac{a}{r}\right)^2\sin^2\theta, \qquad r \geq a \tag{28}$$
$$E_\theta = -\frac{\omega B_0 a}{c}\left(\frac{r}{a}\right)^2\sin\theta\cos\theta.$$

We use these formulas to calculate the expression for the electric field components for the noninertial observer. The result is simple: all components of the electric field for the rotating observer are equal to zero. This result is not surprising because the escape of charged particles transforms the region occupied by the closed magnetic field lines into a perfect conductor, consequently for the noninertial observer there is no reason for the generation of an electric field.

5. *Conclusion*. This paper is a continuation of our investigation of electrodynamical problems in accelerated frame [2]. We have considered a rotating, perfectly conducting sphere and have obtained the electric and magnetic field distribution measured by the rotating observer, using the anholonomic approach. Particularly we calculated the magnetic and electric field components for the following cases: (1) rotating charged spherical shell and (2) uniformly

magnetized rotating sphere. In the first case, our results differ from those of Corum because his tetrad field is not a solution of the FS equations (1) [4].

The second case is interesting since it permits to compare the field distributions in rotating frames obtained by two different approaches of noninertial electrodynamics. Taking into account that $B_{(1)}$ and $B_{(3)}$ are respectively the components in the direction of the acceleration and the axis of rotation, comparison (25), (26) with (21) shows that in the limiting situation $(\omega a/c)^2 << 1$ and $\gamma \approx 1$ the magnetic field distribution is the same for both observers, inertial and noninertial. This result is in agreement with Post's remark that in the Peyram and Kennard experiments "the observations are independent of whether the solenoid generating coaxial B field was stationary or rotating at the same angular velocity as the cylindrical condenser".

When the rotating sphere is surrounded by vacuum the electric distributions for the inertial and rotating observer are completely different. In the inertial frame the radial dependence of the fields components is $1/r^4$ whereas in the rotating frame the decrease with distance is slower, i.e. $1/r^2$. Let us also mention that the angular dependence is also different. When the escape of charged particles is permitted, the electric field components are all equal to zero. This result is in agreement with the Post's remark that no unipolar induction effect exists for the noninertial observer rotating with the sphere.

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К ЭЛЕКТРОДИНАМИКЕ ПУЛЬСАРОВ В ВРАЩАЮЩИХСЯ СИСТЕМАХ

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В этой статье мы рассмотрели вращающуюся, проводящую сферу и вычислили распределения электрических и магнитных полей, измеряемые вращающимся наблюдателем, в неголономном приближении. Вычисления проводились в следующих двух случаях: (1) вращающейся заряженной сферы, (2) равномерно намагниченной сферы. Мы показали, что в предельном случае $(\omega a/c)^2 << 1$ и $\gamma \approx 1$, распределения магнитного поля одинаковы для инерциального и неинерциального наблюдателей. Выражения для компонент электрического поля во вращающейся системе сравнены с соответствующими выражениями в инерциальной системе, где наблюдатель находится в состоянии покоя. Некоторые результаты находятся в согласии с приближением Поста неинерциальной электродинамики.

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