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LIGHT RAYS IN GRAVITATING AND REFRACTIVE MEDIA: A COMPARISON OF THE FIELD TO PARTICLE AND HAMILTONIAN APPROACHES

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The field to particle method of H.P.Robertson as applied by Noonan, in order to obtain the general relativistic equations describing the trajectory of a photon in a refractive medium, is compared with Synge's general relativistic Hamiltonian theory of waves and rays. For a photon in vacuum it is known that both approaches yield the same equation for the trajectory i.e. a null geodesic. However for a photon in a medium, in contradistinction to the Hamiltonian theory, the field to particle method (a) yields equations of the photon trajectory valid only in a non-dispersive medium, (b) the time component u^0 of the tangent to the ray remains an undetermined quantity, (c) agreement with the Hamiltonian theory is achieved by substituting into Noonan's equations the Hamiltonian expression for u^0.

1. *Introduction*. The properties of light rays in the presence of both gravitation and refraction may be investigated by applying Synge's Hamiltonian theory of rays and waves [1]. This general relativistic theory of geometrical optics has received wide physical applications. It is at the base, for instance, of the treatment of the transfer of radiation in a dispersive medium in a curved space-time [2,3]. In this Hamiltonian approach the behavior of light rays is governed by the medium equation

$$\Omega(x,p)=\frac{1}{2}\left[g^{ij}P_iP_j-\left(n^2-1\right)\left(P_iV^i\right)^2\right] \tag{1.1}$$

and the ray equations

$$\frac{\partial x^i}{d\tau}=\frac{\partial\Omega}{\partial P_i}\ \text{(a)},\qquad \frac{\partial P_i}{d\tau}=-\frac{\partial\Omega}{\partial x^i}\ \text{(b)} \tag{1.2}$$

where τ is a parameter, n the refractive index (the reciprocal of the wave speed $\overline{u}$) is a given function of the coordinates and frequency, V^i is the 4 velocity of the medium and P_i the frequency 4 vector which Synge identifies with the 4 momentum of the photon associated with a system of waves. Further, to comply with concepts of causality it is required that the rays should be timelike (or null for a photon in vacuum), mathematically this requirement reads $g_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}\le 0$, with a signature +2 of the metric. However, there exists in the literature a quite different method for obtaining the trajectory of any particle. This method known as "the field to particle method" [4] has been applied by Noonan to obtain the general relativistic equation of motion for a photon in a refractive medium [5]. The gist of this method is to derive the equation of motion from the conservation law for the stress-energy tensor by going to the limit as this tensor can be localized along a single world line. Noonan found that when the electromagnetic stress-energy tensor S^{ij} in a medium is localized into a world line $x^i(\tau)$, the equation describing this world line, when (*i*) the medium is linear (*ii*) the wavelength is short compared with variations in the permittivity and permeability, are

$$\frac{ds^i}{d\tau}+\Gamma^i_{jk}\,u^j s^k+\varphi\zeta(Ln\,n)^i+\zeta\left(1-\frac{1}{n^2}\right)s_j V_l^{jl}=0, \tag{1.3}$$

$$u^i=\frac{\zeta}{\varphi n^2}\left[s^i+\varphi\left(n^2-1\right)V^i\right], \tag{1.4}$$

where $u^i=\dfrac{dx^i}{d\tau}$ along the trajectory $x^i(\tau)$, $s^i=\int\sqrt{|g|}\,S^{i0}\,dV$, $\zeta=u_iV^i$, $\varphi=s_iV^i$, n and V^i having the same meaning as in eq (1.1). For a photon in vacuum the field to particle method and the Hamiltonian theory are equivalent in the sense that they both yield the same equation for the trajectory, i.e. a null geodesic. The fact that the trajectory is a geodesic and the null nature of u^i, in Robertson's method, stems from the symmetry requirement of the electromagnetic stress-energy tensor S^{ij} and the property $S^i_i=0$. However for a photon in a medium only $S^i_i=0$ is true and the equivalence of the two theories remains an open question.

The main purpose of this note is to present such a discussion. We shall assume that we are dealing with an isotropic refractive medium in a static spherically symmetric universe with world lines along the x^0 lines.

In sec.2 we briefly recall the field to particle method as applied by Noonan to the general relativistic Maxwell equations in a medium. We note that his description of the photon in a medium is correct only for a non dispersive medium. In sec.3 we compare the equations of the photon trajectory obtained by application of the field to particle method to those resulting from Synge's Hamiltonian theory. It is shown that agreement between the two sets of equations is achieved only if in Noonan's eqs (1.3, 1.4) we substitute for u^0 the expression given by the Namiltonian theory.

2. *The field to particle method.* Assuming that the medium is linear, i.e. it possesses a scalar electric permittivity ε and a scalar magnetic permeability μ, the skew symmetric tensors F^{ij} and H^{ij} entering in the definition of Maxwell's equations

$$H^{ij}_{;j}=J^i \tag{2.1a}$$

$$F_{ij,k}+F_{jk,i}+F_{ki,j}=0 \tag{2.1b}$$

are related by

$$H^{ij} = C^{ijkl} F_{kl} \tag{2.2}$$

where

$$C^{ijkl} = \varepsilon_0 \left[\frac{\mu_0}{\mu} g^{ik} g^{jl} + \left(\frac{\varepsilon}{\varepsilon_0} - \frac{\mu_0}{\mu} \right) \left(g^{ik} V^j V^l + g^{jl} V^i V^k \right) \right]. \tag{2.3}$$

The semi-colon in eq 2.1a denoting the covariant derivation of H^{ij} with respect to x^j. Interpreting the force density

$$f^i = J^k F^i_k \tag{2.4}$$

as the force on the free charges in the medium, Noonan using eq 2.2 writes eq 2.4 in the form

$$\frac{\partial}{\partial x^j} \left(\sqrt{g}\, S^{ij} \right) = -\sqrt{g} \left(R^i + \Gamma^i_{jk} S^{jk} \right) \tag{2.5}$$

where

$$S^{ij} = F^{ik} H^j_k + \frac{1}{4} g^{ij} F^{kl} H_{kl} \tag{2.6}$$

$$R^i = -W^i - f^i \tag{2.7}$$

with

$$W_i = \frac{1}{4} F_{kl} F_{mn} \left(C^{klmn} \right); i. \tag{2.8}$$

Thus separating in eq (2.5) the contribution of the field derivatives from that of the gradients of permittivity and permeability. When the fields are localized into a particle, the Robertson's method applied to eq (2.5) yields the following result

$$\frac{d\sigma^{i0}}{d\tau} = -u^0 \left(\rho^i + \Gamma^i_{jk} \sigma^{jk} \right) \tag{2.9}$$

$$s^{ij} = u^0 \sigma^{ij} = u^j \sigma^{i0} \tag{2.10}$$

where σ^{ij} and ρ^i denote respectively the volume integrals

$$\sigma^{ij} = \int \sqrt{g}\, S^{ij} dV, \quad \rho^i = \int \sqrt{g}\, R^i dV \tag{2.11}$$

with the aid of eq 2.10, the elimination of σ^{ij} in eq (2.9) gives

$$\frac{ds^i}{d\tau} + \Gamma^i_{jk} u^j s^k = r^i \tag{2.12}$$

where $r^i = u^0 \rho^i$ and $s^i = \sigma^{i0}$. Since a photon has zero charge r^i in eq 2.12 is simply equal to $-w^i$ where

$$w_i = -\varphi\zeta(Ln\, n);\ i - \zeta\left(1 - n^{-2}\right) s_j V^j_{;i} \tag{2.13}$$

is the corpuscular limit of W_i as given by eq 2.8 when there is equipartition of electric and magnetic energy. An important relation characterzing the 4 vectors

u^l and s^l follows from the property $S^l_l = 0$ used in the from $\sigma^l_l = 0$, eq 2.10 gives

$$u^l s_l = 0. \tag{2.14}$$

Eq (2.10) can also be used to give

$$u^l = \frac{u^0}{s^0}\sigma^{0l}. \tag{2.15}$$

As to eq (1.4), it simply states that in the local rest frame (L.R.F) of the medium u^α is parallel to s^α and is defined by

$$u^\alpha = \frac{u^0}{n^2 s^0} s^\alpha \tag{2.16}$$

This relation results from eqs (2.6) and (2.15). Evaluation of eq (2.6) in the LRF gives $S^{0\alpha} = \varepsilon_0\mu_0 S^\alpha$, $S^{\alpha 0} = \varepsilon\mu S^\alpha$, where $\vec{S}$ is the Poynting vector. Therefore in the corpuscular limit $s^\alpha = n^2\sigma^{0\alpha}$, $n^2 = \frac{\varepsilon\mu}{\varepsilon_0\mu_0}$, substitution of this result into eq (2.15) gives eq (2.16). We shall see in the next section that the orthogonality condition (2.14) is restricted to non dispersive media and follows directly from the ray and medium equations (1.1, 1.2a). The components of s^l in the LRF are given by

$$s^0 = \frac{\xi}{c^2}, \quad s^\alpha \doteq \varepsilon\mu\chi N^\alpha \tag{2.17}$$

where $\xi = \frac{1}{2}\int\left(\varepsilon E^2 + \mu H^2\right)dV$ is the corpuscular limit of the electromangnetic energy density and χN^α represents the corpuscular volume integral of the Poynting vector with $\chi = \int\left|\vec{E}\vec{H}\right|dV$ and N^α the unit 3 vector in the direction of propagation. Moreover Noonan shows that the energy flow χN^α and the 3 velocity $v^\alpha = \frac{u^\alpha}{u^0}$ of the light corpuscule satisfy the relations

$$\begin{aligned} v^\alpha &= \frac{\chi}{\xi} N^\alpha \qquad \text{(a)} \\ v^\alpha &= \frac{\xi}{\varepsilon\mu\chi} N^\alpha \qquad \text{(b)} \end{aligned} \tag{2.18}$$

the product of which yields

$$v^2 = \frac{1}{\varepsilon\mu} = \frac{c^2}{n^2}. \tag{2.19}$$

Consequently within the framework of Noonan's theory the transport of energy, in the LRF, occurs with a 3 velocity v equal to the phase velocity. Noonan's theory is therefore restricted to non dispersive media, a fact already mentioned in connection with the condition $u^l s_l = 0$. To connect Noonan's work with the Hamiltonian theory it is useful to write, with the aid of eq (2.18), the components of s^i in the following from:

$$s^0 = \frac{\xi}{c^2} \quad s^\alpha = \frac{n}{c^2}\xi N^\alpha \tag{2.20}$$

with the special relation (2.21) holding at each event within the medium

$$s^i s_i = \frac{\xi^2}{c^4}\left(1 - n^2\right) \tag{2.21}$$

3. *Comparison of the fields to particle and Hamiltonjan approaches.* The equations of the trajectory for a photon in a medium supplied by Noonan's theory are valid only when the medium is non dispersive. To compare with Synge's theory we shall therefore consider a non dispersive isotropic medium in a static universe with a spherically symmetric metric

$$ds^2 = g_{00}\left(dx^0\right)^2 + g_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{3.1}$$

and with world lines along the x^0 lines so that its 4 velocity V^i satisfies

$$V^a = 0, \qquad V^0 = \sqrt{|g_{00}|} \tag{3.2}$$

where the coefficients g_{ij} may be written in the form

$$g_{00} = \varepsilon c^\Phi \quad g_{11} = \varepsilon e^\Lambda \quad g_{22} = \varepsilon r^2 \quad g_{33} = \varepsilon r^2 \sin^2\theta \tag{3.3}$$

$$\varepsilon = \begin{cases} -1 \text{ for a signature}(-+++) \\ +1 \text{ for a signature}(+---) \end{cases}$$

with Φ and Λ functions of r only.

The ray equations now read

$$u^i \equiv \frac{dx^i}{d\tau} = g^{ij}P_j - \left(n^2-1\right)V^i\left(P_j V^j\right) \tag{3.4a}$$

$$\frac{dP_i}{d\tau} = -\frac{1}{2}g^{mn}{}_{,i}\, P_m P_n + \frac{1}{2}\partial_i\left[\left(n^2-1\right)\left(P_j V^j\right)^2\right] \tag{3.4b}$$

Since $\Omega(x,P)$ is independent of x^0, P_0 is constant along each ray. From eqs (3.4a,b) we get

$$\begin{aligned} \frac{dP^r}{d\tau} &= -\Gamma^r_{jk}\, P^j P^k + A^r + P_i B^k g^{ir}{}_{,k} \\ A_i &= \frac{1}{2}\partial_i\left[\left(n^2-1\right)\left(P_j V^j\right)^2\right] \\ B^k &= -\left(n^2-1\right)\left(P_j V^j\right)V^k \end{aligned} \tag{3.5}$$

The energy of the photon measured in the instantaneous rest frame of the medium is according to Synge's definition given by the formula

$$E = \varepsilon\left(P_j V^j\right). \tag{3.6}$$

Now, multiplication of eq (3.4a) by P_i gives

$$P_i u^i = 0, \tag{3.7}$$

which implies the spacelike nature of the frequency 4 vector, and from the medium equation (1.1) we obtain

$$P_i P^i = \left(1 - n^2\right) E^2 \qquad n = \bar{u} < 1. \tag{3.8}$$

We thus recover the orthogonality condition (2.14) and the relation (2.21) provided we identify the frequency 4 vector P_i with s^i. This identification will be implicitly understood in the remaining part of this work.

To proceed further in the comparison let us write explicitly the space and time components of eqs (3.3a) and (3.4). We have, limiting to just one space component for P_i e.g, P^1.

$$\begin{aligned} & u^0 = \frac{dx^0}{d\tau} = n^2 P^0 \; (a) \quad \frac{dP^0}{d\tau} = -\frac{d\Phi}{dr} P^0 P^1 \; (c) \\ & u^\alpha = \frac{dx^\alpha}{d\tau} = P^\alpha \quad (b) \quad \frac{dP^1}{d\tau} + \Gamma^1_{\alpha\beta} P^\alpha P^\beta = e^{(\Phi - \Lambda)} \left(P^0\right)^2 \left[n \frac{dn}{dr} - \frac{n^2}{2} \frac{d\Phi}{dr} \right] (d). \end{aligned} \tag{3.9}$$

In the static universe with metric form (3.1) the equations (1.3) and (1.4) describing the trajectory of a photon in a medium, according to Noonan's theory, now read explicitly:

$$\begin{aligned} & u^\alpha \equiv \frac{dx^\alpha}{d\tau} = \frac{u^0}{n^2} \frac{s^\alpha}{s^0} \quad (a) \quad \frac{ds^0}{d\tau} = -\frac{d\Phi}{dr} u^1 s^0 = -\frac{d\Phi}{dr} \frac{u^0 s^1}{n^2} \quad (b) \\ & \frac{ds^1}{d\tau} + \Gamma^1_{\alpha\beta} \frac{u^0}{n^2 s^0} s^\alpha s^\beta = e^{(\Phi - \Lambda)} u^0 s^0 \left[\frac{1}{n} \frac{dn}{dr} - \frac{1}{2n^2} \frac{d\Phi}{dr} \right] \quad (c). \end{aligned} \tag{3.10}$$

By eq (1.4), the tangent to the trajectory lies in the 2-element defined by the 4 vector s^i and the 4 velocity V^i of the medium, as one may expect for isotropy and in agreement with the Hamiltonian theory. However, in contradistinction to the ray equation (1.2a), eq (1.4) does not determine the time component u^0 since for $i=0$ it simply reduces to an identity. On account of this undeterminary of u^0 it is not quite correct to say that eqs (1.3) and (1.4) contain a complete description of the trajectory. Comparison of the two sets of eqs (3.9) and (3.10) shows that we may force eqs (3.10) to reduce to eqs (3.9) by adopting the Hamiltonian relation $u^0 = n^2 s^0$. As a concluding remark it may be interesting to recall how we recover the results for the photon in vacuum. In this case the symmetry property of the electromagnetic stress-energy tensor S^{ij} and eq (2.10) require that s^i be a multiple of u^i

$$s^i = K u^i. \tag{3.11}$$

The substitution of eq (3.11) into eq (2.10) and use of the property $S_i^i = 0$ give $Ku^i u_i = 0$, since $K=0$ presents no physical interest, u^i is a null vector. Inserting eq (3.11) into eq (2.12), with $r'=0$, we obtain the geodesic equation.

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СВЕТОВЫЕ ЛУЧИ В ГРАВИТАЦИОННОЙ И ПРЕЛОМЛЯЮЩЕЙ СРЕДАХ: СРАВНЕНИЕ ТЕОРИИ ВОЛНА-ЧАСТИЦА С ГАМИЛЬТОНОВЫМ ПОДХОДОМ

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Метод волна-частица Робертсона, примененный Нунаном при получении общих релятивистских уравнений для траектории фотона в преломляющей среде, сравнивается с общей релятивистской гамильтоновской теорией Синга для волн и световых лучей. Известно, что для фотона в вакууме оба подхода приводят к одному и тому же уравнению для траектории, дающему нулевую геодезическую линию. Однако для фотона, движущегося в среде, метод волна-частица, в противоположность гамильтоновской теории, (а) приводит к таким уравнениям для траектории фотона, которые справедливы лишь для непреломляющей среды, (б) оставляет неопределенной временную компоненту u^0 касательной к лучу, (в) согласие с гамильтоновской теорией достигается при подстановке в уравнения Нунана гамильтоновского выражения для u^0.

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