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# CONSERVATION LAWS FOR MULTILEVEL TRANSFER PROBLEMS

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We discuss the question whether the way of finding the conservation laws based on the variational formalism is applicable to the multilevel problems of the radiative transfer in a homogeneous atmosphere. For expository reasons, the simplest one-dimensional model case is considered. For the special three-level problem treated in the paper the Lagrangian approach allows to derive not only the H- and K-integrals, but also the nonlinear integral which is an analog of the Q-integrals previously obtained for the classical transfer problems. It is shown that, in general, the constraints imposed by the variational principle on the symmetry properties of the transfer equations are too stringent to be satisfied.

1. *Introduction.* It was first Rybicki's paper [1], which called attention to the quadratic and bilinear integrals of the radiative transfer equation. The concept of "bilinear integrals" was introduced for quadratic integrals that connect the radiation fields of two separate transfer problems referred to the same optical depth. Somewhat more general results for monochromatic, isotropic scattering in the plane-parallel atmosphere was given later by Ivanov [2]. Quite recently the present authors [3,5] applied the Lagrangian formalism [6] to several transfer problems of astrophysical interest, and used the Noether theorem to derive the proper conservation laws. It was shown that these laws correspond to the quadratic integrals obtained by Rybicki in [1], and are the direct consequence of the form-invariance of the Lagrangian with respect to the translation transformation $\tau \to \tau + \Delta\tau$ of the optical depth.

In paper [4] it was suggested a physical interpretation of the resulting Q-relations (we use here the terminology adopted in [1]) that allows one to derive these immediately. The underlying physical reasonings are, as a matter of fact, a further generalization of those in deriving the classical principle of invariance (see e.g., [7]). It was shown that the invariance property of the radiation field is due to the homogeneity assumption made for an atmosphere. This fact sets one thinking that this property may appear to be characteristic of the multilevel problems as well.

The present paper concerns the one-dimensional transfer problems for three-level atoms. This relatively simple configuration of the energy levels is a constituent of any multilevel system, and hence the conclusions we arrive at in the paper are of importance for any multilevel transfer problem. It will be demonstarated that in some special cases the Lagrangian approach is applicable, and leads to the conservation laws similar to those previously obtained in [3-5]. In the more general situations the symmetry constraints imposed on the differential operators, in the classical context of the calculus of variations, are too stringent to be met so that a variational formulation is not possible.

2. *A special problem for three-level atoms.* We start by considering the simplest situation when no radiative transitions occur between the two upper levels (assigned as levels 2 and 3). Instead, they are assumed to be coupled by means of collisional processes, which is the case of multiplets such as the lines $D_1$ and $D_2$ of NaI, and H and K lines of CaII. We shall consider the model problem, in which the effects of the induced emission are neglected. These effects will be treated later on.

The equations of the statistical equilibrium in this case have the form

$$\begin{aligned} n_3(A_{31}+a_{31}+a_{32}) &= n_1(B_{13}J_{13}+b_{13})+n_2b_{23}, \\ n_2(A_{21}+a_{21}+b_{23}) &= n_1(B_{12}J_{12}+b_{12})+n_3a_{32}, \end{aligned} \tag{2.1}$$

where $n_i$ ($i=1,2,3$) is the population of atoms in the $i$th level; $A_{ij}$ and $B_{ij}$ are Einstein's transition coefficients for the spontaneous and radiative absorption processes, respectively; $a_{ij}$ and $b_{ij}$ are the rates of the collisional transitions. In the one-dimensional approximation the quantities $J_{1k}$ ($k=2,3$) are expressed in terms of the intensities $I_{1k}^{\pm}$, in each direction, as follows $J_{1k}=\left(I_{1k}^{+}+I_{1k}^{-}\right)/2$.

The radiation transfer equations for the problem under consideration read

$$\pm\frac{dI_{12}^{\pm}}{q\,d\tau}=-I_{12}^{\pm}+\frac{A_{21}}{B_{12}}\frac{n_2}{n_1};\qquad \pm\frac{dI_{13}^{\pm}}{d\tau}=-I_{13}^{\pm}+\frac{A_{31}}{B_{13}}\frac{n_3}{n_1}, \tag{2.2}$$

where the optical depth $\tau$ is introdused in such a way that

$$d\tau=\frac{h\nu_{13}}{\Delta\nu_{13}}\frac{B_{13}}{c}n_1ds; \tag{2.3}$$

$q=(\nu_{12}\Delta\nu_{13}B_{12}/\nu_{13}\Delta\nu_{12}B_{13})$, $\nu_{1k}$ and $\Delta\nu_{1k}$ are the frequency and the width of the lines resulting from transitions between the level 1 and $k$ ($k=2,3$).

As it follows from Eq.(2.2), $J_{12}'=-qH_{12}$, $J_{13}'=-H_{13}$ (hereafter the primed quantities denote derivatives with respect to $\tau$), where $H_{1k}=\left(I_{1k}^{+}-I_{1k}^{-}\right)/2$ characterizes the radiative flux in each line. Incorporating Eqs.(2.1) and (2.2), we arrive at the following system of differential equations of the second order:

$$\begin{aligned} f\left(J_{12},J_{12}',J_{12}''\right) &\equiv \frac{d^2J_{12}}{d\tau^2}-LJ_{12}+NJ_{13}+J_{12}^{0}=0, \\ g\left(J_{13},J_{13}',J_{13}''\right) &\equiv \frac{d^2J_{13}}{d\tau^2}-KJ_{13}+MJ_{12}+J_{13}^{0}=0, \end{aligned} \tag{2.4}$$

where we introduced the notations

$$
\begin{aligned}
&K = (1/\Delta)\left[(A_{21}+a_{21})(a_{31}+a_{32}) + a_{31}\,b_{23}\right], \quad M = A_{31}B_{12}b_{23}/B_{13}\,\Delta, \\
&L = \left(q^2/\Delta\right)\left[(A_{31}+a_{31})(a_{21}+b_{23}) + a_{21}\,a_{32}\right], \quad N = q^2 A_{21}B_{13}\,a_{32}/B_{12}\,\Delta, \\
&J_{12}^{\circ} = \frac{q^2 A_{21}}{B_{12}\,\Delta}\left[b_{12}(A_{31}+a_{31}+a_{32}) + b_{13}a_{32}\right], \\
&J_{13}^{\circ} = \frac{A_{31}}{B_{13}\,\Delta}\left[b_{13}(A_{21}+a_{21}+b_{23}) + b_{12}\,b_{23}\right], \\
&\Delta = (A_{21}+a_{21})(A_{31}+a_{31}+a_{32}) + b_{23}(A_{31}+a_{31}).
\end{aligned}
\tag{2.5}
$$

It is well known that the system of differential equations of the second order of the type Eq.(2.4) may be regarded as the Euler-Lagrange equations of a variational principle if the following symmetry properties are satisfied (see e.g. [8,9])

$$
\frac{\partial g}{\partial J_{12}'} = -\frac{\partial f}{\partial J_{13}'}, \qquad \frac{\partial g}{\partial J_{12}} = \frac{\partial f}{\partial J_{13}} - \frac{d}{d\tau}\frac{\partial f}{\partial J_{13}'}. \tag{2.6}
$$

One can easily verify that the second condition fails by giving $M=N$. Nevertheless, the symmetry condition will be met if we multiply preliminarily the first of Eqs.(2.4) by $M$, and the second one by $N$. Then the Lagrangian corresponding to this new system of equations may be derived immediately

$$
\begin{aligned}
\mathcal{L} = &\int_0^1 J_{12}\left(\lambda\, MJ_{12}'' - \lambda\, LMJ_{12} + \lambda\, MN\,J_{13} + MJ_{12}^{\circ}\right) d\lambda \, + \\
+ &\int_0^1 J_{13}\left(\lambda\, NJ_{13}'' - \lambda\, KNJ_{13} + \lambda\, MNJ_{12} + NJ_{13}^{\circ}\right) d\lambda,
\end{aligned}
\tag{2.7}
$$

or

$$
\mathcal{L} = MJ_{12}'^{2} + NJ_{13}'^{2} + LMJ_{12}^{2} + KNJ_{13}^{2} - 2\,MNJ_{12}J_{13} - 2\,MJ_{12}^{\circ}J_{12} - 2\,NJ_{13}^{\circ}J_{13}. \tag{2.8}
$$

Since the optical depth does not appear explicitly in the Lagrangian (2.8) Noether's theorem leads to the conservation law of the form

$$
MJ_{12}'^{2} + NJ_{13}'^{2} - LMJ_{12}^{2} - KNJ_{13}^{2} + 2\,MN\,J_{12}J_{13} + 2\,MJ_{12}^{\circ}\,J_{12} + 2\,NJ_{13}^{\circ}J_{13} = \text{const.} \tag{2.9}
$$

The conservation law (2.9), resulting from translational invariance of $\mathcal{L}$ with respect to the variable $\tau$, may be considered as the analog of the Q-integrals of the transfer equation previously derived in [1,3-5]. Using the procedure developed in [4,5], one may derive the so-called 'two-point' integrals, which connect the radiation fields at two different depths in the atmosphere. Moreover, it is easy to obtain also the bilinear integrals relating with each other two different problems of the considered type.

If the collisional processes in the lines $1\to 2$ and $1\to 3$ are neglected we have $J_{12}^{\circ} = J_{13}^{\circ}$, so that now

$$\mathcal{L} = MJ_{12}'^2 + NJ_{13}'^2 + LMJ_{12}^2 + KNJ_{13}^2 - 2\,MN\,J_{12}J_{13}, \tag{2.10}$$

and, as easily seen from Eq.(2.5), $KL = MN$. Consequently the Lagrangian may be represented in the form

$$\mathcal{L} = MJ_{12}'^2 + NJ_{13}'^2 + \frac{N}{K}\left(NJ_{12} - KJ_{13}\right)^2. \tag{2.11}$$

We notice that the Lagrangian (2.11) remains invariant with respect to the infinitesimal transformation of the field variables $J_{1k}$ $(k = 2,3)$:

$$J_{12} \to J_{12} + \varepsilon, \quad J_{13} \to J_{13} - \frac{M}{K}\varepsilon. \tag{2.12}$$

Making $\varepsilon$ an added field variable, and performing the transformation (2.12), we obtain [10]

$$\bar{\mathcal{L}} = \mathcal{L} + 2\left(MJ_{12}' + LJ_{13}'\right)\varepsilon'. \tag{2.13}$$

The Euler-Lagrange equation associated with $\varepsilon$ is

$$\frac{d}{d\tau}\frac{\partial\bar{\mathcal{L}}}{\partial\varepsilon'} - \frac{\partial\bar{\mathcal{L}}}{\partial\varepsilon} = \frac{d}{d\tau}\left(MJ_{12}' + LJ_{13}'\right) = 0, \tag{2.14}$$

hence

$$MJ_{12}' + LJ_{13}' = \text{const}. \tag{2.15}$$

Taking into account Eq.(2.5), we are led to the flux conservation law

$$\bar{H}_{12} + \bar{H}_{13} = \text{const}, \tag{2.16}$$

where $\bar{H}_{1k} = \left(\Delta\nu_{1k}/\nu_{1k}\right)H_{1k}$.

From Eq.(2.15) we find also

$$J_{12} + q^2\left(B_{13}/B_{12}\right)J_{13} = C\tau + D, \tag{2.17}$$

where $C$ and $D$ are the integration constants. Equation (2.17) may be interpreted as the analog of the well-known K-integral.

3. *Multilevel atoms, the general case.* Now the question arises whether or not the Lagrangian approach adopted in this paper may be applied to the more general multilevel problems for finding conservation laws. This section demonstrates that the constraints imposed by the symmetry requirement for a system of equations are very stringent, and may be satisfied only in extremely specific situations. Even for the simple model problem investigated above, in which the radiative transitions between levels 2 and 3 are forbidden, allowance for the induced emission processes render the Lagrangian method unapplicable. In this case the problem becomes essentially nonlinear and is reduced to the solution of the following system of differential equations

$$\frac{d^2\bar{J}_{12}}{d\tau^2}+\bar{b}\frac{\left[\bar{J}_{13}\left(1-\gamma\bar{J}_{12}\right)-(1-\gamma)\bar{J}_{12}\right]\left(\bar{J}_{13}+m\right)}{\left(\bar{J}_{12}+l\right)^2}-\bar{J}'_{12}\left(\ln\frac{\bar{J}_{13}+m}{\bar{J}_{12}+l}\right)=0,$$

$$\frac{d^2\bar{J}_{13}}{d\tau^2}-b\frac{\bar{J}_{13}\left(1-\gamma\bar{J}_{12}\right)-(1-\gamma)\bar{J}_{12}}{\bar{J}_{12}+l}=0, \tag{3.1}$$

for the functions $\bar{J}_{12}=J_{12}/h\nu_{12}$ and $\bar{J}_{12}=J_{12}/h\nu_{12}$, where

$$\gamma=1-\frac{g_2b_{23}}{g_3a_{32}},\quad b=\frac{a_{32}}{A_{31}+a_{32}-\gamma b_{23}},\quad \bar{b}=bq^2\frac{l}{m}\frac{g_3A_{31}^2}{g_2A_{21}},$$

$$m=\frac{A_{21}\left(A_{31}+a_{32}\right)+b_{23}A_{31}}{A_{21}-\left(g_3/g_2\right)\gamma a_{32}},\quad l=\frac{A_{21}\left(A_{31}+a_{32}\right)+b_{23}A_{31}}{A_{21}\left(A_{31}+\gamma a_{32}\right)}. \tag{3.2}$$

It can be checked that even the simplest symmetry condition (the first of Eqs.(2.6)) is not satisfied, therefore the system (3.1) is not derivable from a variational principle. A similar situation is encountered in the more general case when all the radiative transitions are permitted. This conclusion is obviously valid for any multilevel transfer problem with the larger number of levels, for which the three-level problem we investigated can be regarded as a special case. It should be noted however that this fact alone does not exclude the existence of conservation laws. It is easy to show, for instance, that any multilevel source-free problem will admit the flux-conservation laws similar to that given by Eq.(2.16).

It should be emphasized in conclusion that one may generally use a convolution bilinear form, which render the operator symmetric, and provides a variational formulation for every system of differential equations. In this approach the Lagrangian is no longer a function but an operator. For this reason, we limited ourselves to considering the problem within the framework of the classical theory.

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# ЗАКОНЫ СОХРАНЕНИЯ ДЛЯ МНОГОУРОВЕННЫХ ЗАДАЧ ПЕРЕНОСА ИЗЛУЧЕНИЯ

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В работе обсуждается вопрос о том, является ли способ нахождения законов сохранения, основанный на вариационном формализме, применимым в случае многоуровенных задач переноса излучения в однородной атмосфере. Для наглядности рассматривается простейшая одномерная задача. Для одной частной трехуровенной задачи лагранжиановский подход позволяет вывести не только H- и K-интегралы, но и нелинейные интегралы, являющиеся аналогом Q-интегралов, полученных ранее для ряда классических задач переноса. Показывается, что в общем случае ограничения, накладываемые вариационным принципом на свойства симметрии уравнений переноса излучения, являются слишком жесткими и не удовлетворяются.

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