ԳԻՏՈՒԹՅՈՒՆՆԵՐԻ ԱԶԳԱՅԻՆ ԱԿԱԴԵՄԻԱՅԻ ՏԵՂԵԿԱԳԻՐ
PROCEEDINGS OF NATIONAL ACADEMY OF SCIENCES OF ARMENIA

Մեխանիկա Volume 74, Issue 2, 2021 Mechanics

UDC 539.3 Doi- http://doi.org/10.33018/74.2.1

Control of Vibrations of Infinite Membrane Tape with a Moving Edge in a Supersonic Gas Flow

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Key words: membrane, supersonic gas flow, one-dimensional vibrations, boundary action, control problem, harmonics of control function.

Управление колебаниями бесконечной мембранной ленты с подвижным краем в сверхзвуковом потоке газа

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Рассмотрена задача управления колебаниями бесконечной в одном направлении мембранной ленты в сверхзвуковом потоке газа за конечный интервал времени. Один край мембранной ленты жестко защемлен. К другому краю ленты с помощью жесткой прямой линейки прикладывают управляющее воздействие. Математическая краевая задача бесконечной мембранной ленты моделируется как одномерные колебания струны с краевым воздействием.

Задача управления колебаниями бесконечной мембранной ленты, обтекаемой сверхзвуковым потоком газа и с подвижным краем, решается методом разделения переменных, с разложением прогиба колебания ленты на собственные формы и функции ее гармоник в ряд Фурье. Искомая функция краевого управляющего воздействия также представляется в виде ряда Фурье. В случае краевого управления, собственные гармоники колебания мембранной ленты и гармоники управляющего воздействия строятся совместно, после удовлетворения граничным, начальным и финальным условиям задачи.

Рассмотрены частные случаи возможных управляющих функций воздействия и форм колебаний мембранной ленты в разных случаях начальных и конечных условий. Выполнены расчеты на конкретных примерах.

Ключевые слова: мембранная лента, сверхзвуковой поток газа, одномерные колебания, краевое воздействие, задача управления, гармоники управляющего воздействия.

Անվերջ մեմբրանային ժապավենի տատանումների ղեկավարումը շարժվող եզրով՝ գերձայնային գազի հոսքում

Ավետիսյան Արա Ս., Մկրտչյան Մ. Հ.

Դիտարկվում է մեկ ուղղությամբ անվերջ մեմբրանային ժապավենի տատանումների ղեկավարման խնդիրը գերձայնային գազի հոսքում ՝ ժամանակի վերջավոր միջակայքում: Մեմբրանային ժապավենի մի եզրը կոշտ ամրացված է, իսկ մյուս եզրին ազդում է ղեկավարող ազդեցություն՝ օգտագործելով կոշտ ուղիղ քանոն: Խնդրի մաթեմատիկական մոդելը վերածվում է եզրային ղեկավարում ունեցող լարի միաչափ տատանումների ղեկավարման խնդրի:

Ուսումնասիրվող տատանումների ղեկավարման խնդիրը լուծվում է փոփոխականների անջատման մեթոդի միջոցով՝ մեմբրանային ժապավենի տատանումների տեղափոխությունը ներկայացնելով տատանման սեփական ձևերի և դրանց սեփական հարմոնիկների Ֆուրիեի շարքերի միջոցով։ Ղեկավարման ֆունկցիան նույնպես ներկայացվում է Ֆուրիեի շարքերի միջոցով։ Եզրային ղեկավարման համար մեմբրանային ժապավենի տատանումների սեփական հարմոնիկները և ղեկավարման ֆունկցիայի հարմոնիկները կառուցվում են միաժամանակ՝ խնդրի եզրային, սկզբնական և վերջնական պայմանները բավարարելուց հետո։

Դիտարկվում են տարբեր սկզբնական և վերջնական պայմանների համար մեմբրանային ժապավենի հնարավոր ղեկավարման ազդեցության և տատանումների ռեժիմների հատուկ դեպքեր։ Իրականացվում է որոշակի դեպքերի թվային վերլուծություն։

Հիմնաբառեր: մեմբրանային ժապավեն, գերձայնային գազի հոսք, միաչափ տատանումներ, եզրային ազդեցություն, ղեկավարման խնդիր, ղեկավարման ֆունկցիայի հարմոնիկներ

A vibration control problem is considered for an infinite in one direction membrane tape in a supersonic gas flow in a finite time interval is considered. One edge of the membrane tape is rigidly fixed. A control action is applied to the other edge of the tape using a rigid straight ruler. The mathematical model of the problem is reduced to a control problem for one-dimensional vibrations of a string with a boundary control.

The vibration control problem under study is solved via variables separation method by expanding the deflection of the membrane vibration into Fourier series by natural vibration modes of the membrane and the function of its harmonics. The unknown function of the boundary control is also expanded into Fourier series. For the boundary control, the eigenharmonics of the membrane vibrations and the harmonics of the control are constructed simultaneously, after satisfying the boundary, initial and terminal conditions of the problem.

Particular cases of possible control functions and modes of vibrations of the membrane for different initial and terminal conditions are considered. Numerical analysis of particular cases is carried out.

Key words: membrane tape, supersonic gas flow, one-dimensional vibrations, boundary action, control problem, harmonics of control function.

Introduction

Flexible thin-walled structural elements (plates or shells) made from soft materials, which are technically modeled as membranes are often used in modern technology. Naturally, in technical problems of vibration control of membranes, the mathematical boundary value problem is formulated on the basis of boundary conditions and of the state of the membrane.

From this point of view, the proposed control problem of vibrations of an infinite in one direction membrane in a supersonic gas flow when one edge of the membrane is rigidly fixed, and the other edge is controlled, on a finite time interval is a model. Supersonic gas flow streamlines around the membrane along its width, resulting in a membrane to vibrate.

The problem is mathematically modeled as a problem of one-dimensional forced transverse vibrations of a membrane with a boundary control. Formally, it coincides with the problem of boundary control of string vibrations under a distributed transverse external action [1]. Forced vibrations of the membrane and issues of its stability in the aerodynamics of high supersonic gas velocities were investigated in the middle of the last century [2,3]. However, to the best of our knowledge, the issues of controlling such a membrane vibrations were not considered so far.

The solution to the mathematical boundary value problem of damping string vibrations with two control functions are given in the monograph [4].The problem is solved by the method of Fourier series expansion applied to the string deflection. For a string without a distributed transverse load, a similar mathematical boundary value problem is solved in [5], using D'Alembert method. However, D'Alembert method does not allow to solve similar boundary value problems in cases where the general solution of the problem cannot be represented in an integral forms containing the given initial and terminal conditions explicitly. In the problem of boundary control of vibrations of a string with given states at intermediate moments [6], the control of a string with two acting control functions depending on time at the two ends of the string is investigated.

In well-known monographs [7-10], some of the existing methods and those under intensive development can be found for the solution of model problems of controllability of dynamic systems or for the analysis of the nature of control of physicomechanical dynamic processes.

In the proposed work, we seek a solution to the control problem by expanding all functions, including the function of the boundary control in the form of Fourier series with respect to natural modes of vibrations of the membrane and with respect to its natural harmonics. After satisfying the boundary, initial and terminal conditions, the modes and harmonics of the vibrations of the membrane, as well as the corresponding control function are determined.

1 Statement of the problem. Formulation of the mathematical boundary value problem

Consider the possibility of control of an infinite in one direction membrane vibrating in a supersonic gas flow, when one edge of the tape is rigidly fixed, and the other edge is controlled (Fig. 1). The membrane has a width $0 \leq x \leq l$ and a very long length (considered to be infinite).

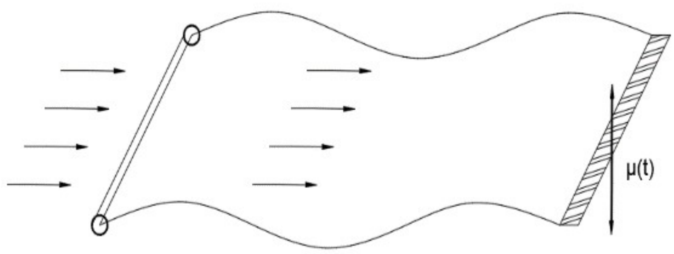


Figure 1: Diagram of supersonic gas flow around the membrane

The supersonic gas stream flows around the membrane along its width, vibrating the membrane. Vibrations of the membrane in a supersonic gas flow are modeled as parallel one-dimensional vibrations of the membrane[1]:

$$\frac{\partial^2 W}{\partial x^2} - \beta \frac{\partial W}{\partial x} - \alpha^2 \frac{\partial^2 W}{\partial t^2} = 0 \ , \quad 0 \leq x \leq l, \quad t \geq 0. \tag{1.1}$$

Here,$\beta = \chi \rho_\infty M N_x^{-1}$ is a physical parameter characterizing the gas flow along the membrane, $\alpha^2 = \rho_0 h N_x^{-1}$is the inverse of the square of the speed, $N_x^{-1}$is the tensile force in the direction of the width of the membrane, ρ_0 is the membrane material density, χ is the aerodynamic constant, ρ_∞ is the gas density, M is the Mach number, h is the membrane thickness.

The edge $x = 0$ of the membrane is rigidly fixed. The induced vibrations of the membrane are controlled by means of a rigid rectilinear ruler applied on the moving edge $x = l$ of the membrane and represented by the function $\mu(t)$ depending only on time. The boundary conditions will therefore be

$$W(0, t) = \mu(t) \ , \quad W(l, t) = 0. \tag{1.2}$$

According to the classical formulation, based on equation (1.1) and boundary conditions (1.2), the boundary control $\mu(t)$ will be considered in the class of functions $\mu(t) \in L_2[0 \leq t \leq T_0]$. It is assumed that at the initial moment $t = 0$, the shape of the membrane deflection and the distribution of the rate of change of the deflection are known:

$$W|_{t=0} = \varphi(x) \ , \quad \left.\frac{\partial W}{\partial t}\right|_{t=0} = \psi(x). \tag{1.3}$$

It is required to find such a boundary control $\mu(t)$ for which equation (1.1) is transmitted from the initial state (1.3) to the terminal state

$$W(x, T_0) = \tilde{\varphi}(x) \ , \quad \left.\frac{\partial W}{\partial t}\right|_{t=T_0} = \tilde{\psi}(x), \tag{1.4}$$

over the interval $t \in [0; T_0]$.

Deflection functions $\varphi(x)$ and $\tilde{\varphi}(x)$, as well as functions of the vibration speed $\psi(x)$ and $\tilde{\psi}(x)$, at the initial moment of time $t = 0$ and at the final moment of time $t = T_0$ respectively, are considered to be elements of $L_2[0 \leq x \leq l]$. The solution of the formulated mathematical boundary value problem is obtained by introducing a new displacement function $V(x, t)$ such that

$$V(x, t) = W(x, t) - \left(1 - \frac{x}{l}\right) \mu(t). \tag{1.5}$$

Substituting(1.5) into (1.1), boundary control $\mu(t)$ will move into the equation of vibration of the membrane. The mathematical boundary value problem in the form of homogeneous equation (1.1) subject to inhomogeneous boundary conditions (1.2) is reduced to inhomogeneous equation of forced vibrations

$$\frac{\partial^2 V(x,t)}{\partial x^2} - \beta \frac{\partial V(x,t)}{\partial x} - \alpha^2 \frac{\partial^2 V(x,t)}{\partial t^2} = -\frac{\beta}{l}\mu(t) + \alpha^2 \left(1 - \frac{x}{l}\right) \ddot{\mu}(t) \qquad (1.6)$$

with "external influence" $-(\beta/l) \cdot \mu(t) + \alpha^2 (1 - x/l) \cdot \ddot{\mu}(t)$, subject to homogeneous boundary conditions for the unknown function of the reduced displacement $V(x,t)$:

$$V(0,t) = 0, \quad V(l,t) = 0. \qquad (1.7)$$

The initial and terminal conditions are respectively reduced to

$$\begin{aligned} V(x,t)|_{t=0} &= \varphi(x) - \left(1 - \frac{x}{l}\right) \cdot \mu(0), \\ \left.\frac{\partial V(x,t)}{\partial t}\right|_{t=0} &= \psi(x) - \left(1 - \frac{x}{l}\right) \dot{\mu}(0). \end{aligned} \qquad (1.8)$$

$$\begin{aligned} V(x,T_0) &= \tilde{\varphi}(x) - \left(1 - \frac{x}{l}\right) \cdot \mu(T_0), \\ \left.\frac{\partial V(x,t)}{\partial t}\right|_{t=T_0} &= \tilde{\psi}(x) - \left(1 - \frac{x}{l}\right) \cdot \dot{\mu}(T_0) \end{aligned} \qquad (1.9)$$

2 Solution of the mathematical boundary value problem

The new formulation of the mathematical boundary value problem in the form of equation (1.6) and homogeneous boundary conditions (1.7) allows the representation of the solution of the problem by the method of variable separation as follows:

$$V(x,t) = X(x) \cdot f(t) = \sum_{n=1}^{\infty} X_{0n}(x) f_n(t), \qquad (2.1)$$

using the expansion of the reduced displacement of the membrane in the form of a Fourier series in terms of its eigenmodes of the vibration.

2.1 Decomposition of the forced vibration of the membrane tape on its own forms

Taking into account homogeneous boundary conditions (1.7) and the homogeneous part of equation (1.6), the deflection of the membrane can be represented as a Fourier series in terms of its eigenmodes as follows:

$$X(x)=\sum_{n=1}^{\infty}X_{0n}(x),\ \text{where}\ X_{0n}(x)=B_n\exp\left(\frac{\beta x}{2}\right)\cdot\sin\left(\frac{n\pi}{l}x\right),\ n\in\mathbb{N} \tag{2.2}$$

with corresponding eigenharmonics

$$f_0(t)=\sum_{n=1}^{\infty}\theta_{0n}(t),\ \text{and}\ \theta_{0n}(t)=A_{0n}\cdot\sin(\omega_{\theta n}t)+B_{0n}\cdot\cos(\omega_{\theta n}t). \tag{2.3}$$

In this case, the eigenvalues of the vibrational motion are defined as

$$\omega_{0n}^2=(n\pi/\alpha l)^2+(\beta/2\alpha)^2,\ n\in\mathbb{N}. \tag{2.4}$$

It is obvious that the frequencies of eigenharmonics are determined by the physical and geometric parameters of the vibratory system: αl, $\beta=\chi\rho_\infty MN_x^{-1}$ and $\alpha^2=\rho_0 hN_x^{-1}$. The maximum value of the eigenfrequency, at a certain value of the tensile force $N_x/l=\left(\chi^2\rho_\infty^2M^2/8\pi^2l\right)^{1/3}$, is achieved for the first eigenmode $X_{01}(x)=B_{01}\exp(\beta x/2)\cdot\sin(\pi x/l)$. Due to the inhomogeneity of equation (1.6), the newly formed vibration modes on the segment $x\in[0;\ l]$ will be represented by the proper vibration modes (2.2), and the dynamics of these forms is already will be represented by another function of time $\theta(t)=\sum\limits_{n=1}^{\infty}\theta_n(t)$. Decomposing also the factors in the terms on the right-hand side of the inhomogeneous equation (1.6) into the Fourier series with respect to eigenmodes (2.2),

$$1=-\sum_{n=1}^{\infty}C_nX_n(x)\ \text{and}\ (1-x/l)=\sum_{n=1}^{\infty}D_nX_n(x), \tag{2.5}$$

we obtain an equation for the n–th form of vibration of the membrane, in the form of a sequential infinite system of ordinary differential equations

$$\ddot{\theta}_n(t)+(\lambda_n^2/\alpha^2)\cdot\theta_n(t)=-D_n\cdot\left[\ddot{\mu}(t)+(\beta/\alpha^2l)\cdot(C_n/D_n)\cdot\mu(t)\right]. \tag{2.6}$$

In expansions (2.5), Fourier coefficients $C_n$andD_n are defined as

$$\begin{aligned}C_n=&[4\exp(-\beta l/2)\cdot(2\exp(\beta l/2)-2(-1)^n)\cdot n\pi]/\left(4n^2\pi^2+l^2\beta^2\right),\\ D_n=&\frac{8\exp(-\beta l/2)\cdot\left[\exp(\beta l/2)\cdot\left(4n^2\pi^2+l^2\beta^2-4l\beta\right)+4ln\pi\beta(-1)^n\right]\cdot n\pi}{\left(4n^2\pi^2+l^2\beta^2\right)^2}\end{aligned} \tag{2.7}$$

2.2 Control of the natural forms oscillations by harmonics of the edge action

The right-hand sides of the ordinary differential equations of the infinite system (2.6) include the boundary control action $\mu(t)$ corresponding to the oscillations of the eigenforms of the true deflection (2.2), one for all orthogonal forms with its secondary derivatives. The introduction of a new designation for the frequency of the edge control action

$$\omega_{\mu n}^2 = \frac{\beta l \cdot C_n}{(l\alpha)^2 \cdot D_n} = \\ = \frac{\beta l \cdot \left(4n^2\pi^2 + (\beta l)^2\right)[1 - (-1)^n \exp(-\beta l/2\ )]}{4(l\alpha)^2 \cdot \left[(n\pi + (\beta l/2)\ )^2 - \beta l - n\pi\beta l \cdot [1 - (-1)^n \cdot \exp(-\beta l/2\ )]\right]} \tag{2.8}$$

the function $\mu(t)$ is also represented as a series of corresponding harmonics

$$\mu(t) = \sum_{n=1}^{\infty} \mu_n(t), \text{ where } \mu_n(t) = A_{n\mu} \cdot \sin(\omega_{\mu n} t) + B_{n\mu} \cdot \cos(\omega_{\mu n} t) \tag{2.9}$$

The infinite system of ordinary differential equations for the vibrations of the membrane (2.6) can be written in the form of an infinite system of equations for forced vibrations

$$\ddot{f}_{\theta n}(t) + \omega_{0n}^2 \cdot f_{\theta n}(t) = -D_n \cdot \left(\omega_{\mu n}^2 - \omega_{0n}^2\right) \cdot \mu_n(t), \tag{2.10}$$

or in the form

$$\ddot{f}_{\mu n}(t) + \omega_{\mu n}^2 \cdot f_{\mu n}(t) = \left(\omega_{\mu n}^2 - \omega_{0n}^2\right) \cdot \theta_{0n}(t)\ . \tag{2.11}$$

with respect to reduced harmonics $f_{\theta n}(t)$ or $f_{\mu n}(t)$ of displacement $V(x,t)$

$$f_{\theta n}(t) = \theta_n(\omega_{\theta n} t) + D_n \cdot \mu_n(\omega_{\mu n} t), \quad f_{\mu n}(t) = \mu_n(\omega_{\mu n} t) + D_n^{-1} \cdot \theta_n(\omega_{\theta n} t) \tag{2.12}$$

This harmonics is the direct composition eigenmodes of the membrane vibrations and the harmonics of the boundary action. It is obvious from the equations (2.10) and (2.11) that the true frequencies of the reduced harmonics of the eigenmodes of the membrane vibrations are formed in different ways.

According to the equation (2.10), the frequencies of the harmonics of the reduced eigenforms of the membrane are formed on the basis of the eigenfrequencies $\omega_{\theta n} = \omega_{0n}$, undergoing the harmonics of the boundary action $\mu_n(t)$ with frequency $\omega_{\mu n}$.

According to the equation (2.11), the frequencies of the harmonics of the reduced natural forms of the membrane are formed on basis of the "eigenfrequencies" of the boundary action $\omega_{\mu n}$, undergoing the influence of eigenharmonics of the membrane vibrations. From equations (2.10) and (2.11) it is also obvious that the vibration of the membrane subjected to boundary control will be stable or unstable depending on the values of the frequencies $\omega_{\theta n} < \omega_{\mu n}$ or $\omega_{\theta n} > \omega_{\mu n}$.

The general solution of (2.10) for n-th harmonic $f_n(\omega_{\theta n} t)$ is obtained by the method of variation of parameters in the form of addition of harmonics of eigen and forced vibrations of the membrane:

$$f_{\theta n}(\omega_{\theta n}t) = A^*_{\theta n}\cdot\sin(\omega_{\theta n}t) + B^*_{\theta n}\cdot\cos(\omega_{\theta n}t) + \\ + D_n\cdot[A_{n\mu}\cdot\sin(\omega_{\mu n}t) + B_{n\mu}\cdot\cos(\omega_{\mu n}t)] \tag{2.13}$$

Similarly, the general solution of (2.11) for the n-th harmonic,$f_n(\omega_{\mu n}t)$,is obtained as

$$f_{\mu n}(\omega_{\mu n}t) = A^*_{\mu n}\cdot\sin(\omega_{\mu n}t) + B^*_{\mu n}\cdot\cos(\omega_{\mu n}t) + A_{\theta n}\cdot\sin(\omega_{\theta n}t) + B_{\theta n}\cdot\cos(\omega_{\theta n}t) \tag{2.14}$$

It is evident from expressions (2.4) and (2.8) that the frequency characteristics of the system, $\omega_{\theta n}$ and $\omega_{\mu n}$, are determined by physical and mechanical parameters $\beta l = \chi\rho_\infty lMN_x^{-1}$ and $\alpha^2 = \rho_1 hN_x^{-1}$. Formally, these frequencies can be equal under the condition

$$[1-(-1)^n\cdot\exp(-\beta l/2\,)] = \left[4\,(n\pi\cdot(-1)^n - 1) + l\beta/4\alpha^2\,\right] + \left(n^2\pi^2\right)/\left(\alpha^2\beta l\right) \tag{2.15}$$

It follows from equations (2.10), (2.11) and from the corresponding general solutions (2.13), (2.14) that in this case the system vibrates with the eigenfrequency of the reduced forms

$$f_{\theta n}(\omega_{\theta n}t) = (A^*_{\theta n} + D_n\cdot A_{n\mu})\cdot\sin(\omega_{\theta n}t) + (B^*_{\theta n} + D_n\cdot B_{n\mu})\cdot\cos(\omega_{\theta n}t) \tag{2.16}$$

In that case, neither a control problem nor a resonance of vibrations of the membrane occur. According to (1.5), (2.1), (2.2)and (2.12), for the deflection function $W(x,t)$ we obtain

$$W(x,t) = \sum_{n=1}^{\infty} B_n\left[\theta_n(\omega_{\theta n}t) + D_n\cdot\mu_n(\omega_{\mu n}t)\right]\;\cdot\exp\left(\frac{\beta x}{2}\right)\cdot\sin\left(\frac{n\pi x}{l}\right).$$

In order to fulfill initial and terminal conditions (1.3) and(1.4),respectively, functions $\varphi(x)$ and $\psi(x)$, as well as $\tilde{\varphi}(x)$ and $\tilde{\psi}(x)$are also expanded into Fourier series as follows:

$$\varphi(x) = \sum_{n=1}^{\infty}\gamma_n\cdot X_n(x),\ \psi(x) = \sum_{n=1}^{\infty}\delta_n\cdot X_n(x) \tag{2.17}$$

$$\tilde{\varphi}(x) = \sum_{n=1}^{\infty}\tilde{\gamma}_n\cdot X_n(x),\ \tilde{\psi}(x) = \sum_{n=1}^{\infty}\tilde{\delta}_n\cdot X_n(x) \tag{2.18}$$

Taking into account the representation (2.17) of the deflection function $W(x,t)$ and expansions (2.18) and (2.19), initial and terminal conditions (1.8) and (1.9) for vibrations of the membrane with boundary control are written in the form of an infinite system of four algebraic equations for the amplitudes of the harmonics of the membrane vibrations and the boundary control,

$$\begin{cases}\theta_n(0) + D_n\;\cdot\mu_n(0) = \gamma_n\\ \dot{\theta}_n(0) + D_n\cdot\;\dot{\mu}_n(0) = \delta_n\\ \theta_n(\omega_{\theta n}T_0) + D_n\cdot\;\mu_n(\omega_{\mu n}T_0) = \tilde{\gamma}_n\\ \dot{\theta}_n(\omega_{\theta n}T_0) + D_n\;\cdot\dot{\mu}_n(\omega_{\mu n}T_0) = \tilde{\delta}_n\end{cases}$$

In the case of general solution (2.13), the infinite system of algebraic inhomogeneous equations (2.20) can be written in an expanded form with respect to four unknown harmonic coefficients $A^*_{\theta n}$, $B^*_{\theta n}$, $A_{n\mu}$ and $B_{n\mu}$:

$$\begin{cases} B^*_{\theta n} + D_n \cdot B_{n\mu} = \gamma_n \\ A^*_{\theta n} + (\omega_{\mu n}/\omega_{0n})\ D_n \cdot A_{n\mu} = \dfrac{\delta_n}{\omega_{\theta n}} \\ \sin(\omega_{\theta n} T^0_{\theta n}) \cdot A^*_{\theta n} + \cos(\omega_{\theta n} T^0_{\theta n}) \cdot B^*_{\theta n} + \\ + D_n \cdot \sin(\omega_{\mu n} T^0_{\theta n}) \cdot A_{n\mu} + D_n \cdot \cos(\omega_{\mu n} T^0_{\theta n}) \cdot B_{n\mu} = \tilde{\gamma}_n \\ \cos(\omega_{\theta n} T^0_{\theta n}) \cdot A^*_{\theta n} - \sin(\omega_{\theta n} T^0_{\theta n}) \cdot B^*_{\theta n} + \\ (\omega_{\mu n}/\omega_{\theta n})\ D_n \cdot \cos(\omega_{\mu n} T^0_{\theta n}) \cdot A_{n\mu} - (\omega_{\mu n}/\omega_{\theta n})\ D_n \cdot \sin(\omega_{\mu n} T^0_{\theta n}) \cdot B_{n\mu} = \dfrac{\tilde{\delta}_n}{\omega_{\theta n}} \end{cases}$$

Evaluating these four unknown constant coefficients $A^*_{\theta n}$, $B^*_{\theta n}$, $A_{n\mu}$ and $B_{n\mu}$, it will become an easy problem to determine the boundary control function $\mu(t)$ according to (2.9) and the deflection function $W(x,t)$ of the membrane vibrating in a supersonic gas flow according to (2.17) on the finite interval $t \in [0; T_{0\theta}]$.

In the case of general solution (2.14), infinite system of algebraic inhomogeneous equations (2.20) can be reduced to an expanded form with respect to four unknown harmonic coefficients $A_{\theta n}$, $B_{\theta n}$, $A^*_{n\mu}$ and $B^*_{n\mu}$:

$$\begin{cases} B_{\theta n} + B^*_{\mu n} = \gamma_n \\ A_{\theta n} + (\omega_{\mu n}/\omega_{\theta n})\ A^*_{\mu n} = \dfrac{\delta_n}{\omega_{\theta n}} \\ \sin(\omega_{\theta n} T^0_{\mu n}) \cdot A_{\theta n} + \cos(\omega_{\theta n} T^0_{\mu n}) \cdot B_{\theta n} + \sin(\omega_{\mu n} T^0_{\mu n}) \cdot A^*_{\mu n} + \\ \qquad + \cos(\omega_{\mu n} T^0_{\mu n}) \cdot B^*_{\mu n} = \tilde{\gamma}_n \\ \cos(\omega_{\theta n} T^0_{\mu n}) \cdot A_{\theta n} - \sin(\omega_{\theta n} T^0_{\mu n}) \cdot B_{\theta n} + \\ + (\omega_{\mu n}/\omega_{\theta n})\ \cdot \cos(\omega_{\mu n} T^0_{\mu n}) \cdot A^*_{\mu n} - (\omega_{\mu n}/\omega_{\theta n})\ \cdot \sin(\omega_{\mu n} T^0_{\mu n}) \cdot B^*_{\mu n} = \dfrac{\tilde{\delta}_n}{\omega_{\theta n}} \end{cases}$$

Finding four unknown constant coefficients $A_{\theta n}$, $B_{\theta n}$, $A^*_{\mu n}$ and $B^*_{\mu n}$, it will be an easy task to build the boundary control $\mu(t)$ according to (2.9) and the deflection function $W(x,t)$ of the membrane tape vibrating in a supersonic gas flow according to (2.17) on the finite interval $t \in [0; T_{0\mu}]$. In each of these cases, the required time of the boundary control is defined as

$$T_{\theta 0} = \max\left\{T^0_{\theta n} = \frac{2\pi}{\omega_{\theta n}}\right\}, \quad T_{\mu 0} = \max\left\{T^0_{\mu n} = \frac{2\pi}{\omega_{\mu n}}\right\}.$$

3 Numerical analysis for different initial and terminal states

Consider a membrane infinite in one direction vibrating in a supersonic gas flow which streamlines the membrane along its width. One edge of the membrane is rigidly fixed, while the other edge is controlled in the direction parallel to the deflection of the membrane (Fig 1). In numerical calculations, in order to determine the physical and geometric characteristics of the dynamic system, the following characteristics of the membrane material and the gas flow are consider: $\rho_0 = 1500\ kg/m^3$, $N_x = 1/50\ N/m$, $\chi = 0.32$, $M = 2.0$, $l = 2\ m$, $h = 0.0001\ m$,$\rho_\infty = 0.01\ kg/m^3$, $\alpha = 2.738613$, $\beta l = 0.64$. Obviously, depending on the physical and geometric characteristics of the system,the behavior of the fundamental harmonics $\theta_{0n}(t)$ of eigenforms of the membrane tapeand the corresponding harmonics of the boundary action $\mu_n(t)$ will be different.

The boundary control problem a)

In the case when the membrane is transmitted from the initial state

$$W|_{t=0} = \varphi(x) \ = \sin(10x)\ , \quad \dot{W}\Big|_{t=0} = \psi(x) = \cos(10x) \tag{3.1}$$

to the terminal state of rest,

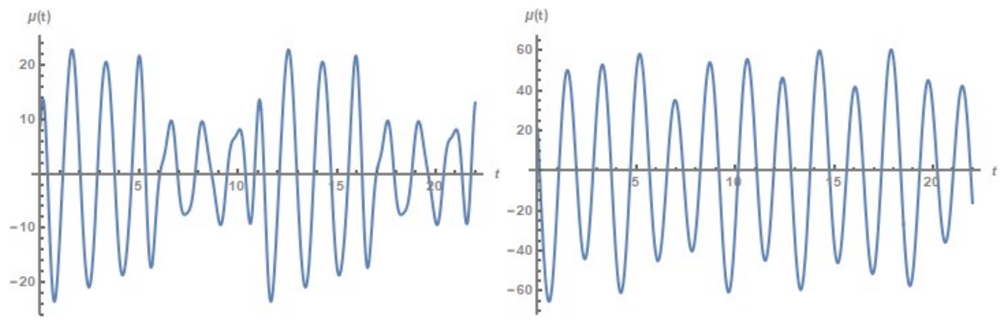
$$W(x, T_0) = \tilde{\varphi}(x) \equiv 0\ , \ \dot{W}\Big|_{t=T_0} = \tilde{\psi}(x) \equiv 0 \tag{3.2}$$

for the boundary control $\mu(t)$ corresponding to general solution (2.13), in the case of $n = 15$, we obtain

$$\begin{aligned} &\mu_1\,(t) = -0.08858 \cdot \sin\left[0.57654 \cdot t\right] - 0.348088 \cdot \sin\left[1.14863 \cdot t\right] - \\ &\ - 0.792535 \cdot \sin\left[1.72171 \cdot t\right] - 0.05522 \cdot \cos\left[0.57654 \cdot t\right] - \\ &\ - 0.550381 \cdot \cos\left[1.14863 \cdot t\right] - 0.5624 \cdot \cos\left[1.72171 \cdot t\right] + ... + \\ &\ + 1.15872 \cdot \sin\left[7.456687 \cdot t\right] + 0.88594 \cdot \sin\left[8.030245 \cdot t\right] + \\ &\ + 0.944 \cdot \sin\left[8.6038 \cdot t\right] + 1.23084 \cdot \cos\left[7.456687 \cdot t\right] - \\ &\ - 0.59297 \cdot \cos\left[8.030245 \cdot t\right] + 1.0624 \cdot \cos\left[8.6038 \cdot t\right] \end{aligned} \tag{3.3}$$

On the other hand, for the boundary control $\mu(t)$ corresponding to the general solution (2.14), in case of $n = 15$, we obtain

$$\begin{aligned} &\mu_2\,(t) = -0.024935 \cdot \sin\left[0.57654 \cdot t\right] + 0.011172 \cdot \sin\left[1.14863 \cdot t\right] - \\ &\ - 0.01416 \cdot \sin\left[1.72171 \cdot t\right] + 0.01911 \cdot \cos\left[0.57654 \cdot t\right] + \\ &\ + 0.030467 \cdot \cos\left[1.14863 \cdot t\right] - 0.0299 \cdot \cos\left[1.72171 \cdot t\right] + ... + \\ &\ - 0.0266 \cdot \sin\left[7.456687 \cdot t\right] + 0.034766 \cdot \sin\left[8.030245 \cdot t\right] + \\ &\ + 0.022 \cdot \sin\left[8.6038 \cdot t\right] + 0.03 \cdot \cos\left[7.456687 \cdot t\right] - \\ &\ - 0.01449837 \cdot \cos\left[8.030245 \cdot t\right] + 0.01334 \cdot \cos\left[8.6038 \cdot t\right] \end{aligned} \tag{3.4}$$



(a) membrane width $l = 2$ m (b) membrane width $l = 5$ m

Figure 2: Edge control functions in the case of damping vibrations of the membrane: transition of the system from state (3.1) to state (3.2)

Despite the difference of expressions (3.3) and (3.4), in the case of the given physical and geometric characteristics of the dynamic system, their graphical representations match exactly (Fig. 2a). In the case of a wide tape, when $l = 5$ m, the boundary control $\mu(t)$ for $n = 15$ has the following form:

$$\begin{aligned}
\mu_3(t) = &-0.02999 \cdot \sin[0.23675 \cdot t] - 0.035520 \cdot \sin[0.46256 \cdot t] - \\
&- 0.234953 \cdot \sin[0.69076 \cdot t] + 0.004589 \cdot \cos[0.23675 \cdot t] - \\
&- 0.04559 \cdot \cos[0.46256 \cdot t] + 0.025341 \cdot \cos[0.69076 \cdot t] - \\
&- 3.634929 \cdot \sin[2.98315 \cdot t] - 1.711715 \cdot \sin[3.21254 \cdot t] - \\
&- 48.79411 \cdot \sin[3.44193 \cdot t] + 1.867593 \cdot \cos[2.98315 \cdot t] + \\
&+ 1.075299 \cdot \cos[3.21254 \cdot t] + 16.53303 \cdot \cos[3.44193 \cdot t]
\end{aligned} \tag{3.5}$$

Boundary control problem b)

In the case when the vibrating membrane is transmitted from the initial state

$$W|_{t=0} = \varphi(x) = \sin(10x), \quad \dot{W}\Big|_{t=0} = \psi(x) \equiv 0 \tag{3.6}$$

to the terminal state

$$W(x, T_0) = \tilde{\varphi}(x) = \sin(2x), \; \dot{W}\Big|_{t=T_0} = \tilde{\psi}(x) = 2\cos(2x) \tag{3.7}$$

for the boundary control $\mu(t)$ corresponding to general solution (2.13) when $n = 15$ we obtain

$$\begin{aligned}
\mu_4(t) = &\ 1.4382355 \cdot \sin[0.57654 \cdot t] - 7.03202 \cdot \sin[1.14863 \cdot t] - \\
&- 1.68865 \cdot \sin[1.72171 \cdot t] - 0.054924 \cdot \cos[0.57654 \cdot t] - \\
&- 19.36217 \cdot \cos[1.14863 \cdot t] - 0.6583 \cdot \cos[1.72171 \cdot t] - \\
&- ... - 0.4129 \cdot \sin[7.456687 \cdot t] + 1.3702 \cdot \sin[8.030245 \cdot t] - \\
&- 0.3794 \cdot \sin[8.6038 \cdot t] + 0.71193 \cdot \cos[7.456687 \cdot t] - \\
&- 1.2402 \cdot \cos[8.030245 \cdot t] + 0.6841 \cdot \cos[8.6038 \cdot t]
\end{aligned} \tag{3.8}$$

On the other hand, for the boundary control $\mu(t)$ corresponding to the general solution (2.14) when $n = 15$,we obtain

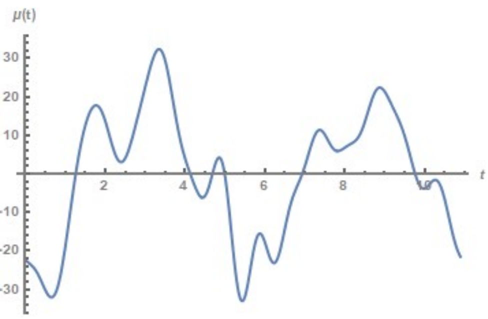
$$\begin{aligned} \mu_5(t) &= -0.024935 \cdot \sin[0.57654 \cdot t] + 0.011172 \cdot \sin[1.14863 \cdot t] - \\ &- 0.01416 \cdot \sin[1.72171 \cdot t] + 0.01911 \cdot \cos[0.57654 \cdot t] + \\ &+ 0.030467 \cdot \cos[1.14863 \cdot t] - 0.0299 \cdot \cos[1.72171 \cdot t] + ... + \\ &- 0.0266 \cdot \sin[7.456687 \cdot t] + 0.034766 \cdot \sin[8.030245 \cdot t] + \\ &+ 0.022 \cdot \sin[8.6038 \cdot t] + 0.03 \cdot \cos[7.456687 \cdot t] - \\ &- 0.01449837 \cdot \cos[8.030245 \cdot t] + 0.01334 \cdot \cos[8.6038 \cdot t] \end{aligned} \tag{3.9}$$

Graphical representations of control edge actions $\mu_4(t)$ and $\mu_5(t)$ are shown in Figures 3a and 3b, respectively.

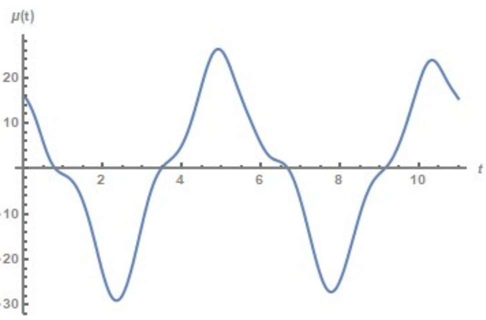
Boundary control problem c)

The problem of boundary control of a membrane changes significantly in the case when the supersonic gas flow is not taken into consideration. Then, the physical parameter $\beta = \chi\rho_\infty M N_x^{-1} = 0$. Therefore, the reduced inhomogeneous equation of forced vibrations (1.6) and the differential equation of the fundamental harmonics of the membrane vibration (2.6) are considerably simplified. As a result, we have forced vibrations of a stretched membrane with boundary excitation $\ddot{\mu}(t)$.

Considering that, the dimensionless physical parameter βl in basic calculations is taken equal to 0.64, then for much smaller values of this parameter, we will have a weak flow around the membrane tape.



(a) boundary control $\mu_4(t)$ in the case of general solution (2.13)



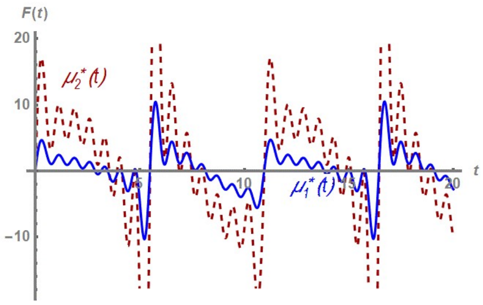
(b) boundary control $\mu_5(t)$ in the case of general solution (2.14)

Figure 3: Boundary controls in the case when the system is transited from state (3.6) to state (3.7)

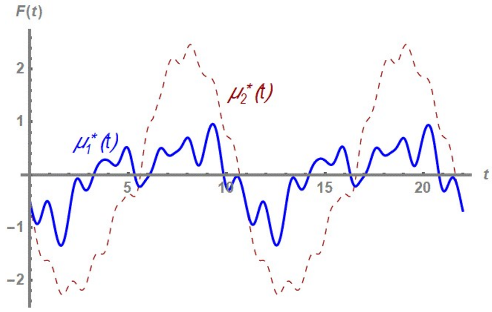
In the case, when the vibrating membrane is transmitted from the initial state

$$W|_{t=0} = \varphi(x) = \sin(10x) \ , \quad \dot{W}\Big|_{t=0} = \psi(x) \equiv 0 \tag{3.10}$$

to the terminal state



(a) The case of the general solution (2.13)
- boundary control $\mu_1^*(t)$ in case of parameter $\beta l\ = 0.64$,
- boundary control $\mu_2^*(t)$ in case of parameter $\beta l\ = 0.0001$,



(b) The case of the general solution (2.14)
- boundary control $\mu_1^*(t)$ in case of parameter $\beta l\ = 0.64$,
- boundary control $\mu_2^*(t)$ in case of parameter $\beta l\ = 0.0001$,

Figure 4: The functions of edge control of the vibration of the membrane tape in the case of the system transition from the state (3.10) to the state (3.11)

$$W(x, T_0) = \tilde{\varphi}(x)\ = \sin(2x)\ ,\ \dot{W}\Big|_{t=T_0} = \tilde{\psi}(x) = 2\cos(2x) \tag{3.11}$$

The boundary control $\mu_4(t)$ for $\beta l\ = 0.64$ and $\mu_5(t)$ for $\beta l\ = 0.0001$, both corresponding to the general solution (2.13), is shown in the figure 4a.

The boundary control $\mu_1^*(t)$ for $\beta l\ = 0.64$ and $\mu_2^*(t)$ for $\beta l\ = 0.0001$, both corresponding to the general solution (2.14) are shown in the figure 4b.

Conclusions

In the control problem by edge action of oscillations of the infinite in one direction membrane tape in a supersonic gas flow, both of the deflection of the membrane tape and the equivalent efforts of the edge control action are decomposed into a Fourier series in terms by eigenforms of the vibration of tape. Mathematically, the problem is reduced to an infinite system of the boundary value problems of ordinary differential equations with matching conditions to the initial and final states of the tape, relative to the true harmonics by the oscillations of the eigenforms of the membrane tape and the corresponding harmonics of the edge action.

The characteristic frequencies both of the true vibration and the control action in a supersonic flow have been determined. The edge control action, as well as the behavior (the law of deflection change) of the oscillating belt under the given initial and final conditions, are found.

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Поступила в редакцию 21.05.2021