АСТРОФИЗИКА

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SOME NEW NONLINEAR RELATIONS OF THE RADIATIVE TRANSFER THEORY

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The paper presents a part of new results concerning the generalization and physical interpretation of Rybicki's quadratic and bilinear relations. The fundamental equations obtained on the base of Ambartsumian's invariance principle and regarded as its extension to all depths in the atmosphere, imply the Q- and R- relations with more general structure than those known up to the present. These equations admit a simple probabilistic interpretation. Some bilinear relations are derived to connect the transfer problems of different sorts. For the sources distributed in the semi-infinite atmosphere by exponential law, the separate Q-and R-relations are obtained.

1. *Introduction.* As it was shown in the G.B. Rybicki's paper [1], for some cases the transfer equation admits integrals that involve quadratic moments of the radiation field. They permit one to generalize to all depths in the atmosphere some surface results as the Hopf-Bronstein equation or the $\sqrt{1-\lambda}$ -law. The cases treated encompass the exponential and power laws for distribution of internal sources of energy. The more general conception of "bilinear integrals" were introduced for quadratic integrals that connect the radiation fields of two separate transfer problems. Although the bilinear integrals were not derived in [1], they obviously were known to the author by that time. Later we shall see that the quadratic and bilinear integrals follow from the simple relations between well-defined physical quantities, in deriving of which no integration is required so that in further discussion we shall prefer the term "relations" to "integrals".

Some generalization of G.B. Rybicki's results for the plane-parallel medium was given by V.V. Ivanov [2], The so-called "two-point" relations were found that couple the intensities of equally directed radiation at two different depths in the atmosphere. While the author calls them also "bilinear relations", by its sense they are out of keeping the definition given in [1]. Similar results were obtained in [3].

Regardless of the new results, the main question on the physical nature of the such kind quadratic relations remains still abstruse. This point is of particular concern also for the second-order escape probability methods recently developed for computational radiative transfer [4].

It is the purpose of this paper to demonstrate the profound connection of the quadratic and bilinear relations with the invariance principle, what suggests a statistical interpretation for these relations. For the first time, there derived two fundamental equations that enable to generalize the major part of hitherto known results for the semi-infinite atmosphere. Bilinear relations for some important transfer problems are given.

2. *The invariance principle and basic equations.* We start with treating the case of monochromatic, isotropic scattering in a semi-infinite, plane-parallel atmosphere. Assume also that the atmosphere is homogeneous and does not contain energy sources. One of the most important characteristics of an atmosphere is its reflectivity that involves also an information on the internal field of radiation established in the presence of initial energy sources.

It is well known [5-7] that the function ρ referred to as «reflection coefficient» or "reflection function" can be found from the separate functional equation obtained on the basis of the Ambartsumian's invariance principle (hereafter we are concerned with the azimuth-averaged quantities)

$$(\eta+\xi)\rho(\eta,\xi)=(\lambda/2)\varphi(\eta)\varphi(\xi) \tag{1}$$

where

$$\varphi(\eta)=1+\eta\int_0^1\rho(\eta,\eta')\,d\eta' \tag{2}$$

is known as Ambartsumian's φ-function. The incidence and reflection angles $\cos^{-1}\xi$ and $\cos^{-1}\eta$ are referenced correspondingly from inward and outward normal directions. The reflection coefficient possesses a symmetry property $\rho(\eta,\xi)=\rho(\xi,\eta)$, resulting from the reciprocity principle of the optical phenomena. Taking together, equations (1) and (2) lead to the separate equation for the function $\varphi(\eta)$

$$\varphi(\eta)=1+(\lambda/2)\eta\int_0^1\varphi(\eta)\varphi(\eta')\,d\eta'/(\eta+\eta') \tag{3}$$

which implies the zero-moment of $\varphi(\eta)$:

$$\alpha_0 = \int_0^1 \varphi(\eta)\, d\eta = 2[1-\sqrt{1-\lambda}]/\lambda. \tag{4}$$

Making use (4), equations (2) and (1) can be rewritten as follows

$$\sqrt{1-\lambda}\varphi(\eta) = 1 - \int_0^1 \rho(\eta,\eta')\eta' d\eta' \tag{5}$$

and

$$(1-\lambda)\rho(\eta,\xi) = (\lambda/2)[1-\int_0^1 \rho(\eta,\eta')\eta' d\eta'][1-\int_0^1 \rho(\xi,\eta')\eta' d\eta']. \tag{6}$$

Now let us introduce into consideration the function $P(\tau,\eta,\mu)$ to denote the surface value of the Green function (i.e. one of its depth arguments is taken to be zero); [8]. In the probabilistic language $P(\tau,\eta,\mu)$ characterizes the probability of the photon exit from the atmosphere in direction μ, if originally it was moving at depth τ with directional cosine η. All the angles are referenced from outward directed normal to the surface of a medium. The symmetry property of the P-function ensues from the reciprocity principle and can be written in the form:

$$|\eta| P(\tau,\eta,\mu) = |\mu| P(\tau,-\mu,-\eta) = |\mu| \widetilde{P}(\tau,\mu,\eta). \tag{7}$$

Here we introduced for brevity the function P with angular arguments referenced from the inward normal direction. This function also admits a probabilistic interpretation such that $\widetilde{P}(\tau,\mu,\eta)\, d\eta$ is the probability that a photon incident on the atmosphere with the directional cosine μ will move (in general, as a result of multiple scatterings) at depth τ within the directional interval $(\eta, \eta + d\eta)$. Keeping in mind the probabilistic meaning of the reflection coeefficient $\eta\rho(\eta,\xi)d\eta$ gives the reflection probability for the photon with the angle of incidence $\cos^{-1}\xi$), we see that $\widetilde{P}(0,\mu,\eta) = \eta\rho(\eta,\mu)$

It is obvious that [6]

$$P(\tau,-\eta,\mu) = \int_0^1 P(\tau,\eta',\mu)\rho(\eta',\eta)\eta' d\eta' \tag{8}$$

$$\widetilde{P}(\tau,\mu,-\eta) = \eta\int_0^1 \widetilde{P}(\tau,\mu,\eta')\rho(\eta,\eta')\, d\eta' \tag{9}$$

Multiplying equation (1) by $P(\tau,\xi,\mu)P(\tau',\eta,\mu')$ and integrating over ξ and η from 0 to 1, we arrive at the first fundamental result

$$\int_{-1}^{+1} P(\tau,\varsigma,\mu)\,P(\tau',-\varsigma,\mu')\,d\varsigma = (\lambda/2)\Big(\int_{-1}^{+1} P(\tau,\varsigma,\mu)\,d\varsigma\Big)\Big(\int_{-1}^{+1} P(\tau',\varsigma,\mu')\,d\varsigma\Big) \qquad (10)$$

in which the relations (8) and symmetry property of the reflection coefficient ρ were used. We shall see later that this formula implies, in particular, all the Q-quadratic and bilinear relations given in [1,2]. To make clear the probabilistic meaning of this equation, we rewrite it in the form

$$\chi(\tau,\mu;\tau',\mu') = \lambda/2 \qquad (11)$$

where

$$\chi(\tau,\mu;\tau',\mu') = \int_{-1}^{+1} P(\tau,\varsigma,\mu)\,P(\tau',-\varsigma,\mu')\,d\varsigma \Big/ \Big(\int_{-1}^{+1} P(\tau,\varsigma,\mu)\,d\varsigma\Big)\Big(\int_{-1}^{+1} P(\tau',\varsigma,\mu')\,d\varsigma\Big)$$

It is seen that χ can be regarded as the correlation coefficient of two random events so that this result can be stated in the following probabilistic language.

Two random events consisting in two photons exit from the semi-infinite atmosphere in certain fixed (diverse, in general) directions, if originally they were moving at some different optical depths in opposite directions, are correlated with the correlation coefficient equaled to $\lambda/2$.

The second fundamental result, generating all the R-quadratic and bilinear relations, can be found by similar manner from equation (6). Multiplying (6) by $\widetilde{P}(\tau,\mu,\eta)\,\widetilde{P}(\tau',\mu',\xi)$ and integrating over η and ξ in the range (0,1), we use equation (9) to obtain

$$(1-\lambda)\int_{-1}^{+1} \widetilde{P}(\tau,\mu,\varsigma)\,\widetilde{P}(\tau',\mu',-\varsigma)\,d\varsigma =$$
$$= (\lambda/2)\Big(\int_{-1}^{+1} \widetilde{P}(\tau,\mu,\varsigma)\varsigma\,d\varsigma/|\varsigma|\Big)\Big(\int_{-1}^{+1} \widetilde{P}(\tau',\mu',\varsigma)\varsigma\,d\varsigma/|\varsigma|\Big) \qquad (12)$$

To assign a probabilistic sense to this result, we rewrite (12) in the form

$$\chi(\tau,\mu;\tau',\mu') = \lambda\kappa(\tau,\mu)\kappa(\tau',\mu')/(1-\lambda) \qquad (13)$$

where

$$\kappa(\tau,\mu) = \left(\int_{-1}^{+1} \tilde{P}(\tau,\mu,\varsigma)\varsigma\, d\varsigma / |\varsigma|\right) \Big/ \left(\int_{-1}^{+1} \tilde{P}(\tau,\mu,\varsigma)\, d\varsigma\right).$$

Thus, it is seen that c also can be regarded as the correlation coefficient for the ingoing radiation, and now it is not constant as for upward directed radiation, but is given by much more complex expression. As should be expected, the upward and inward di-rections are not tantamount. So, this result also can be formulated in probabilistic language.

Two random events, consisting in that two photons incident on the semi-infinite atmosphere in certain fixed (diverse, in general) directions, will move at some (different) depths in opposite directions, are correlated with the correlation coefficient given by (13).

Utilizing the reciprocity principle (7) in (10) and (12), one can write another pair of equations for the functions $\tilde{P}$ and P:

$$\int_{-1}^{+1} \tilde{P}(\tau,\mu,\varsigma)\, \tilde{P}(\tau',\mu',-\varsigma)\, d\varsigma / \varsigma^2 =$$

$$= (\lambda/2)\left(\int_{-1}^{+1} \tilde{P}(\tau,\mu,\varsigma)\, d\varsigma / |\varsigma|\right)\left(\int_{-1}^{+1} \tilde{P}(\tau',\mu',\varsigma)\, d\varsigma / |\varsigma|\right) \tag{14}$$

and

$$(1-\lambda)\int_{-1}^{+1} P(\tau,\varsigma,\mu)\, P(\tau',-\varsigma,\mu')\varsigma^2\, d\varsigma =$$

$$= (\lambda/2)\left(\int_{-1}^{+1} P(\tau,\varsigma,\mu)\varsigma\, d\varsigma\right)\left(\int_{-1}^{+1} P(\tau',\varsigma,\mu')\varsigma\, d\varsigma\right) \tag{15}$$

Equations (10), (12) (alongside with (14) and (15)) involve four free parameters, so that they are pithy and have many consequences. In particular, letting in these equations $\tau=\tau'$ and $\mu=\mu'$, we obtain quadratic relations that are the prototypes of those obtained in [1].

3. *Some consequences and applications*. This section demonstrates how can be found and generalized the existing results and presents some new results.

i. *Ambartsumian's invariance equation*. Setting in (10) $\tau=\tau'$ and taking into account the condition $P(0,\xi,\mu)=\delta(\xi-\mu)$ (δ is the Dirac δ-function), we arrive at the well-known invariance functional equation (1) for the reflection coefficient ρ. To elucidate the similarity of equations (1) and (10), it is expedient to rewrite them as follows:

$$(\eta+\xi)\rho(\eta,\xi) = (2/\lambda)p(0,\eta)p(0,\xi), \tag{16}$$

$$\int_{-1}^{+1} P(\tau,\varsigma,\eta)\,P(\tau',-\varsigma,\xi)\,d\varsigma = (2/\lambda)\,p(\tau,\eta)\,p(\tau',\xi) \tag{17}$$

where $p(\tau,\eta)$ is the photon exit probability [7] designed for the photon *absorbed* at optical depth τ. It is seen that these two relations are constructed by similar fashion with the latter having more general meaning, so that relation (17) (i.e. (10)) can be regarded as the extension of the Ambartsumian's invariance equation to the all depths in the atmospheres concerned with two separate transfer problems.

ii. *The problem of diffuse reflection.* Suppose that the atmosphere is illuminated from outside by parallel beam of radiation of unit intensity (what does not impose any restriction) with directional cosine μ. Using the superscripts + and - to denote the intensity with angular argument $+\eta$ and $-\eta$, respectively, and taking into account the probabilistic interpretation of the function $\tilde{P}$, one can write

$$I^+(\tau,\eta,\mu) = \tilde{P}(\tau,\mu,-\eta)/\eta, \quad I^-(\tau,\eta,\mu) = \tilde{P}(\tau,,\mu,\eta)/\eta. \tag{18}$$

Then equations (14) and (12) correspondingly yield:

$$Q(\tau,\mu;\tau',\mu') = \lambda\,J(\tau,\mu)\,J(\tau',\mu') \tag{19}$$

and

$$(1-\lambda)\,R(\tau,\mu;\tau',\mu') = \lambda\,H(\tau,\mu)\,H(\tau',\mu') \tag{20}$$

in which the notations of [1] are used:

$$J(\tau,\mu) = (1/2)\int_0^1 [I^+(\tau,\eta,\mu) + I^-(\tau,\eta,\mu)]\,d\eta, \tag{21}$$

$$H(\tau,\mu) = (1/2)\int_0^1 [I^+(\tau,\eta,\mu) - I^-(\tau,\eta,\mu)]\,d\eta, \tag{22}$$

$$Q(\tau,\mu;\tau',\mu') =$$

$$= (1/2)\int_0^1 [I^+(\tau,\eta,\mu)\,I^-(\tau',\eta,\mu') + I^+(\tau',\eta,\mu')\,I^-(\tau,\eta,\mu)]\,d\eta \tag{23}$$

$$R(\tau,\mu;\tau',\mu') =$$

$$= (1/2)\int_0^1 [I^+(\tau,\eta,\mu)\, I^-(\tau',\eta,\mu') + I^+(\tau',\eta,\mu')\, I^-(\tau,\eta,\mu)]\eta^2\, d\eta \tag{24}$$

The derivation of the corresponding quadratic relations is straightforward. The angular arguments m and m', which specify the directions of incident radiation in two diverse problems, enter into (19) and (20) as parameters, so that the relations of this type can be written for arbitrary angular distribution of illuminating radiation. Note also that equations (19) and (20) are the further generalization of proper equations of [2] and those given in [9]. They connect the radiation fields at diverse depths pertaining two separate problems of diffuse reflection. Letting $\mu=\mu^1$, we obtain the Ivanov's results; Rybicki's results assume $\tau=\tau'$.

iii. *Uniformly distributed sources.* As in [1,2], the initial sources of energy are assumed to be due to thermal emission, therefore the source function has a form

$$S(\tau) = \lambda J(\tau) + (1-\lambda)\, B \tag{25}$$

where $B=\text{const}$ is related to the Planck function. This problem, as it was shown in [10], is closely connected with that of diffuse reflection treated in the previous subsection. Especially simple relationship exists between radiation fields in the atmosphere with uniformly distributed sources such that (25) holds, and the atmosphere illuminated by isotropic radiation. The plain probabilistic considerations, based on the fact that the photon, moving somewhere in the semi-infinite atmosphere, will either be destroyed or escape it, enable one to write

$$I_*^{\pm}(\tau,\eta) = \int_0^1 I_*^{\pm}(\tau,\eta,\mu)\, d\mu = 1 - I_*^{\pm}(\tau,\eta,B)/B \tag{26}$$

where the intensities relevant to the problem of diffuse reflection are supplied by asterisk. It is also convenient to mark explicitly the internal source in arguments of the proper intensities.

Integrating equations (19) and (20) over μ and μ' from 0 to 1, and incorporating the formulas (26) applied to the two separate problem with different values of sources B and B', after some algebra one can obtain

$$\begin{aligned} Q(\tau,B;\tau',B') &= \lambda J(\tau,B) J(\tau',B') + \\ &+ (1-\lambda)[BJ(\tau',B') + B' J(\tau,B)] - (1-\lambda)\, BB', \end{aligned} \tag{27}$$

$$(1-\lambda)\,R(\tau,B;\tau',B') = \lambda\,H(\tau,B)\,H(\tau',B') + \\ +(1-\lambda)[BK(\tau',B') + B'\,K(\tau,B)] - 1/3\,(1-\lambda)\,BB' \tag{28}$$

where

$$K(\tau,B) = 1/2\int_0^1 [I^+(\tau,\eta,B) + I^-(\tau,\eta,B)]\eta^2\,d\eta, \tag{29}$$

and other quantities are given by (21)-(24) with μ and μ' replaced by B and B', respectively. In terms of the source function equation (27) takes a form:

$$\lambda Q(\tau,B;\tau',B') = S(\tau,B)\,S(\tau',B') - (1-\lambda)\,BB' \tag{30}$$

Again, as in the previous subsection, the bilinear relations are obtained that connect the radiation fields at different depths in two separate problems, what also generalize the existing results. Inasmuch as at the surface ($\tau=\tau'=0$) Q and R vanish, equation (30) yields $S(0) = B\sqrt{1-\lambda}$, where as equation (28) leads to the simple relation between the first- and second-order moments of the φ-function.

It is evident that we could integrate equations (19) and (20) merely over one of two angular variables to obtain relations, connecting radiation fields pertaining two diverse problems, namely, the problem of diffuse reflection and that for an atmosphere with uniformly distributed sources. For instance, integrating (19) and (20) over μ' from 0 to 1 and utilizing (26), one can write

$$Q(\tau,\mu;\tau',B) = J(\tau,\mu)\,S(\tau',B), \tag{31}$$

$$(1-\lambda)\,R(\tau,\mu;\tau',B) = \lambda\,H(\tau,\mu)\,H(\tau',B) + (1-\lambda)\,BK(\tau,B), \tag{32}$$

where Q and R are given by (23) and (24), respectively, with μ' replaced by B.

iv. *Exponential source distribution.* Having bilinear relations for the problem of diffuse reflection, one can readily derive appropriate relations in the case of an atmosphere that contains energy sources with exponential variation over depth. As in [1], the sources of the form $b(\tau,m) = B(1-\lambda)\exp(-m\tau)$ will be treated. However, we start with considering the case in which the initial sources distributed according to the formula $(\lambda/2)\exp[-(\tau/\mu)]$. It is clear physically that the radiation field in such atmosphere differs from that in the atmosphere illuminated by parallel beam of radiation merely by the contribution of non-scattered quanta into downward directed flux. Therefore, one can write

$$I^+(\tau,\eta,\mu) = I_i^+(\tau,\eta,\mu);\; I^-(\tau,\eta,\mu) = I_i^-(\tau,\eta,\mu) + \delta(\eta-\mu)\exp(-\tau/\mu) \tag{33}$$

where I^+ correspond to the case of internal sources. Insertion (33) into (19) leads to the following Q-relation

$$\lambda Q_l(\tau,\mu;\tau',\mu') = S_l(\tau,\mu)S_l(\tau,\mu') + \\ +\lambda/2\left\{I_l^+(\tau,\mu,\mu')\exp(-\tau'/\mu') + I_l^+(\tau',\mu,\mu')\exp(-\tau/\mu)\right\} \tag{34}$$

where $Q_l(\tau,\mu;\tau',\mu')$ is given by (23) with I_l^+ taken in place of I^+, and $S_l(\tau,\mu) = \lambda J_l(\tau,\mu) + \lambda/2\exp(-\tau/\mu)$. The final result for the sources distributed as $b(\tau,m)$ can be found formally by replacing the second arguments of I_l^+ in (48) as follows $\mu \to 1/m$, $\mu' \to 1/m'$, and multiplying both sides of this relation by $(2/\lambda)^2(1-\lambda)^2\, BB'$. It is apparent that such replacement is justified physically only for $m, m' \geq 1$. Remind also that μ and μ' as the third arguments of I_l^+ (and as the second ones of S_l or J_l) specify the problem at hand so that must be replaced by m and m', respectively. As a result the following Q-relation is obtained:

$$\lambda Q(\tau,m;\tau',m') = S(\tau,m)S(\tau',m') - \\ -[B(\tau,m)I^+(\tau',1/m,m') + B(\tau',m')I^+(\tau,1/m',m) \tag{35}$$

where $b(\tau',m') = -B'(1-\lambda)\exp(-m'\tau') - \tau/\mu, S(\tau,m) = \lambda J(\tau,m) + b(\tau,m)$; and $Q(\tau,m;\tau',m')$ is given according to the formula (23) with m and m' substituted for μ and μ', respectively. By analogous manner one can derive R-relation for the same sources, viz.

$$\lambda(1-\lambda)R(\tau,m;\tau',m') = [\lambda H(\tau,m) - b(\tau,m)][\lambda H(\tau',m') - b(\tau',m')] - \\ -(1-\lambda)[(1/m'^2)b(\tau',m')I^+(\tau',1/m',m) + (1/m^2)b(\tau,m)I^+(\tau',1/m,m') \tag{36}$$

where the notation R is adopted for that of (24) with μ, μ' replaced by m, m'.

The derivation of appropriate "two-point" and quadratic relations on the base of (35) and (36) is straightforward. Note that in the specific case of $m = m'$ and $\tau = \tau'$, these equations enable one to exclude $I^+(\tau,1/m,m)$, yielding the combined quadratic relation obtained in [1]. The latter is obviously valid for any value of m. The procedure alluded to above can be carried out with respect to only one triad of variables to give the Q- and R-relations, connecting the problem at issue with that of diffuse reflection.

Thus, it is seen that there exists a class of transfer problems pertaining to the semi-infinite atmosphere, which are related by means of bilinear equations. It can be shown that this class can be extended to include the sources with the power-law distribution in the atmosphere.

НЕКОТОРЫЕ НОВЫЕ НЕЛИНЕЙНЫЕ СООТНОШЕНИЯ ТЕОРИИ ПЕРЕНОСА ИЗЛУЧЕНИЯ

А.Г.НИКОГОСЯН

В статье представлена часть новых результатов, касающихся обобщения и физической интерпретации квадратичных и билинейных интегралов Райбики. Полученные на основе принципа инвариантности Амбарцумяна фундаментальные уравнения могут рассматриваться как распространение этого принципа на все глубины среды. Они позволяют вывести Q- и R-соотношения более общей структуры, чем те, которые извесны в настоящее время, и допускают простое вероятностное истолкование. Получено несколько соотношений, связывающее между собой задачи переноса излучения различного типа. Дается вывод отдельных Q- и R- уравнений для случая, когда источники распределены в полубесконечной среде по экспоненциальному закону.

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