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## A NUMERICAL METHOD FOR INVESTIGATION OF SPHERICAL SYMMETRIC SUPERDENSE BODY EVOLUTION


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**For fully. relativistic strong gravitation fields self consistent equations of hydrodynamics and field equations that describe all kinds of waves in the matter (from small perturbation to strong shock wave) are obtained. An algorithm of numerical method for the solution of these equations in the case of special equations of state for degenerate matter is described.**


1. *Introduction.* After the discovery of quasars (1960) it became necessary to investigate the evolution of supermassive and superdense bodies in terms of fully relativistic hydrodynamics. In the early works of Zeldovich and Novikov [1], Podurets [2], an attempt was made to solve the problem as the problem of dust matter evolution is solved: analytic solution was found in the co–moving frame. But the co–moving frame is unsuitable for numerical calculations because in this frame the gravitational field in the vacuum depends on time.

Today we have a long list of works in the field of numerical investigation of star evolution such as Fackerell Ipser and Thorne [4], Ipser and Thorne [5], Katz and Horvitz [6]. The series of publications by Shapiro and Teukolskey [3] are devoted to numerical solution of Einstein's equations for the dynamic evolution of the collisionless gas of particles and the equations of hydrodynamics are used as consequence from the General Relativity Theory field equations.

All this studies are based on solution of time equations of gravitational field.

Another approach is suggested in the works of Grigorian, Sadoyan [7], Gourgoulhon [8], where the dynamic equations are the hydrodynamic ones and gravitation equations add to the completeness of the system of equations. This approach is preferable because, on the one hand, the hydrodynamic equations are generally independent of the theory of gravitation, on the other hand, it corresponds to the formalism of dynamics in classical mechanics.

Our aim is to investigate the "inner evolution" of spherical superdense matter configuration. The "inner state" can be described by state equation $P(\rho)$ ($P$ is pressure) of matter and two independent functions — by matter density $\rho(r,t)$, and radial velocity $v(r,t)$.

We try to represent these equations with the method of shock waves representation, because this approach solves two difficulties in numerical calculations:

1. The catastrophic increase of numercal errors,

2. The physical and numerical unstability confusion, both occurring during the numerical observation of wave propagation in the matter.

The obtained equations enable to investigate the evolution of all kinds of waves (from small perturbations to strong shock waves) in case of strong gravitational fields.

2. *Basic Equations.* Essentially the algorithm is a chain of calculations starting with the given density and velocity distribution at zero point of time to a new values of this physical quantities at the next moment, using the relativistic hydrodynamic equations.

It is obvious that during the evolution the configuration remains spherically symmetric and the hydrodynamic equation has the following form

$$\nabla \cdot T = 0 \,, \tag{1}$$

where the viscosity in matter is ignored. Here $T = (\rho + P)\, u \otimes u - Pg$ is the energy momentum tensor, $u$ is the 4–dimensional velocity, $g$ is the metric tensor of space time, $P$ and $\rho$ are the pressure and density of the matter in co–moving frame $(c = G = 1)$, $\nabla$ is a covariant differential operator corresponding to the connection of space–time manifold.

We can write the equation (1) in the following form

$$\begin{cases} u\ [\rho] + (\rho + P)\nabla\cdot u = 0 \\ n\ [P] - (\rho + P) < n, \nabla u >= 0 \end{cases} \tag{2}$$

where $u\ [\rho] \equiv \nabla_u \rho$, $n\ [P] \equiv \nabla_n P$.

The vector $n$ is a 4–dimensional vector orthogonal to the vector $u$, to isobar spheres and is normalized ($u \cdot n = 0$ and $n \cdot n = -1$). In the coordinate frame ( $t, r, \theta, \varphi$ )

$$u = u^t \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial r} \right) ,$$

$$n = u^t \left(v\frac{\partial}{\partial t} + \frac{\partial}{\partial r}\right) ,$$

where $v = \frac{dr}{dt}$ is the coordinate velocity, $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial r}$ are orts of the coordinate system, $u^t$ is the component of 4–dimensional velocity.

The most common type of metric in the spherical symmetric case is

$$ds^2 = e^{2\Phi} dt^2 - e^{2\psi} (dr + \beta dt)^2 - r^2 e^{2\chi}(d\theta^2 + \sin^2\theta\, d\varphi^2) \qquad (3)$$

We use the thetrad basis

$$\begin{aligned} \omega^o &= e^{\Phi} dt , \\ \omega^1 &= e^{\psi}(dr + \beta dt) , \\ \omega^2 &= re^{\chi} d\theta , \\ \omega^3 &= re^{\chi} \sin\theta\, d\varphi . \end{aligned} \qquad (4)$$

In the conjugate basis $(e_\mu)$ the vectors $u$ and $n$ are of the form: $u = \gamma (e_o + w\, e_1)$, $n = \gamma (w\, e_o + e_1)$ ,where $w = (v - \beta)\, e^{\Phi - \psi}$ and $\gamma = (1 - w^2)^{-1/2}$, where $w$ is the actual velocity on the sphere for an observer in the infinity, and $\beta$ is the velocity of the frame of an observer in the infinity. It is easy to see that for the 1–form of connection $\omega = \Gamma^{\alpha}_{\beta\gamma} \omega^{\gamma} E^{\beta}_{\alpha}$

$$\begin{aligned} \omega = \alpha\, (E^o_1 + E^1_o) + a\, [\omega^2(E^o_2 + E^2_o) + \omega^3(E^o_3 + E^3_o)] \, + \\ + b\, [\omega^2(E^1_2 - E^2_1) + \omega^3(E^1_3 - E^3_1)] + c\, \omega^3(E^2_3 - E^3_2) \end{aligned} \qquad (5)$$

where $E^{\nu}_{\mu}$ is the basis for $4 \times 4$ matrix algebra, $\alpha = f\, \omega^o + h\, \omega^1$ is a 1–form and $a, b, c$ are scalar coefficients. So for such a choice of metric (4) the mentioned coefficients are

$$a = e_o[\chi] \ , \ b = e_1[\chi + \ln(r)] \ , \ c = e_2[\ln(sin\,\theta)] \ ,$$
$$f = e_1[\Phi] \ , \ h = e_o[\psi] - e^{\psi - \Phi} e_1[\beta] \ .$$

Let us return to the system (2) and put there $w = th\,(\zeta)$. We obtain

$$\nabla \cdot u = n\ [\zeta] + \alpha\,(n) + 2u\ [\chi + ln(r)]$$

$$<u, \nabla_u u> = -u[\zeta] - \alpha(u) \tag{6}$$

where $\alpha(u)$ and $\alpha(n)$ are the values of 1–form $\alpha$ on $u$ and $n$ vector fields. The overall information about the gravitational field is contained in the 1–form $\alpha$. $\beta$ and $\chi$ depends on the coordinate frame of the observer.

If we put the expression (6) in the system (2) we obtain two first order hyperbolic differential equations. The tangent vectors of characteristic lines of this equations are $D_{\pm} = u \pm cn$, where $c = \left(\frac{\partial P}{\partial \rho}\right)_s^{1/2}$ is the velocity of sound in stellar matter.

Now the system (2) can be written in the following form

$$\begin{cases} u\,[I] + cn\,[\zeta] + \alpha(cn) + 2cu\,[\chi + \ln(r)] = 0 \\ cn\,[I] + u\,[\zeta] + \alpha(u) = 0 \end{cases}$$

where

$$I(\rho) = \int \frac{c\,d\rho}{\rho + P}\,. \tag{7}$$

By adding and subtracting these equations we obtain

$$D_{\pm}[J_{\pm}] + \alpha(D_{\pm}) \pm 2cu[\chi + \ln(r)] = 0 \tag{8}$$

where $J_{\pm} = \zeta \pm I(\rho)$.

This system of equations is a system for shock spherical waves. Remembering that $\alpha = f\omega^0 + h\omega^1$ and $u = ch(\zeta)\,e_0 + sh(\zeta)\,e_1$, $n = sh(\zeta)\,e_0 + ch(\zeta)\,e_1$, $\alpha(D_{\pm}) = (h \pm cf)\,sh(\zeta) + (f \pm ch)\,ch(\zeta)$. We have

$$D_{\pm} = (ch(\zeta) \pm c\,sh(\zeta))\,e^{-\Phi}\frac{\partial}{\partial t} + (sh(\zeta) \pm c\,ch(\zeta))\,e^{-\psi}\frac{\partial}{\partial r} =$$
$$= \gamma(1 \pm wc)\,e^{-\Phi}\left[\frac{\partial}{\partial t} + (w \pm V_c^{\pm})\frac{\partial}{\partial r}\right]. \tag{9}$$

So

$$v_c^{\pm} = c\,\frac{1 - w^2}{1 \pm wc}\,e^{\Phi - \psi} \tag{10}$$

is the real velocity of sound i the direction of the flow (+) and opposite direction (–) in the frame connected with the center of the star. It is not difficult to see, that the propagation of sound depends not only upon the gravitation (like the red shift effect of light), but also the flow of the matter (Doppler efect), which results in anisotropy of propagation.

3. *Boundary Conditions.* Up to now we did not fix the gravitational field equations and the observer's coordinate frame to describe the hydrodynamics independently of the gravitational field theory. It is necessary to have conditions for gravitational field in space section for each moment. We assume that spherical pulsing stars have no gravitational radiation, thus the $f$, $h$, $\Phi - \Psi$ functions in (9) can be defined for given matter state (density and velocity distributions).

In this work we illustrate our method by chossing Einstein GR theory. In GRT dynamic equations (8) are consequences of the field equations

$$G = 8\pi T \ . \tag{11}$$

The rest of this system gives us equations for connection. Tensor G in field equation (11) defines by 2–form of space–time manifold curvature

$$\Omega = d\alpha\,(\overset{o}{E}_1 + \overset{1}{E}_o) + \bar{\beta}\wedge[\overset{2}{\omega}(\overset{o}{E}_2 + \overset{2}{E}_o) + \overset{3}{\omega}(\overset{o}{E}_3 + \overset{3}{E}_o)] +$$
$$+\bar{\gamma}\wedge[\overset{2}{\omega}(\overset{1}{E}_2 - \overset{2}{E}_1) + \overset{3}{\omega}(\overset{1}{E}_3 - \overset{3}{E}_1)] - (\kappa^2 + a^2 - b^2)\,\overset{2}{\omega}\wedge\overset{3}{\omega}(\overset{2}{E}_3 - \overset{3}{E}_2)$$

where

$$A\,\overset{1}{\omega}\wedge\overset{o}{\omega} \equiv d\,\alpha = (f_{,1} - h_{,o} + f^2 - h^2)\,\overset{1}{\omega}\wedge\overset{o}{\omega} \ ,$$
$$B\,\overset{o}{\omega} + C\,\overset{1}{\omega} \equiv \bar{\beta} = -\,da + b\,\alpha - a^2\,\overset{o}{\omega} - ab\,\overset{1}{\omega} \ ,$$
$$C\,\overset{o}{\omega} + D\,\overset{1}{\omega} \equiv \bar{\gamma} = -\,db + a\,\alpha - b^2\,\overset{1}{\omega} - ab\,\overset{o}{\omega} \ ,$$
$$E + b^2 - a^2 \equiv k = e^{-2\chi}/r^2 \ ,$$
$$G = (E + 2D)\,\overset{o}{\omega}\otimes\overset{o}{\omega} + (2B - E)\,\overset{1}{\omega}\otimes\overset{1}{\omega} + 2C\,(\overset{o}{\omega}\otimes\overset{1}{\omega} + \overset{1}{\omega}\otimes\overset{o}{\omega}) +$$
$$+ (A + B - D)\,(\overset{2}{\omega}\otimes\overset{2}{\omega} + \overset{3}{\omega}\otimes\overset{3}{\omega}) \ .$$

We choose the Schwarzschild coordinates, where $\chi = 0$ and $\beta = 0$, that enables us to find analytic static solutions of the gravitational field in the external region of the star (Birchoff theorem). So we have

$$\begin{cases} E + 2D = 8\pi\,\varepsilon \equiv 8\pi\,(\rho ch^2(\zeta) + P sh^2(\zeta)) \ , \\ 2B - E = 8\pi\Pi \equiv 8\pi\,(\rho sh^2(\zeta) + P ch^2(\zeta)) \ , \\ C = -4\pi\,(\rho + P)\, sh\,(\zeta)\ ch\,(\zeta) \ . \end{cases}$$

This system can be solved analytically.

$$h = C/b\,,\ f = B/b\,,\ b^2 = r^{-2} - E\,,\ b = 1/(re^{\psi})$$

we obtain

$$2rb\,b' + 3b^2 - r^{-2} = -8\pi\varepsilon \text{ or}$$
$$r^2 b^2 = e^{-2\psi} = 1 - 2mr^{-1} \tag{12}$$

here $m$ is the accumulated mass

$$m = 4\pi \int \varepsilon r^2 dr \tag{13}$$

Using (12) we have

$$\psi' = \frac{4\pi \varepsilon r^3 - m}{r\,(r - 2m)}$$

and

$$\Phi' = \frac{4\pi \Pi r^3 + m}{r\,(r - 2m)}$$

(generalized Oppenheimer–Volkoff equation).

Subtracting the last two equations and integrating we have

$$\Phi - \psi = \int \frac{2m + 4\pi\,(P - \rho)\,r^3}{r\,(r - 2m)}\,dr \tag{14}$$

and

$$\begin{cases} f = \dfrac{4\pi\,\Pi r^3 + m}{r^3 b} \\ h = -\,4\pi\,(P + \rho)\,sh\,(2\zeta)/b \end{cases} \tag{15}$$

Finally, the equations (8) can be rewritten as

$$\hat{D}_{\pm}\,[J_{\pm}] + \frac{r^2\,\tilde{\alpha}}{r - 2m} \pm \frac{2sh\,(\zeta)}{r}\,c = 0 \tag{16}$$

where

$$\tilde{\alpha} = (\tilde{h} \pm c\tilde{f})\,sh\,(\zeta) + (\tilde{f} \pm c\tilde{h})\,ch\,(\zeta)\,,$$
$$\tilde{h} = -4\pi\,(P + \rho)\,sh\,(2\zeta)\,,\ \tilde{f} = 4\pi\Pi + mr^{-3} \quad \text{and}$$
$$\hat{D}_{\pm} \equiv \left(ch\,(\zeta) \pm c\,sh\,(\zeta)\right) e^{\psi - \Phi}\,\partial/\partial t + (sh\,(\zeta) \pm c\,ch\,(\zeta))\,\partial/\partial r$$

The equations (16) with conditions (13) and (14) became a complete dynamic system for $J_+$ and $J_-$ functions. So the unknown functions for dynamic problems can be obtained using

$$\begin{cases} w = th\left((J_+ + J_-)/2\right) \\ \rho = I^{-1}\left((J_+ - J_-)/2\right) \\ P = P(\rho) \end{cases} \tag{17}$$

Here $I^{-1}$ is the opposite function for integral I (7).

4. *Numerical Algorithm.* For such nonlinear partial differential coupled set of hyperbolic equations it is impossible to find an analytic solution. That is why we need to offer an effective numerical method to solve such equations. Numerical algorithms for solving hyperbolic partial differential equations are well developed and have been implemented in many computer codes. One of such codes designed for solving dynamic problems in astrophysics is ZEUS [9] that uses hydrodynamic equations for nonrelativistic mechanics, taking into account the newtonian weak gravitational field based on the method of finite differences. In our problem we use the method of characteristics. As for the case of spherical symmetry the center of the configuration is a singular point of coordinate system and we must have a special boundary conditions. The functions must satisfy the conditions of regularity in the center of configuration at all moments. It means that for each moment the functions $\rho$ and w near the center have the following behavior

$$\rho(r, t) \simeq \rho_c(t) + \tilde{\rho}(t) r^2 + \ldots$$
$$w(r, t) \simeq \tilde{w}(t) r + \ldots$$

The physical meaning of these conditions is that in the center of the star every type of motion disappears except the change of density and gravitational potential $\Phi$. In other words, we can say that for spherically distributed configuration this very close region of center is a homogeneous core and its density increases or decreases respectively to the direction of the flux of matter. These conditions can be repeated for wave functions $J_\pm$. In the region of the center $J_+(0, t) = -J_-(0, t)$, that means that for each incoming wave an opposite outgoing wave in the core is generated and the flux of these waves must be equal to each other: $J_+(r_o, t) - J_+(0, t) = J_-(r_o, t) - J_-(0, t)$. So the central boundary conditions can be written as follows

$$\begin{cases} J_+(0\,,t)=\frac{1}{2}\left(J_+(r_o\,,t)-J_-(r_o\,,t)\right) \\ J_-(0\,,t)=\frac{1}{2}\left(J_-(r_o\,,t)-J_+(r_o\,,t)\right) \end{cases}$$

where $r_0$ is the radius of the homogeneous core. Other boundary conditions determine the surface of configuration where $P=\rho=0$. It is known, that the vacuum solution of the gravitational field does not depend on time and for $r>R$ we have

$$\Phi(r\,,t)=-\Psi(r\,,t)=\ln\left(1-\frac{2M}{r}\right) \tag{18}$$

where $M=m(R)$ is the total mass of the star and $R$ — is the radius. The continuity of the inner and outer solutions of the gravitational field is demanded though the wave functions can be interrupted. It is easy to see that integral I (7) is null out of the matter distribution and we can put $J_\pm(r\,,t)=0$ for $r>R$. Using these conditions we can find the right values for the functions (13), (14) in the center

$$\Psi(0\,,t)=0\,,\ \ \Phi(0\,,t)=\ln\left(1-\frac{2M}{R}\right)+$$
$$+\int_R^0\frac{2m+4\pi(P-\rho)r^3}{r(r-2m)}\,dr\,,\ \ m(0\,,t)=0\ . \tag{19}$$

Our algorithm consists of the following steps:
1) for $t=0$ we determine the functions $J_+(r\,,0)$ and $J_-(r\,,0)$ corresponding to the given physical problem;
2) our aim is to determine the functions $J_+$, $J_-$ for each moment $t$ satisfying the boundary conditions (18–19);
The exact code must limit the time step to satisfy the numerical stability condition. It can be understood as a limitation of the distance that information can travel in one time step to be smaller than one step of space network

$$\Delta t\leq\min(\Delta r)/(|w|+c)\ .$$

The transfer from $t_1\to t_2=t_1+\Delta t$ is realized towards the characteristics of operators $\hat{D}_\pm$. Characteristics are described by parameters $s_+$ and $s_-$ of $D_+$ and $D_-$ correspondingly, and for this short interval $\Delta t$ assumed to be linear approximated. Due to (9) for each event $(r_o\,,t_o)$ we can find spheres with radius $r_\pm=r_o+(w\pm V_o^\pm)\,\Delta t$ where the waves $J_+$ and $J_-$ propagates

$$\Delta s_{\pm} = s_{\pm}(t) - s_{\pm}(t_o)\Big|_{r=r} = \left(ch\,(\zeta) \pm c\,sh\,(\zeta)\right)^{-1} e^{\psi - \Phi} \;.$$

Finally,

$$J_{\pm}(r^{\pm}, t) = J_{\pm}(r_o, t_o) + D_{\pm}\,[J_{\pm}]((r_o, t_o)\,\Delta s_{\pm} \;.$$

4) To determine the functions from the equations (8) the transformations (17) must be used.

5. *Special Cases.* The initial conditions of the problem are:
a) for given physical problem we need to have the state equations of matter as a functional connection between pressure and density $P = P(\rho)$;
b) the initial state of the configuration is determined by two functions $\rho(r, 0)$ and $w(r, 0)$ for an arbitrary current moment.

Let's consider two equations of state for which we can simplify the procedure of transformation (17). Equations of state of degenerated relativistic ideal gas are of the form

$$\begin{cases} \rho = k\,(shT - T) \\ P = \dfrac{k}{3}\left(sh\,T - 8sh\,\dfrac{T}{2} + 3T\right) \end{cases}$$

where $k$ is the coefficient describing the sort of matter, $T$ is the parameter proportional to Fermi momentum of barions in degenerate matter. The speed of the sound in this case is

$$c = \frac{1}{3}\,th^2\,T/4$$

and integral

$$I(T) = T/4\sqrt{3}.$$

The equation of state of polytrops is

$$P = k\rho^{1+1/n}$$

where k is a coefficient, n is the index of the polytrops. The speed of the sound can be obtained by $c = \left(k\,(1 + 1/n)\,\rho^{1/n}\right)^{1/2}$. To obtain the connection with the wave functions we denote a new parameter $\Theta$:

$$\rho = \Theta^{2n}/\sqrt{k}\,,\;\; P = \sqrt{k}\cdot\Theta^{2(n+1)}$$

and integral (7) will have the form

$$I(\Theta) = 2\sqrt{n(n+1)} \cdot arctn\,\Theta$$

Finally the connections (16) between wave functions and the couple of the velocity and the density of the matter is of the form

$$\text{ideal gas — } \; T = 2\sqrt{3}(J_+ - J_-); \; w = th\left((J_+ + J_-)/2\right)$$
$$\text{polytrops — } \Theta = tn\left((J_+ - J_-)/4\sqrt{n(n+1)}\right);$$
$$w = th\left((J_+ + J_-)/2\right)$$

The next special case is presented by the configurations consisting of the dust matter, for which the pressure and consequently the speed of the sound are equal to zero. For this case we need to change our system (17), because it is invalid for c = 0. However we can obtain the equations

$$\begin{cases} u[\xi] + \dfrac{r^2}{r-2m}\alpha(u) = 0 \\ u[\xi] + u\,[ln(\rho + P)] + \dfrac{r^2}{r-2m}\alpha(n) + \dfrac{2sh\,\xi}{r} = 0 \end{cases}$$

when c tends to zero.
This equations are valid for the regions of phase transitions in any equation of state too ($P = \text{const}, c = 0$). For dust matter $P = 0$,

$$\begin{cases} \alpha(u) = -\rho sh2\xi + \rho sh^2\xi ch\xi + \dfrac{m}{r^3}ch\xi \\ \alpha(n) = \left(\rho sh^2\xi + \dfrac{m}{r^3}\right)sh\xi - \rho sh2\xi ch\xi \end{cases}$$

We will present the numerical results calculated by this algorithm in a separate paper.

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## ЧИСЛЕННЫЙ МЕТОД ДЛЯ ИЗУЧЕНИЯ ЭВОЛЮЦИИ СФЕРИЧЕСКИ–СИММТЕРИЧЕСКИХ СВЕРХПЛОТНЫХ НЕБЕСНЫХ ТЕЛ

О.А.ГРИГОРЯН, А.А.САДОЯН

Для сильных гравитационных полей получены самосогласованные уравнения гидродинамики и поля, которые описывают распространение всех видов волн в материи (от малых колебаний до ударных волн). Получен алгоритм численного решения этих уравнений для некоторых уравнений состояния вырожденного вещества.

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