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ON ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF n-TH ORDER EMDEN-FOWLER TYPE DIFFERENCE EQUATIONS WITH ADVANCED ARGUMENT

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Abstract. We study oscillatory properties of solutions of the Emden-Fowler type difference equation $\Delta^{(n)}u(k) + p(k)\,\big|u(\sigma(k))\big|^{\lambda}\operatorname{sign} u(\sigma(k)) = 0$, where $n \geq 2$, $0 < \lambda < 1$, $p : \mathbb{N} \to \mathbb{R}_+$, $\sigma : \mathbb{N} \to \mathbb{N}$ and $\sigma(k) \geq k + 1$ for $k \in \mathbb{N}$. Sufficient conditions of new type for oscillation of solutions of the above equation are established. Analogous results for linear ordinary and nonlinear functional differential equations see in [1–8].

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1. Introduction

This work is dedicated to the study of oscillatory properties of the difference equation

$$\Delta^{(n)}u(k) + p(k)\,\big|u(\sigma(k))\big|^{\lambda}\operatorname{sign} u(\sigma(k)) = 0, \tag{1.1}$$

where $n \geq 2$, $p : \mathbb{N} \to \mathbb{R}_+$, $\sigma : \mathbb{N} \to \mathbb{N}$ and

$$0 < \lambda < 1, \quad \sigma(k) \geq k + 1 \quad \text{for} \quad k \in \mathbb{N}. \tag{1.2}$$

Here $\Delta^{(1)}u(k) = u(k+1) - u(k)$, $\Delta^{(i)} = \Delta^{(1)} \circ \Delta^{(i-1)}$ $(i = 2, \dots, n)$. It will always be assumed that the condition

$$p(k) \geq 0 \quad \text{for} \quad k \in \mathbb{N} \tag{1.3}$$

is fulfilled. The following notation will be used throughout the work:

Let $k_0 \in \mathbb{N}$. By $\mathbb{N}^+_{k_0}$ ($\mathbb{N}^-_{k_0}$) we denote the set of natural number $\mathbb{N}^+_{k_0} = \{k_0, k_0 + 1, \dots\}$ ($\mathbb{N}^-_{k_0} = \{1, 2, \dots, k_0\}$).

Definition 1.1. Let $k_0 \in \mathbb{N}$. We will call a function $u : \mathbb{N}^+_{k_0} \to \mathbb{R}$ a proper solution of the equation (1.1), if it satisfies (1.1) on $\mathbb{N}^+_{k_0}$ and

$$\sup\left\{\big|u(i)\big| : i \in \mathbb{N}^+_k\right\} > 0 \ \text{ for any } \ k \in \mathbb{N}^+_{k_0}.$$

Definition 1.2. We say that a proper solution $u : \mathbb{N}^+_{k_0} \to \mathbb{R}$ of equation (1.1) is oscillatory, if for any $k \in \mathbb{N}^+_{k_0}$ there exist $k_1; k_2 \in \mathbb{N}^+_k$ such that $u(k_1)\,u(k_2) \leq 0$. Otherwise the solution is called nonoscillatory.

Definition 1.3. We say that equation (1.1) has Property **A** if any its proper solutions is oscillatory when n is even and either is oscillatory or satisfies

$$|\Delta^{(i)}u(k)| \downarrow 0 \quad \text{as} \quad k \uparrow +\infty, \quad k \in \mathbb{N} \quad (i = 0, \dots, n-1), \quad \text{when } n \text{ is odd.} \tag{1.4}$$

Some results analogous to those of the paper are given without proofs in [9–11]. The problem of establishing sufficient conditions for the oscillation of all solutions to the second order linear and nonlinear difference equations see in [12–16].

2. On some classes of nonoscillatory discrete functions

Lemma 2.1. *Let $n \geq 2$, $k_0 \in \mathbb{N}$, $u : \mathbb{N}^+_{k_0} \to \mathbb{R}$ and $u(k) > 0$, $\Delta^{(n)}u(k) \leq 0$ for $k \in \mathbb{N}^+_{k_0}$, $\Delta^{(n)}u(k) \not\equiv 0$ for any $s \in \mathbb{N}^+_{k_0}$ and $k \in \mathbb{N}^+_s$. Then there exist $k_1 \in \mathbb{N}^+_{k_0}$ and $\ell \in \{0, \dots, n\}$ such that $\ell + n$ is odd and*

$$\begin{gathered} \Delta^{(i)}u(k) > 0 \quad \text{for} \quad k \in \mathbb{N}^+_{k_1} \quad (i = 0, \dots, \ell), \\ (-1)^{i+\ell}\Delta^{(i)}u(k) > 0 \quad \text{for} \quad k \in \mathbb{N}^+_{k_1} \quad (i = \ell, \dots, n-1), \\ \Delta^{(n)}u(k) \leq 0 \quad \text{for} \quad k \in \mathbb{N}^+_{k_1}. \end{gathered} \tag{2.1}$$

Proof. The Lemma follows immediately from the fact that, if $u(k) > 0$ and $\Delta^{(2)}u(k) \leq 0$ for $k \in \mathbb{N}^+_{k_0}$, then there exist $k_1 \in \mathbb{N}^+_{k_0}$, such that $\Delta^{(1)}u(k) > 0$ for $k \in \mathbb{N}_{k_1}$. $\square$

Remark 2.1. It is obvious that if $u; v : \mathbb{N} \to \mathbb{R}$ and $\Delta^{(i)}u(k_0) = \Delta^{(i)}v(k_0)$ $(i = 0, \dots, m-1)$ and $\Delta^{(m)}u(k) = \Delta^{(m)}v(k)$ for $k \in \mathbb{N}^+_{k_0}$ (for $k \in \mathbb{N}^-_{k_0}$). Then $u(k) = v(k)$ for $k \in \mathbb{N}^+_{k_0}$ (for $k \in \mathbb{N}^-_{k_0}$).

Lemma 2.2. *Let $u : \mathbb{N} \to \mathbb{R}$, $m; s \in \mathbb{N}$. Then*

$$\begin{aligned} \Delta^{(i)}u(k) = {} & \sum_{j=i}^{m-1} \frac{\Delta^{(j)}u(s)}{(j-i)!} \prod_{r=1}^{j-i}(k-s-r+1) + \frac{1}{(m-i-1)!} \\ & \times \sum_{j=s}^{k} \prod_{r=1}^{m-i-1} (k-j-r+1)\Delta^{(m)}u(j-1), \quad i = 0, \dots, m-1, \quad \text{for} \ k \in \mathbb{N}^+_s, \end{aligned} \tag{2.2}$$

where

$$\Delta^{(m)}u(s-1) = 0, \quad \prod_{r=1}^{0}(k-s-r+1) = 1, \tag{2.3}$$

and

$$\begin{aligned} \Delta^{(i)}u(k) = {} & \sum_{j=i}^{m-1} \frac{\Delta^{(j)}u(s)}{(j-i)!} \prod_{r=1}^{j-i}(k-s-r+1) - \frac{1}{(m-i-1)!} \\ & \times \sum_{j=k}^{s} \prod_{r=1}^{m-i-1} (k-j-r+1)\Delta^{(m)}u(j), \quad i = 0, \dots, m-1 \ \text{for} \ k \in \mathbb{N}^-_s, \end{aligned} \tag{2.4}$$

where

$$\Delta^{(m)}u(s) = 0, \quad \prod_{r=1}^{0}(k-s-r+1) = 1. \tag{2.5}$$

Proof. Denote

$$u_1(k) = \Delta^{(i)}u(k), \tag{2.6}$$

$$\begin{aligned} u_2(k) &= \sum_{j=i}^{m-1} \frac{\Delta^{(j)}u(s)}{(j-i)!} \prod_{r=1}^{j-i}(k-s-r+1) \\ &+ \frac{1}{(m-i-1)!} \sum_{j=s}^{k} \prod_{r=1}^{m-i-1}(k-j-r+1)\Delta^{(m)}u(j-1), \quad k \in \mathbb{N}_s^+. \end{aligned} \tag{2.7}$$

Since

$$\begin{aligned} \Delta^{(1)} \prod_{r=1}^{j-i}(k-s-r+1) &= \prod_{r=1}^{j-i-i}(k+2-r-s) - \prod_{r=1}^{j-i}(k+1-r-s) \\ &= \prod_{r=0}^{j-i-1}(k+1-r-s) - \prod_{r=1}^{j-i}(k+1-r-s) = (j-i)\prod_{r=1}^{j-i-1}(k+1-r-s), \end{aligned}$$

according to (2.3), (2.6) and (2.7) we get $\Delta^{(j)}u_1(s) = \Delta^{(j)}u_2(s)$ $(j = 0, \dots, m-i-1)$ and $\Delta^{(m-i)}u_1(k) = \Delta^{(m-i)}u_2(k)$ for $k \in \mathbb{N}_s^+$. Therefore, the conditions of Remark 2.1 are fulfilled, which proves that the equality (2.2) is valid.

By (2.5), similarly we can prove that the equality (2.4) is valid, which proves the lemma. □

Lemma 2.3. *Let $u : \mathbb{N} \to \mathbb{R}$, $m; s \in \mathbb{N}$. Then the equality holds*

$$\begin{aligned} \sum_{i=s}^{k} i^{m-j-1}\Delta^{(m)}u(i) &= \sum_{i=j}^{m-1}(-1)^{m+i-1}\Delta^{(i)}u(k+1)\Delta^{(m-i-1)}(k+i+1-m)^{m-j-1} \\ &- \sum_{i=j}^{m-1}(-1)^{m+i-1}\Delta^{(i)}u(s+1)\Delta^{(m-i-1)}(s+i+1-m)^{m-j-1} \quad \textit{for} \quad k \in \mathbb{N}_s^+, \end{aligned} \tag{2.8}$$

where

$$\Delta^{(m)}u(s) = 0 \tag{2.9}$$

and

$$\begin{aligned} &-\sum_{i=k}^{s}(i+1)^{m-j-1}\Delta^{(m)}u(i+1) \\ &= \sum_{i=j}^{m-1}(-1)^{m+i-1}\Delta^{(i)}u(k+1)\Delta^{(m-i-1)}(k+i+1-m)^{m-j-1} \\ &- \sum_{i=j}^{m-1}(-1)^{m+i-1}\Delta^{(i)}u(s+1)\Delta^{(m-i-1)}(s+i+1-m)^{m-j-1} \quad \textit{for} \quad k \in \mathbb{N}_s^-, \end{aligned} \tag{2.10}$$

where

$$\Delta^{(m)}u(s+1) = 0. \tag{2.11}$$

Proof. Let $u, v : \mathbb{N} \to \mathbb{R}$, then $\Delta^{(1)}\big[u(k)\, v(k)\big] = v(k+1)\Delta^{(1)}u(k) + u(k)\Delta^{(1)}v(k)$. Therefore

$$\begin{aligned}
&\Delta^{(1)}\Big(\sum_{i=j}^{m-1}(-1)^{m+i-1}\Delta^{(i)}u(k+1)\Delta^{(m-i-1)}(k+i+1-m)^{m-j-1}\Big)\\
&= \sum_{i=j}^{m-1}(-1)^{m+i-1}\Delta^{(i+1)}u(k+1)\Delta^{(m-i-1)}(k+i+2-m)^{m-j-1}\\
&+ \sum_{i=j}^{m-1}(-1)^{m+i-1}\Delta^{(i)}u(k+1)\Delta^{(m-i)}(k+i+1-m)^{m-j-1}.
\end{aligned}$$

Since $\Delta^{(m-j)}(k+i+1-m)^{m-j-1} = 0$, then

$$\Delta^{(1)}\Big(\sum_{i=j}^{m-1}(-1)^{m+i-1}\Delta^{(i)}u(k+1)\Delta^{(m-i-1)}(k+i+1-m)^{m-j-1}\Big) = (k+1)^{m-j-1}\Delta^{(m)}u(k+1).$$

By (2.9), ((2.11)) the equality (2.8) (the equality (2.10)) holds. $\square$

Lemma 2.4. *Let $u : \mathbb{N} \to \mathbb{R}$, $k_0; n \in \mathbb{N}$ and*

$$(-1)^i\Delta^{(i)}u(k) > 0 \ \ (i = 0, \dots, n-1), \ \ (-1)^n\Delta^{(n)}u(k) \geq 0 \ \ \textit{for} \ \ k \in \mathbb{N}^+_{k_0}. \tag{2.12}$$

Then

$$\sum_{k=1}^{+\infty} k^{n-1}\big|\Delta^{(n)}u(k)\big| < +\infty, \tag{2.13}$$

$$\big|\Delta^{(i)}u(k)\big| \geq \frac{1}{(n-i-1)!}\sum_{j=k}^{+\infty}\prod_{r=1}^{n-i-1}(j-k+r-1)\big|\Delta^{(n)}u(j)\big| \quad \textit{for} \ \ k \in \mathbb{N}^+_{k_0}, \ \ (i = 0, \dots . n-1), \tag{2.14}$$

$$u(k) \geq u(s) + \sum_{j=1}^{n-1}\frac{\big|\Delta^{(j)}u(s)\big|}{j!}\prod_{r=1}^{j}(j-k+r-1) \ \ \textit{for} \ \ s \geq k. \tag{2.15}$$

Proof. Let $k_0 \leq k < s$. It can be assumed without loss of generality that $\Delta^{(n)}u(s) = 0$. Let $m = n$, according to (2.12) from (2.4) with $s \to +\infty$, we can readily obtain (2.13) and (2.14). As to (2.15), it is immediate consequence of (2.4). $\square$

Lemma 2.5. *Let $u : \mathbb{N} \to \mathbb{R}$ and for some $k_1 \in \mathbb{N}$ and $\ell \in \{1, \dots, n-1\}$, (2.1) be fulfilled. Then*

$$\sum_{k=1}^{+\infty} k^{n-\ell-1}\big|\Delta^{(n)}u(k)\big| < +\infty, \tag{2.16}$$

there exists $k_2 \in \mathbb{N}^+_{k_1}$ *such that*

$$\big|\Delta^{(i)}u(k)\big| \geq \frac{1}{(n-i-1)!}\sum_{j=k}^{+\infty}\prod_{r=1}^{n-i-1}(j+r-k-1)\big|\Delta^{(n)}u(j)\big| \quad \text{for } k \in \mathbb{N}^+_{k_2} \ (i=\ell,\dots,n-1), \tag{2.17}$$

$$\Delta^{(i)}u(k) \geq \Delta^{(i)}u(k_2) + \frac{1}{(\ell-i-1)!(n-\ell-1)!}\sum_{s=k_2}^{k-1}\prod_{r=1}^{\ell-i-1}(k+r-(1+s))$$
$$\times\sum_{j=s}^{+\infty}\prod_{r=1}^{n-\ell-1}(j+r-s-1)\big|\Delta^{(n)}u(j)\big|, \quad \text{for } k \in \mathbb{N}^+_{k_2+1} \ (i=0,\dots,\ell-1). \tag{2.18}$$

If in addition

$$\sum_{k=1}^{+\infty}k^{n-\ell}\big|\Delta^{(n)}u(k)\big| = +\infty, \tag{2.19}$$

then

$$\frac{u(k)}{\prod\limits_{i=0}^{\ell-1}(k-i)}\downarrow, \qquad \frac{u(k)}{\prod\limits_{i=1}^{\ell-1}(k-i)}\uparrow, \tag{2.20}$$

for large k

$$u(k) \geq \frac{1+o(1)}{\ell!}\,k^{\ell-1}\Delta^{(\ell-1)}u(k) \tag{2.21}$$

and

$$\Delta^{(\ell-1)}u(k) \geq \frac{k}{(n-\ell-1)!}\sum_{i=k}^{+\infty}i^{n-\ell-1}\big|\Delta^{(n)}u(i)\big|$$
$$+\frac{1}{(n-\ell-1)!}\sum_{i=k_2}^{k}i^{n-\ell}\big|\Delta^{(n)}u(i)\big| \quad \text{for } k \in \mathbb{N}^+_{k_2}. \tag{2.22}$$

Proof. Let $s;k \in N^+_{k_2}$ and $s<k$. Assumed that (2.9) be fulfilled. By virtue of (2.1), from the equality (2.8) with $j=\ell$ and $m=n$ we have

$$\sum_{i=s}^{k}(-1)^{n+\ell}i^{n-\ell-1}\Delta^{(n)}u(i) = \sum_{i=\ell}^{n-1}(-1)^{\ell+i}\Delta^{(i)}u(s+1)\Delta^{(n-i-1)}(s+i+1-n)^{n-\ell-1}$$
$$-\sum_{i=\ell}^{n-1}(-1)^{\ell+i}\Delta^{(i)}u(k+1)\Delta^{(n-i-1)}(k+i+1-n)^{n-\ell-1}.$$

Therefore

$$\sum_{i=s}^{k}i^{n-\ell-1}\big|\Delta^{(n)}u(i)\big| \leq \sum_{i=\ell}^{n-1}\big|\Delta^{(i)}u(s+1)\big|\Delta^{(n-i-1)}(s+i+1-n)^{n-\ell-1} \quad \text{for } k \in \mathbb{N}^+_s.$$

The last inequality with $k \to +\infty$ we obtain (2.16). The equality (2.10) also implies the inequality

$$\sum_{i=\ell}^{n-1} \left|\Delta^{(i)} u(k+1)\right| \Delta^{(n-i-1)} (k+i+1-n)^{n-\ell-1}$$

$$\geq \sum_{i=k}^{+\infty} i^{n-\ell-1} \left|\Delta^{(n)} u(i+1)\right| \text{ for } k \in \mathbb{N}_{k_2}^{+}. \tag{2.23}$$

On account of (2.1) and (2.16), from (2.4) we obtain (2.17).

Analogously, equality (2.2) with $s = k_2$ and $m = \ell$, gives

$$\Delta^{(i)} u(k) \geq \Delta^{(i)} u(k_2) + \frac{1}{(\ell-i-1)!} \sum_{j=k_2}^{k} \prod_{r=1}^{\ell-i-1} (k-j+r-1) \Delta^{(\ell)} u(j-1)$$
$$(i = 0, \dots, \ell-1) \text{ for } k \in \mathbb{N}_{k_2}^{+}.$$

Hence, by (2.17) we obtain (2.18). Using (2.1), from (2.8) with $j = \ell-1$ and $m = n$, for $s = k_2$ we have

$$\Delta^{(\ell-1)} u(k) = \frac{1}{(n-\ell)!} \sum_{i=k_2}^{k} i^{n-\ell} \left|\Delta^{(n)} u(i)\right|$$
$$+ \frac{1}{(n-\ell)!} \sum_{i=\ell}^{n-1} \left|\Delta^{(i)} u(k+1)\right| \Delta^{(n-i-1)} (k+i+1-n)^{n-\ell}$$
$$+ \frac{1}{(n-\ell)!} \sum_{i=\ell-1}^{n-1} (-1)^{n+i-1} \Delta^{(i)} u(k_2+1) \Delta^{(n-i-1)} (k_2+i+1-n)^{n-\ell}.$$

Therefore, according to (2.19) there exist $k^* > k_2$ such that

$$\Delta^{(\ell-1)} u(k+1) \geq \frac{1}{(n-\ell)!} \sum_{i=k_*}^{k} i^{n-\ell} \left|\Delta^{(n)} u(i)\right|$$
$$+ \frac{1}{(n-\ell)!} \sum_{i=\ell}^{n-1} \left|\Delta^{(i)} u(k+1)\right| \Delta^{(n-i-1)} (k+i+1-n)^{n-\ell} \text{ for } k \in \mathbb{N}_{k^*}^{+}.$$

From the last inequality by (2.19) we have

$$\Delta^{(\ell-1)} u(k+1) - (k+\ell+1-n) \Delta^{(\ell)} u(k+1) \to +\infty \text{ for } k \to +\infty \tag{2.24}$$

and by (2.23) the inequality (2.22) holds.

Let $k_0 \in \mathbb{N}$ and for any $k \in \mathbb{N}_{k_0}^{+}$ and $i \in \{1, \dots, \ell\}$ put

$$\rho_i(k) = i \Delta^{(\ell-i)} u(k) - (k+1-i) \Delta^{(\ell-i+1)} u(k), \tag{2.25}$$

$$\gamma_i(k) = (k-i) \Delta^{(\ell-i+1)} u(k) - (i-1) \Delta^{(\ell-i)} u(k). \tag{2.26}$$

Applying (2.24) and L'opital rule, we have

$$\lim_{k \to +\infty} \frac{\Delta^{(\ell-i)} u(k)}{\prod_{j=1}^{i-1} (k-j)} = +\infty \quad (i = 1, \dots, \ell). \tag{2.27}$$

(Here it is meant that $\prod\limits_{j=1}^{0}(k-j)=1$). Since

$$\Delta^{(1)}\left(\frac{\Delta^{(\ell-i)}u(k)}{\prod\limits_{j=1}^{i-1}(k-j-1)}\right)=\frac{\gamma_i(k)}{\prod\limits_{j=0}^{i-1}(k-j-1)}\,,$$

by (2.27) there exist $k_\ell>\cdots>k_1>k_0$ such that $\gamma_i(k_i)>0$ $(i=1,\dots,\ell)$. Therefore, by (2.24) $\rho_1(k)\to+\infty$ as $k\to+\infty$, $\Delta^{(1)}\rho_{i+1}(k)=\rho_i(k)$, $\Delta^{(1)}\gamma_{i+1}(k)=\gamma_i(k)$ and $\gamma_1(k)=(k-1)\Delta^{(\ell)}u(k)>0$ for $k\in\mathbb{N}^+_{k_0}$ $(i=1,\dots,\ell-1)$, we find that $\rho_i(k)\to+\infty$ as $k\to+\infty$, and $\gamma_i(k)>0$ for $k\in\mathbb{N}^+_{k_i}$ $(i=1,\dots,\ell)$. These fact along with (2.24)–(2.27) prove (2.20).

On the other hand, since $\rho_i(k)\to+\infty$, by (2.25) for large k, $i\Delta^{(\ell-i)}u(k)>(k+1-i)\Delta^{(\ell-i+1)}u(k)$ $(i=1,\dots,\ell)$, which implies (2.21). □

3. Necessary condition for existence of solutions of type 2.1

The results of this section play an important role in establishing sufficient conditions for equation (1.1) to have Property **A**.

Let $k_0\in N$ and $\ell=\{1,\dots,n-1\}$. By U_{ℓ,k_0} we denote the set of all solutions of equation (1.1) satisfying the condition (2.1).

Theorem 3.1. *Let condition* (1.2), (1.3) *be fulfilled,* $\ell\in\{1,\dots,n-1\}$ *with* $\ell+n$ *odd and*

$$\sum_{k=1}^{+\infty}k^{n-\ell}\big(\sigma(k)\big)^{\lambda(\ell-1)}p(k)=+\infty. \tag{3.1}$$

If, moreover, for some $k_0\in\mathbb{N}$, $U_{\ell,k_0}\neq\varnothing$, *then for any* $\delta\in[0,\lambda]$ *and* $i\in\mathbb{N}$ *we have*

$$\sum_{k=1}^{+\infty}k^{n-\ell-1+\lambda-\delta}\big(\sigma(k)\big)^{\lambda(\ell-1)}\big(\rho_{i,\ell}(\sigma(k))\big)^{\delta}p(k)<+\infty, \tag{3.2}$$

where

$$\rho_{1,\ell}(k)=\left(\frac{1-\lambda}{\ell!(n-\ell)!}\sum_{i=1}^{k-1}\sum_{j=i}^{+\infty}j^{n-\ell-1}\big(\sigma(j)\big)^{\lambda(\ell-1)}p(j)\right)^{\frac{1}{1-\lambda}}, \tag{3.3}$$

(3.4)

$$\rho_{s,\ell}(k)=\frac{1}{\ell!(n-\ell)!}\sum_{i=1}^{k-1}\sum_{j=i}^{+\infty}j^{n-\ell-1}\big(\sigma(j)\big)^{\lambda(\ell-1)}p(j)\big(\rho_{s-1,\ell}(\sigma(j))\big)^{\lambda}\ (s=2,3,\dots).$$

Proof. Let $k_0\in\mathbb{N}$ and $U_{\ell,k_0}\neq\varnothing$. By definition of the set U_{ℓ,k_0}, equation (1.1) has a proper solution $u\in U_{\ell,k_0}$ satisfying the condition (2.1). By (2.1) and (3.1) it is clear that the condition (2.19) holds. Thus by Lemma 2.5, (2.20)–(2.22) are fulfilled and by (1.1) and (2.21), (2.22) we have

$$\Delta^{(\ell-1)}u(k) \geq \frac{k}{\ell!(n-\ell)!}\sum_{i=k}^{+\infty} i^{n-\ell-1}\sigma^{\lambda(\ell-1)}(i)\big(\Delta^{(\ell-1)}u(\sigma(i))\big)^{\lambda}p(i)$$

$$+\frac{1}{\ell!(n-\ell)!}\sum_{i=k_*}^{k} i^{n-\ell}\sigma^{\lambda(\ell-1)}(i)\big(\Delta^{(\ell-1)}u(\sigma(i))\big)^{\lambda}p(i) \quad \text{for} \quad k\in\mathbb{N}_{k_*}^{+}, \tag{3.5}$$

where k_* it is sufficiently large natural number. By the identity

$$\sum_{i=k_*}^{k} u(i)\Delta^{(1)}v(i) = u(k)v(k+1) - u(k_*-1)v(k_*) - \sum_{i=k_*}^{k} v(i)\Delta^{(1)}u(i-1)$$

we have

$$\begin{aligned}
&\sum_{i=k_*}^{k} i^{n-\ell}\sigma^{\lambda(\ell-1)}(i)\big(\Delta^{(\ell-1)}u(\sigma(i))\big)^{\lambda}p(i)\\
&\quad= -\sum_{i=k_*}^{k} i\Delta^{(1)}\sum_{s=i}^{+\infty} s^{n-\ell-1}\sigma^{\lambda(\ell-1)}(s)\big(\Delta^{(\ell-1)}u(\sigma(s))\big)^{\lambda}p(s)\\
&\quad= -k\sum_{s=k}^{+\infty} s^{n-\ell-1}\sigma^{\lambda(\ell-1)}(s)\big(\Delta^{(\ell-1)}u(\sigma(s))\big)^{\lambda}p(s)\\
&\qquad+(k_*-1)\sum_{s=k_*}^{+\infty} s^{n-\ell-1}\sigma^{\lambda(\ell-1)}(s)\big(\Delta^{(\ell-1)}u(\sigma(s))\big)^{\lambda}p(s)\\
&\qquad+\sum_{i=k_*}^{k}\sum_{s=i}^{+\infty} s^{n-\ell-1}\sigma^{\lambda(\ell-1)}(s)\big(\Delta^{(\ell-1)}u(\sigma(s))\big)^{\lambda}p(s)
\end{aligned}$$

Therefore, from (3.5) we get

$$\Delta^{(\ell-1)}u(k) \geq \frac{1}{\ell!(n-\ell)!}\sum_{i=k_*}^{k}\sum_{s=i}^{+\infty} s^{n-\ell-1}\sigma^{\lambda(\ell-1)}(s)\big(\Delta^{(\ell-1)}u(\sigma(s))\big)^{\lambda}p(s) \tag{3.6}$$

for $k\in\mathbb{N}_{k*}^{+}$. Denote

$$x(k) = \frac{1}{\ell!(n-\ell)!}\sum_{i=k_*}^{k-1}\sum_{s=i}^{+\infty} s^{n-\ell-1}\sigma^{\lambda(\ell-1)}(s)\big(\Delta^{(\ell-1)}u(\sigma(s))\big)^{\lambda}p(s).$$

Since $\Delta^{(\ell-1)}u(k)$ is nondecreasing and $\sigma(k)\geq k+1$, by (3.6) we have

$$\begin{aligned}
\Delta^{(1)}x(k) &\geq \frac{\big(\Delta^{(\ell-1)}u(k+1)\big)^{\lambda}}{\ell!(n-\ell)!}\sum_{s=k}^{+\infty} s^{n-\ell-1}\sigma^{\lambda(\ell-1)}(s)p(s)\\
&\geq \frac{x^{\lambda}(k+1)}{\ell!(n-\ell)!}\sum_{s=k}^{+\infty} s^{n-\ell-1}\sigma^{\lambda(\ell-1)}(s)p(s) \quad \text{for} \quad k\in\mathbb{N}_{k*}^{+}.
\end{aligned}$$

Therefore

$$\sum_{j=k_*}^{k-1}\frac{\Delta^{(1)}x(j)}{x^{\lambda}(j+1)} \geq \frac{1}{\ell!(n-\ell)!}\sum_{j=k_*}^{k-1}\sum_{i=j}^{+\infty} i^{n-\ell-1}\sigma^{\lambda(\ell-1)}(i)p(i). \tag{3.7}$$

Since
$$\sum_{j=k_*}^{k-1}\frac{\Delta^{(1)}x(j)}{x^{\lambda}(j+1)}=\sum_{i=k_*}^{k-1}x^{-\lambda}(j+1)\int_{x(j)}^{x(j+1)}dt$$
and $x^{-\lambda}(j+1)\le t^{-\lambda}$ when $x(j)\le t\le x(j+1)$, we have
$$\sum_{j=k_*}^{k-1}\frac{\Delta^{(1)}x(j)}{x^{\lambda}(j+1)}\le\sum_{j=k_*}^{k-1}\int_{x(j)}^{x(j+1)}t^{-\lambda}dt=\int_{x(k_*)}^{x(k)}t^{-\lambda}dt.$$
That's why, from (3.7) we get

$$x(k)\ge\left(\frac{1-\lambda}{\ell!(n-\ell)!}\sum_{j=k_*}^{k-1}\sum_{i=j}^{+\infty}i^{n-\ell-1}\sigma^{\lambda(\ell-1)}(i)p(i)\right)^{\frac{1}{1-\lambda}}. \tag{3.8}$$

I.e.

$$\Delta^{(\ell-1)}u(k)\ge\rho_{1,\ell,k_*}(k)\ \text{ for }\ k\in\mathbb{N}^+_{k_*}, \tag{3.9}$$

where
$$\rho_{1,\ell,k_*}(k)=\left(\frac{1-\lambda}{\ell!(n-\ell)!}\sum_{j=k_*}^{k-1}\sum_{i=j}^{+\infty}i^{n-\ell}\sigma^{\lambda(\ell-1)}(i)p(i)\right)^{\frac{1}{1-\lambda}}.$$
Thus, by (3.6), (3.9) we get

$$\Delta^{(\ell-1)}u(k)\ge\rho_{s,\ell,k_*}(k)\ \text{ for }\ k\in\mathbb{N}^+_{k_*}\ \ (s=2,3,\dots), \tag{3.10}$$

where
$$\rho_{s,\ell,k_*}(k)=\frac{1}{\ell!(n-\ell)!}\sum_{j=k_*}^{k-1}\sum_{i=j}^{+\infty}i^{n-\ell-1}\sigma^{\lambda(\ell-1)}(i)p(i)\big(\rho_{s-1,\ell,k_*}(\sigma(i))\big)^{\lambda}.$$
On the other hand, by (1.2), (2.1), (3.9) and (3.10) from (3.6) for any $\delta\in[0,\lambda]$ we have
$$\Delta^{(\ell-1)}u(k+1)\ge\frac{1}{\ell!(n-\ell)!}\sum_{i=k_*}^{k}\sum_{j=i}^{+\infty}j^{n-\ell-1}\sigma^{\lambda(\ell-1)}(j)p(j)$$
$$\times\big(\rho_{s,\ell,k_*}(\sigma(j))\big)^{\delta}|\big(\Delta^{(\ell-1)}u(\sigma(i))\big)^{\lambda-\delta},\ \ s=1,2,\dots$$
and
$$\Delta^{(\ell-1)}u(k+1)\ge\frac{k-k_*}{\ell!(n-\ell)!}\sum_{j=k}^{+\infty}j^{n-\ell-1}\sigma^{\lambda(\ell-1)}(j)p(j)$$

$$\times\big(\rho_{s,\ell,k_*}(\sigma(j))\big)^{\delta}\big(\Delta^{(\ell-1)}u(\sigma(j))\big)^{\lambda-\delta},\ \ s=1,2,\dots \tag{3.11}$$

If $\delta=\lambda$, then from the last inequality we get
$$\sum_{j=k}^{+\infty}j^{n-\ell-1}\sigma^{\lambda(\ell-1)}(j)p(j)\big(\rho_{s,\ell,k_*}(\sigma(j))\big)^{\lambda}\le\frac{\ell!(n-\ell)!(k+1)}{k-k_*}\cdot\frac{\Delta^{(\ell-1)}u(k+1)}{k+1}.$$
By first condition of (2.20) we have

$$\sum_{j=k}^{+\infty}j^{n-\ell-1}\sigma^{\lambda(\ell-1)}(j)p(j)\big(\rho_{s,\ell,k_*}(\sigma(j))\big)^{\lambda}<+\infty\ \ (s=1,2,\dots). \tag{3.12}$$

Let $\delta \in [0,\lambda)$. Then from (3.11) implies

$$\frac{\Delta^{(\ell-1)}u(k+1)}{\sum\limits_{j=k}^{+\infty} j^{n-\ell-1}\sigma^{\lambda(\ell-1)}(j)p(j)\big(\rho_{s,\ell,k_*}(\sigma(j))\big)^{\delta}\big(\Delta^{(\ell-1)}u(\sigma(j))\big)^{\lambda-\delta}} \ge \frac{k-k_*}{\ell!(n-\ell)!}$$

for $k \in \mathbb{N}^+_{k_*}$. Therefore

$$\frac{\big(\Delta^{(\ell-1)}u(k+1)\big)^{\lambda-\delta}k^{n-\ell-1}p(k)\sigma^{\lambda(\ell-1)}(k)\big(\rho_{s,\ell,k_*}(\sigma(k))\big)^{\delta}}{\Big(\sum\limits_{j=k}^{+\infty} j^{n-\ell-1}\sigma^{\lambda(\ell-1)}(j)p(j)\big(\rho_{s,\ell,k_*}(\sigma(j))\big)^{\delta}\big(\Delta^{(\ell-1)}u(\sigma(j))\big)^{\lambda-\delta}\Big)^{\lambda-\delta}}$$

$$\ge \Big(\frac{k-k_*}{\ell!(n-\ell)!}\Big)^{\lambda-\delta}k^{n-\ell-1}p(k)\sigma^{\lambda(\ell-1)}(k)\big(\rho_{s,\ell,k_*}(\sigma(k))\big)^{\delta}. \tag{3.13}$$

Denote

$$a_k=\sum_{j=k}^{+\infty} j^{n-\ell-1}\sigma^{\lambda(\ell-1)}(j)p(i)\big(\rho_{s,\ell,k_*}(\sigma(j))\big)^{\delta}\big(\Delta^{(\ell-1)}u(j+1)\big)^{\lambda-\delta}. \tag{3.14}$$

Since $\Delta^{\ell-1)}u(k)$ is nondecreasing function, according to (3.14), from (3.13) we get

$$\frac{a_k-a_{k+1}}{a_k^{\lambda-\delta}} \ge \Big(\frac{k-k_*}{\ell!(n-\ell)!}\Big)^{\lambda-\delta}k^{n-\ell-1}p(k)\sigma^{\lambda(\ell-1)}(k)\Big(\rho_{s,\ell,k_*}(\sigma(k))\Big)^{\delta}.$$

Thus, from the last inequality we get

$$\sum_{i=k_*}^{k}\frac{a_i-a_{i+1}}{a_i^{\lambda-\delta}} \ge \Big(\frac{1}{\ell!(n-\ell)!}\Big)^{\lambda-\delta}\sum_{i=k_*}^{k}(i-k_*)^{\lambda-\delta}i^{n-\ell-1}p(i)\sigma^{\lambda(\ell-1)}(i)\big(\rho_{s,\ell,k_*}(\sigma(i))\big)^{\delta}. \tag{3.15}$$

Since

$$\sum_{i=k_*}^{k}\frac{a_i-a_{i+1}}{a_i^{\lambda-\delta}}=\sum_{i=k_*}^{k}a_i^{\delta-\lambda}\int_{a_{i+1}}^{a_i}dt \le \sum_{i=k_*}^{k}\int_{a_{i+1}}^{a_i}t^{\delta-\lambda}dt \le \int_0^{a_{k_*}}t^{\delta-\lambda}dt=\frac{a_{k_*}^{1+\delta-\lambda}}{1+\delta-\lambda},$$

from (3.15) we get

$$\sum_{i=k_*}^{k}(i-k_*)^{\lambda-\delta}i^{n-\ell-1}p(i)\sigma^{\lambda(\ell-1)}(i)\big(\rho_{s,\ell,k_*}(\sigma(i))\big)^{\delta} \le \frac{a_{k_*}^{1+\delta-\lambda}\big(\ell!(n-\ell)!\big)^{\lambda-\delta}}{1+\delta-\lambda}. \tag{3.16}$$

Without loss of generality, by (3.14) we can assume that $a_{k_*} \le 1$. Thus from (3.16) we have

$$\sum_{i=k_*}^{k}(i-k_*)^{\lambda-\delta}i^{n-\ell-1}p(i)\sigma^{\lambda(\ell-1)}(i)\big(\rho_{s,\ell,k_*}(\sigma(i))\big)^{\delta} \le \frac{\big(\ell!(n-\ell)!\big)^{\lambda-\delta}}{1+\delta-\lambda}. \tag{3.17}$$

According to (3.12) and (3.17), for any $\delta \in [0,\lambda]$ and $s \in \mathbb{N}$ we have

$$\sum_{i=k_*}^{k} i^{n-\ell-1+\lambda-\delta}p(i)\sigma^{\lambda(\ell-1)}(i)\big(\rho_{s,\ell,k_*}(\sigma(i))\big)^{\delta} < +\infty. \tag{3.18}$$

Since $\frac{\rho_{s,\ell}(k)}{\rho_{s,\ell,k_*}(k)} \longrightarrow 1$ for $k \to +\infty$, by (3.18) it is obvious that (3.2) holds, which proves the validity of the theorem. □

4. Sufficient conditions of nonexistence of solutions of type (2.1)

Theorem 4.1. *Let conditions* (1.2), (1.3), (3.1) *be fulfilled,* $\ell \in \{1, \dots, n-1\}$ *with* $\ell + n$ *odd and for some* $\delta \in [0, \lambda]$ *and* $s \in \mathbb{N}$

$$\sum_{k=1}^{+\infty} k^{n-\ell-1+\lambda-\delta} \sigma^{\lambda(\ell-1)}(k) \big(\rho_{s,\ell}(\sigma(k))\big)^{\delta} p(k) = +\infty, \tag{4.1}$$

where $\rho_{s,\ell}$ *is defined by* (3.3) *and* (3.4)*. Then* $U_{\ell,k_0} = \varnothing$ *for any* $k_0 \in \mathbb{N}$.

Proof. Assume the contrary. Let there exists $k_0 \in \mathbb{N}$ such that $U_{\ell,k_0} \neq \varnothing$. Thus equation (1.1) has a proper solution $u : \mathbb{N}^+_{k_0} \to (0, \infty)$ satisfying the condition (2.1). Since conditions of Theorem 3.1 are fulfilled, (3.2) holds for any $\delta \in [0, \lambda]$ and $s \in \mathbb{N}$, which contradicts (4.1). The obtained contradiction proves the validity of the theorem. $\square$

Theorem 4.2. *Let conditions* (1.2), (1.3) *be fulfilled,* $\ell \in \{1, \dots, n-1\}$ *with* $\ell + n$ *odd and for some* $\alpha \in (1, +\infty)$ *and* $\gamma \in (\lambda, 1)$

$$\liminf_{k\to+\infty} k^{\gamma} \sum_{j=k}^{+\infty} j^{n-\ell-1} \sigma^{\lambda(\ell-1)}(j) p(j) > 0, \quad \liminf_{k\to+\infty} \frac{\sigma(k)}{k^{\alpha}} > 0. \tag{4.2}$$

If moreover, at last one of the conditions

$$\alpha\lambda \geq 1, \tag{4.3}$$

or if $\alpha\lambda < 1$*, for some* $\varepsilon > 0$

$$\sum_{k=1}^{+\infty} k^{n-\ell-1+\frac{\alpha\lambda(1-\gamma}{1-\alpha\lambda}-\varepsilon} \big(\sigma(k)\big)^{\lambda(\ell-1)} p(k) = +\infty \tag{4.4}$$

holds, then $U_{\ell,k_0} = \varnothing$ *for any* $k_0 \in \mathbb{N}$.

Proof. It suffices to show that the condition (4.1) is satisfies for some $s \in \mathbb{N}$ and $\delta = \lambda$. Indeed, according to (4.2) there exist $\alpha > 1$, $\gamma \in (\lambda, 1)$, $c > 0$ and $k_0 \in \mathbb{N}$ such that

$$k^{\gamma} \sum_{j=k}^{+\infty} j^{n-\ell-1} \big(\sigma(j)\big)^{\lambda(\ell-1)} p(j) \geq c \quad \text{for} \quad k \in \mathbb{N}^+_{k_0} \tag{4.5}$$

and

$$\sigma(k) \geq c k^{\alpha} \quad \text{for} \quad k \in \mathbb{N}^+_{k_0}. \tag{4.6}$$

By (3.3) and (4.2) it is obvious that $\lim_{k\to+\infty} \rho_{1,\ell}(k) = +\infty$. Therefore, without loss of generality we can assume that $\rho_{1,\ell}(k) \geq 1$ for $k \in \mathbb{N}^+_{k_0}$. Thus, by (4.6) from (3.4) we get

$$\rho_{2,\ell}(k) \geq \frac{c}{\ell!(n-\ell)!} \sum_{i=k_0}^{k-1} i^{-\gamma} = \frac{c}{\ell!(n-\ell)!} \sum_{i=k_0}^{k-1} i^{-\gamma} \int_i^{i+1} dt$$

$$\geq \frac{c}{\ell!(n-\ell)!}\sum_{i=k_0}^{k-1}\int_i^{i+1} t^{-\gamma}dt = \frac{c}{\ell!(n-\ell)!}\int_{k_0}^{k} t^{-\gamma}dt = \frac{c}{\ell!(n-\ell)!(1-\gamma)}(k^{1-\gamma}-k_0^{1-\gamma}).$$

We can choose $k_1 \in \mathbb{N}_{k_0}^+$ such that $\rho_{2,\ell}(k) \geq \frac{c}{2\ell!(n-\ell)!(1-\gamma)}k^{1-\gamma}$ for $k \in \mathbb{N}_{k_1}^+$. Thus, by (4.6) from (3.4), for $s=3$ we have

$$\rho_{3,\ell}(k) \geq \left(\frac{c}{2\ell!(n-\ell)!(1-\gamma)}\right)^{1+\lambda} \cdot k^{(1-\gamma)(1+\alpha\lambda)} \text{ for } k \in \mathbb{N}_{k_2}^+.$$

where $k_2 \in \mathbb{N}_{k_1}^+$ is a sufficiently large natural number. Therefore, for any $s \in \mathbb{N}$ there exists $k_s \in \mathbb{N}$ such that for $k \in \mathbb{N}_s^+$

(4.7)
$$\rho_{s,\ell}(k) \geq \left(\frac{c}{2\ell!(n-\ell)!(1-\gamma)}\right)^{1+\lambda+\cdots+\lambda^{s-2}} k^{(1-\gamma)(1+\alpha\lambda+\cdots+(\alpha\lambda)^{s-2})}, \quad k \geq k_s.$$

Assume that (4.3) be fulfilled. Choose $s_0 \in \mathbb{N}$ such that $(1-\gamma)(s_0-1) \geq \frac{1}{\lambda}$. Then, according to (4.7), $\rho_{s_0,\ell}(k) \geq c_0 k$ for $k \in \mathbb{N}_{k_{s_0}}$, where $c_0 > 0$. Therefore, by (4.7) it is obvious that (4.1) hold, for $\delta = \lambda$ and $s = s_0$. In the case, when (4.3) holds, the validity of the theorem has been already proved.

Assume now that $0 < \alpha\lambda < 1$ and (4.4) holds. Let $\varepsilon > 0$ and by (4.7), choose $s_0 \in \mathbb{N}$ such that $\rho_{s_0,\ell}(k) \geq c_1 k^{\frac{\alpha\lambda(1-\gamma)}{1-\alpha\lambda}-\varepsilon}$ for $k \in \mathbb{N}_{k_{s_0}}^+$, where $c_1 > 0$. Therefore, by (4.4), (4.1) holds for $s = s_0$. The proof of the theorem is proved. $\square$

5. Difference equations with property **A**

Theorem 5.1. *Let the conditions* (1.2), (1.3) *be fulfilled and for any* $\ell \in \{1,\ldots,n-1\}$ *with* $\ell+n$ *odd, let* (3.1) *as well as* (4.1) *hold for some* $\delta \in [0,\lambda]$ *and* $s \in \mathbb{N}$. *Let moreover*

$$\sum_{k=1}^{+\infty} k^{n-1}p(k) = +\infty, \tag{5.1}$$

when n *is odd, then equation* (1.1) *has Property* **A**.

Proof. Let equation (1.1) have a proper nonoscillatory solution $u : \mathbb{N}_{k_0} \to (0,+\infty)$ (the case $u(k) < 0$ is similar). Then by (1.1), (1.3) and Lemma 1.1, there exist $\ell \in \{0,\ldots,n-1\}$ such that $\ell+n$ is odd and the condition (2.1) holds. Since conditions of the Theorem 4.1 are fulfilled, for any $\ell \in \{1,\ldots,n-1\}$ with $\ell+n$ odd, we have $\ell \notin \{1,\ldots,n-1\}$. Therefore, n is odd and $\ell = 0$. Then we will show that the conditions (1.5) hold. If that is not the case, there exists $c > 0$ such that $u(k) \geq c$ for sufficiently large k. According to 2.1, with $\ell = 0$, from (1.1) we have

$$\sum_{j=k_0}^{k} j^{n-1}\Delta^{(n)}u(j) + c\sum_{j=k_0}^{k} j^{n-1}p(j) \leq 0, \tag{5.2}$$

where $k \in \mathbb{N}$ is sufficiently large natural number. On the other hand in view of identity

$$\sum_{j=k_0}^{k} j^{n-1}\Delta^{(n)}u(j) = k^{n-1}\Delta^{(n-1)}u(k+1) - (k_0-1)^{n-1}\Delta^{(n-1)}u(k_0) - \sum_{j=k_0}^{k} \Delta^{(n-1)}u(j)\Delta(j-1)^{n-1}$$

it is easy to show that

$$\sum_{j=k_0}^{k} j^{n-1}\Delta^{(n)}u(j) = \sum_{j=0}^{n-1}(-1)^j\Delta^{(j)}(k-j)^{n-1}\Delta^{(n-j-1)}u(k+1) - \sum_{j=0}^{n-1}(-1)^j(k_0-j-1)^{(n-j-1)}\Delta^{(n-j-1)}u(k_0).$$

From (5.2), by (2.1) with $\ell = 0$

$$c\sum_{j=k_0}^{k} j^{n-1}p(j) \le \sum_{j=0}^{n-1}(k_0-j-1)^{n-j-1}\big|\Delta^{(n-j-1)}u(k_0)\big|.$$

Therefore $\sum_{j=1}^{+\infty} j^{n-1}p(j) < +\infty$, which contradict the condition (5.1). Therefore, equation (1.1) has Property **A**. □

From this theorem, with $\delta = 0$, immediately follow

Theorem 5.1′. *Let the conditions* (1.2), (1.3) *be fulfilled and for any* $\ell \in \{1,\dots,n-1\}$ *with* $\ell + n$ *odd,* (3.1) *as well as*

$$\sum_{k=1}^{+\infty} k^{n-\ell-1+\lambda}\sigma^{\lambda(\ell-1)}(k)p(k) = +\infty \tag{5.3}$$

holds. Then in the case of odd n *condition* (5.1) *is sufficient for equation* (1.1) *to have Property* **A**.

Theorem 5.2. *Let the condition* (1.2), (1.3) *as well as* (5.1) *be fulfilled for odd* n *and*

$$\liminf_{k\to+\infty} \frac{\sigma^{\lambda}(k)}{k} > 0. \tag{5.4}$$

Then the condition

$$\sum_{k=1}^{+\infty} k^{n-2+\lambda}p(k) = +\infty, \tag{5.5}$$

for even n *and the condition*

$$\sum_{k=1}^{+\infty} k^{n-3+\lambda}\big(\sigma(k)\big)^{\lambda}p(k) = +\infty, \tag{5.6}$$

for odd n *is sufficient for equation* (1.1) *to have property* **A**.

Proof. It is obvious that, according to (5.4)–(5.6), for any $\ell = \{1, \dots, n-1\}$, where $\ell+n$ odd, the conditions (5.3) hold. Therefore, all conditions of the Theorem 5.1′ hold, which proves the validity of the theorem. □

Theorem 5.3. *Let the conditions* (1.2), (1.3) *be fulfilled and let*

$$\limsup_{k\to+\infty} \frac{\sigma^\lambda(k)}{k} < +\infty. \tag{5.7}$$

Then for equation (1.1) *to have Property* **A** *it is sufficient that*

$$\sum_{k=1}^{+\infty} k^\lambda \big(\sigma(k)\big)^{\lambda(n-2)} p(k) = +\infty. \tag{5.8}$$

Proof. It is obvious that, according to (5.7), (5.8) and first condition of (1.2), the condition (5.1) and for any $\ell = \{1, \dots, n-1\}$, where $\ell+n$ is odd, the conditions (5.3) hold. Therefore, all conditions of the Theorem 5.1′ hold, which proves the validity of the theorem. □

Theorem 5.4. *Let the conditions* (1.2), (1.3), (4.3), (4.6) *and* (5.4) *or if* $0 < \alpha\lambda < 1$, *for some* $\varepsilon > 0$

$$\sum_{k=1}^{+\infty} k^{n-2+\frac{\alpha\lambda(1-\gamma)}{1-\alpha\lambda}-\varepsilon} p(k) = +\infty \tag{5.9}$$

be fulfilled. If moreover, there exist $\gamma \in (\lambda, 1)$ *such that*

$$\liminf_{k\to+\infty} k^\gamma \sum_{j=k}^{+\infty} j^{n-2} p(j) > 0, \tag{5.10}$$

then equation (1.1) *has Property* **A**.

Proof. Let equation (1.1) have a proper nonoscillatory solution $u : \mathbb{N}_{k_0} \to (0, +\infty)$ (the case $u(k) < 0$ is similar). Then by(1.1), (1.3) and Lemma 1.1, there exist $\ell \in \{0, \dots, n-1\}$ such that $\ell+n$ is odd and the condition (2.1) holds. Since by (4.3), (4.6), (5.4), (5.9) and (5.10) all conditions of the Theorem 4.2 are fulfilled. So for any $\ell \in \{1, \dots, n-1\}$ with $\ell+n$ odd, we have $\ell \notin \{1, \dots, n-1\}$. Therefore, n is odd and $\ell = 0$. It is obvious that, since $\gamma \in (0, 1)$, by (5.10) satisfying the condition (5.1). Therefore, analogously Theorem 5.1, we can proved the condition (1.5) hold. That is equation (1.1) has Property **A**. The proof of the theorem is complete. □

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