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PERIODIZED WAVELET PACKETS ON BOUNDED SUBSETS OF $\mathbb{R}$

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Abstract. In this paper, we study Daubechies periodized wavelet packets (DPWP) and give a necessary condition for it. Also, some properties of DPWP are discussed. Further, we give an estimate for the approximation error related to DPWP. Finally, we use the thresholding technique to study compression errors.

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1. Introduction

The notion of wavelet packets was introduced by Coifman et al. [1] as a family of orthonormal bases for discrete functions in $\mathbb{R}^n$. They split a generalization of the procedure of MRA and constitute the whole set of subband coded decomposition. The “best basis"selection can be easily done, since wavelet packets give quick access to a rich library of orthonormal bases. It proves to be more flexible and useful in application of pyramid algorithm to an image in order to reduce the information into lesser number of coefficients (see [29]).

Zhang and Wu [42] gave a novel image compression technique using wavelet packets and directional decomposition to exploit the image redundancy efficiently and thereby giving high compression ratio. Klappenecker [25] observed that employing periodized wavelet packet transform on quantum computer is much better and economical than the periodized wavelet transform. Kasaei et al. [15] introduced a novel compression algorithm using wavelet packets and lattice vector quantization for fingerprint analysis. Yoon and Vaidyanathan [41] defined a customized thresholding function which significantly improved the performance of powerful wavelet-based denoising scheme known as VisuShrink which uses a single threshold for all the scales. Joseph [14] used wavelet packets for spoken digit compression and employed Malyalam spoken digit for the same.

Later on, Khanna et al. [22] defined the orthogonal Coifman wavelet packet systems and biorthogonal Coifman wavelet packet systems which have good approximation properties with exponential decay and gave wavelet packet approximation theorem. The problem of inadequacy of a wavelet function to study both the symmetries of an asymmetric signal has been addressed by defining wavelets associated with Riesz projectors [23]. Also, wavelet packets and their moments were studied by Khanna et al. [13, 24]. Recently, Khanna and Kaushik [17] gave wavelet packet approximation theorem for H^r type norm which can measure difference of the (weak) derivatives. Uniform approximation of wavelet packet expansions have been studied in [19]. For litrature related to wavelets and wavelet packets one may consult [2], [4 - 13], [16 - 18], [20 - 24], [26 - 30], [32 - 34], [36].

Overview. Inspired from the work of Daubechies [3, 4], Restrepo et al. [35] introduced periodized wavelets by restricting the wavelets on bounded subsets of $\mathbb{R}$. In Section 3, we define wavelet packets associated with the wavelets introduced by Daubechies [3, 4] and called them Daubechies periodized wavelet packets (DPWP) and obtain a necessary condition for it. Also, we give some properties of DPWP. In Section 4, we define and obtain an estimate of the approximation error of a function in $L^2([0,1]) \cap C^g(\mathbb{R})$ $(g > 1)$. Finally, in Section 5, we discuss compression errors using hard thresholding techniques.

2. Preliminaries

In [12], Multiresolution analysis (MRA), is defined as an increasing sequence of closed subspaces $(V_j)_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ satisfying

$$V_j \subseteq V_{j+1}, \text{ for all } j \in \mathbb{Z}, \tag{2.1}$$

$$f \in V_j \text{ if and only if } f(2(\cdot)) \in V_{j+1}, \text{ for all } j \in \mathbb{Z}, \tag{2.2}$$

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}, \tag{2.3}$$

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}). \tag{2.4}$$

$$\begin{gathered}\text{There exists a function } \phi \in V_0 \text{ such that } \phi(\cdot - k) : k \in \mathbb{Z}\} \\ \text{is an orthonormal basis for } V_0.\end{gathered} \tag{2.5}$$

The function ϕ whose existence is asserted in (2.5) is called a scaling function of the given MRA. The scaling function ϕ solves the dilation equation

$$\phi(x) = \sum_{p\in\mathbb{Z}} u_p \, \phi(2x - p) \tag{2.6}$$

with $|\widehat{\phi}(0)| = 1$. But it is convenient to choose the phase of ϕ so that $\int_{\mathbb{R}} \phi(x)\, dx = 1$ and the associated function ψ is defined by

$$\psi(x) = \sum_{p\in\mathbb{Z}} v_p\ \phi(2x-p). \tag{2.7}$$

Note that only finitely many u_p and v_p are non-zero for Daubechies wavelet system. A family of functions $\omega_n,\ n = 0, 1, 2, ...$ defined by

$$\omega_{2n}(x) = \sum_{p=0}^{2g-1} u_p\ \omega_n(2x-p), \tag{2.8}$$

$$\omega_{2n+1}(x) = \sum_{p=0}^{2g-1} v_p\ \omega_n(2x-p), \tag{2.9}$$

where $\omega_1 = \psi$ and $\omega_0 = \phi$ often called mother and father wavelets, are called Daubechies wavelet packets with genus g (see [36]).

Also, the set $\{\omega_n(x-k) : k \in \mathbb{Z},\ n = 0, 1, 2, ...\}$ is an orthonormal basis of $L^2(\mathbb{R})$. The family of wavelet packets $\{\omega_n\}$ define the family of subspaces of $L^2(\mathbb{R})$ corresponding to some orthonormal scaling function $\phi = \omega_0$ given by

$$U_{n,j} = \overline{span}\{\omega_n(2^j x-k) : k \in \mathbb{Z}\},\ j \in \mathbb{Z},\ n = 0, 1, 2, ... \,. \tag{2.10}$$

Note that $U_{0,j} = V_j$ and $U_{1,j} = W_j$ so that the orthogonal decomposition $V_{j+1} = V_j \oplus W_j$ can be re-written as $U_{0,j+1} = U_{0,j} \oplus U_{1,j},\ j \in \mathbb{Z}$. In general, the above expression is given by $U_{n,j+1} = U_{2n,j} \oplus U_{2n+1,j}$, for $n = 1, 2, 3, ...; j \in \mathbb{Z}$, where $U_{n,j}$ is defined by (2.10).

Proposition 2.1. [36] *Let $\omega_n, n \in \mathbb{N}_0$ be wavelet packets associated with scaling function ω_0. Then, for $j, k, l, m \in \mathbb{Z}$ with $m \geqslant 0$ and $\omega_{j,n,k}(x) = 2^{j/2}\, \omega_n(2^j x-k)$, we have*

(i) $\langle \omega_{j,n,k}, \omega_{j,n,l}\rangle = \delta_{k,l}$,
(ii) $\langle \omega_{j,n,k}, \omega_{j,m,l}\rangle = \delta_{m,n}\ \delta_{k,l}$.

3. Periodized Daubechies wavelet packets

Heretofore, we have seen that the functions which were defined on $\mathbb{R}$ as in some applications such as audio signal processing, where the length of the signal is arbitrarily long and unknown prior to the desistance of its activity. Nevertheless, for many applications, the time domain is a finite interval. One may notice such example in case of data fitting problems, image processing of signal, etc. These problems can be worked out efficiently with the introduction of periodized wavelet packets. Significantly, wavelet packets which are defined in general can be periodized with a technique of Poisson summation and give rise to periodic wavelet packets.

Analogously, to the construction of non-periodic wavelet packets given in [1, 12], the periodized wavelet packets have an exception that they wraps over the edges of the domain, but in computation for large value of j, they reduced to the non-periodic forms. Thus, due to compact support and the construction by the scaling property of the non-periodic functions, many of the properties of wavelet packets are preserved in the periodic case. For various details related to periodized wavelets and wavelet packets, one may refer [4, 12, 31, 35], [37] - [40].
Next, we give the definition of Daubechies periodized wavelet packets (DPWP).

Definition 3.1. The wavelet packets $\omega_n \in L^2(\mathbb{R})$ $(n \in \mathbb{N}_0)$ obtained from scaling function using multiresolution analysis are said to be periodized in the sense of Daubechies (Daubechies periodized wavelet packets) (DPWP) if

$$\omega^{per}_{j,n,k}(x) = \sum_{l=-\infty}^{\infty} \omega_{j,n,k}(x+l), \tag{3.1}$$

where $j, k \in \mathbb{Z}$ and $x \in \mathbb{R}$.

Periodized wavelet packets unlike non-periodic ones, must be first dialated before periodization as periodization does not commute with dialation.

Next, we give a necessary condition for Daubechies wavelet packets associated with scaling function ω_0 such that $\hat{\omega}_1(0) = 0$. More preciously, we prove the following result.

Proposition 3.1. *Let $\omega_n,\ n \in \mathbb{N}_0$ be Daubechies wavelet packets associated with scaling function ω_0 and let $\hat{\omega}_1(0) = 0$. Then*

$$\hat{\omega}_{4q+1}(4\pi r) = 0,\ \textit{for } r \in \mathbb{Z},\ q \in \mathbb{N}. \tag{3.2}$$

Proof. Taking Fourier transform of $\omega_{2n}(x)$ and using (2.8), we compute

$$\begin{aligned} \hat{\omega}_{2n}(\eta) &= \int_{\mathbb{R}} \omega_{2n}(x)\ e^{-i\eta x}\ dx \\ &= \frac{1}{2}\sum_{p=0}^{2g-1} u_p\ e^{-\frac{i\eta p}{2}} \int_{\mathbb{R}} \omega_n(x)\ e^{-\frac{i\eta x}{2}}\ dx = F(\frac{\eta}{2})\ \hat{\omega}(\frac{\eta}{2}), \end{aligned} \tag{3.3}$$

where

$$F(\eta) = \frac{1}{2}\sum_{p=0}^{2g-1} u_p\ e^{-i\eta p}. \tag{3.4}$$

Applying (3.3) k-times, we have

$$\hat{\omega}_{2n}(\eta) = \prod_{j=1}^{k} F(\frac{\eta}{2^j})\ \hat{\omega}_{\frac{n}{2^k}}(\frac{\eta}{2^k}). \tag{3.5}$$

Since $\hat{\omega}_0(0) = 1$, we have

$$\sum_{p=0}^{2g-1} u_p = 2. \tag{3.6}$$

Using (3.4) and (3.6), we obtain $-1 \leqslant F(\eta) \leqslant 1$ and so the product converges as $k \to \infty$. This yields

$$\hat{\omega}_{2n}(\eta) = \prod_{j=1}^{\infty} F(\frac{\eta}{2^j})\ \hat{\omega}_0(0), \eta \in \mathbb{R}.$$

This further gives

$$\hat{\omega}_{2n}(2\pi r) = \prod_{j=1}^{\infty} F(\frac{2\pi r}{2^j}),\ \ r \in \mathbb{Z}.$$

If $r = 0$, then using (3.4) and (3.6), we have $\hat{\omega}_{2n}(0) = 1$. Let $r \in \mathbb{Z} \setminus \{0\}$ be such that $r = 2^s M$, where $s \in \mathbb{N}_0$ and M is odd integer. Then

$$\begin{aligned}\hat{\omega}_{2n}(2\pi r) &= \prod_{j=1}^{\infty} F(\frac{2^{s+1} M\pi}{2^j})\\ &= F(2^s M\pi)\ F(2^{s-1} M\pi) \dots F(M\pi) \dots = 0.\end{aligned}$$

This gives $\hat{\omega}_{2n}(2\pi r) = \delta_{0,r},\ r \in \mathbb{Z}$. Using Proposition 2.1, we get $\sum\limits_{p=0}^{2g-1} u_p\ v_l = 0$. Note that v_p can be expressed in terms of u_p as

$$v_p = (-1)^p\ u_{2g-1-p},\ p = 0, 1, ..., 2g-1. \tag{3.7}$$

Using (2.9), we obtain $\hat{\omega}_{2n+1}(\eta) = G(\frac{\eta}{2})\ \hat{\omega}(\frac{\eta}{2})$, where

$$G(\eta) = \frac{1}{2} \sum_{p=0}^{2g-1} v_p\ e^{-i\eta p},\ \eta \in \mathbb{R}. \tag{3.8}$$

Also, using (3.7) in (3.8), we evaluate

$$\begin{aligned}G(\eta) &= \frac{1}{2} \sum_{p=0}^{2g-1} (-1)^p\ u_{2g-1-p}\ e^{-i\eta p}\\ &= \frac{1}{2} e^{-i(2g-l)(\eta+\pi)} \sum_{q=0}^{2g-1} u_q\ e^{iq(\eta+\pi)} = e^{-i(2g-1)(\eta+\pi)}\ \overline{F(\eta+\pi)}.\end{aligned}$$

This gives

$$\hat{\omega}_{2n+1}(\eta) = e^{-i(2g-1)(\frac{\eta}{2}+\pi)}\ \overline{F(\frac{\eta+\pi}{2})}\ \hat{\omega}_n(\frac{\eta}{2}). \tag{3.9}$$

Taking $n = 2s,\ s \in \mathbb{N}_0$ and $\eta = 4\pi r$, we have

$$\begin{aligned}\hat{\omega}_{4s+1}(4\pi r) &= e^{-i(2g-1)(2\pi r+\pi)}\ \overline{F(2\pi r + \pi)}\ \hat{\omega}_{2s}(2\pi r)\\ &= \begin{cases} 0, & \text{if } r \neq 0;\\ e^{-i(2g-1)\pi}\ \overline{F(\pi)}, & \text{if } r = 0.\end{cases}\end{aligned} \tag{3.10}$$

Since $\hat{\omega}_1(0) = 0$, it follows that

$$0 = \sum_{p=0}^{2g-1} v_p \int_{\mathbb{R}} \omega_0(2x-p)\ dx = \frac{1}{2} \sum_{l=0}^{2g-1} (-1)^l\ u_l. \tag{3.11}$$

Using (3.4) and (3.11) in (3.10), we finally get $\hat{\omega}_{4s+1}(4\pi r) = 0$. □

In the following result, we give some properties of the Daubechies periodized wavelet packets.

Theorem 3.1. *Let $\omega_n \in L^2(\mathbb{R})$ $(n \in \mathbb{N})$ be wavelet packets. Then*

(i) *for any $j,\ k \in \mathbb{Z}$, $\omega^{per}_{j,n,k}$ is 1-periodic.*

(ii) *for $j \leqslant -1$, $k \in \mathbb{Z}$, $s \in \mathbb{N}_0$ and $x \in \mathbb{R}$, $\omega^{per}_{j,4s+1,k}(x) = 0$, but for $j = 0$, $\omega^{per}_{j,4s+1,k}(x)$ is neither zero nor any constant for odd choice of k.*

(iii) *for $j > 0$, $\omega^{per}_{j,n,k}$ is periodic in the shift parameter with period 2^j.*

(iv) *for $j > j' \geqslant \lceil log_2(2g-1)\rceil$ and $x \in [0,1]$ with ω_n having compact support $[0, 2g-1]$,*

$$\omega^{per}_{j,n,k}(x) = \begin{cases} \omega_{j,n,k}(x), & \text{if } x \in I_{j,k} \cap [0,1]; \\ \omega_{j,n,k}(x+1), & \text{if } x \in [0,1] \text{ and } x \notin I_{j,k}. \end{cases} \tag{3.12}$$

Proof. (i) Let $j, k \in \mathbb{Z}$ and $x \in \mathbb{R}$. Then

$$\omega^{per}_{j,n,k}(x+1) = \sum_{l=-\infty}^{\infty} \omega_{j,n,k}(x+l+1) = \omega^{per}_{j,n,k}(x).$$

Thus $\omega^{per}_{j,n,k}(x)$ is 1-periodic.

(ii) Let $k \in \mathbb{Z}$ and $x \in \mathbb{R}$. Then

$$\omega^{per}_{j,n,k}(x+1) = \sum_{l=-\infty}^{\infty} \omega_{j,n,k}(x+l) = \omega^{per}_{j,n,0}(x).$$

Since $\omega^{per}_{j,n,0}(x)$ is a 1-periodic function, it can be expanded using Fourier series expansion, i.e.,

$$\omega^{per}_{j,n,0}(x) = \sum_{r=-\infty}^{\infty} a_r\ e^{2\pi i r x},\ x \in \mathbb{R}, \tag{3.13}$$

where the Fourier coefficients a_r are given by

$$\begin{aligned} a_r &= \int_0^1 \omega^{per}_{j,n,0}(x)\ e^{-2\pi i r x}\ dx \\ &= \int_0^1 \sum_{p=-\infty}^{\infty} \omega_{j,n,0}(x+p)\ e^{-2\pi i r x}\ dx = 2^{-\frac{j}{2}}\ \hat{\omega}_n(2\pi r 2^{-j}),\ r \in \mathbb{Z}. \end{aligned} \tag{3.14}$$

This yields

$$\omega^{per}_{j,n,k}(x) = \sum_{r=-\infty}^{\infty} 2^{\frac{-j}{2}}\ \hat{\omega}_n(2\pi r 2^{-j})\ e^{2\pi i r x},\ x \in \mathbb{R},\ r \in \mathbb{Z}. \tag{3.15}$$

Using (3.10) in (3.15), we have

$$\omega^{per}_{j,4s+1,k}(x) = 0,\ j \leqslant -1,\ k \in \mathbb{Z},\ s \in \mathbb{N}_0 \text{ and } x \in \mathbb{R}.$$

If $j = 0$ in (3.14), then using (3.9), we have $\hat{\omega}_{4s+1}(2\pi k) \neq 0$ for odd k, so $\omega^{per}_{0,4s+1,k}(x)$ is neither 0 nor any constant for such value of k.

(iii) Let $j > 0,\ m \in \mathbb{Z}$ and $0 \leqslant k \leqslant 2^j - 1$. Then

$$\begin{aligned} \omega^{per}_{j,n,k+2^j m}(x) &= \sum_{l=-\infty}^{\infty} \omega_{j,n,k+2^j m}(x+l) \\ &= 2^{\frac{j}{2}} \sum_{l=-\infty}^{\infty} \omega_n(2^j(x+l-m)-k) = \omega^{per}_{j,n,k}(x),\ x \in \mathbb{R}. \end{aligned}$$

(iv) Let $2^j \geqslant 2g-1$. Then, using (3.1), we get

$$\begin{aligned} \omega^{per}_{j,n,k}(x) &= 2^{\frac{j}{2}} \sum_{l=-\infty}^{\infty} \omega_n(2^j x + 2^j l - k) \\ &= 2^{\frac{j}{2}} \sum_{l=-\infty}^{\infty} \omega_n(2^j x - (k - 2^j l)) = \sum_{l=-\infty}^{\infty} \omega_{j,n,k-2^j l}(x). \end{aligned} \tag{3.16}$$

Since ω_n is compactly supported, it follows that the supports of the terms in the above sum do not overlap for sufficiently large value of 2^j. Choose smallest $j' \in \mathbb{Z}$ such that $2^{j'} \geqslant 2g-1$. Now, $supp(\omega_{j,n,k}) = I_{j,k}$, where $I_{j,k} = \left[\frac{k}{2^j}, \frac{k+2g-1}{2^j}\right]$ and for $j > j'$ the width of $I_{j,k} \leqslant 1$ and thus, (3.16) implies that for $x \in [0,1]$, periodized wavelet packets can be expressed as

$$\omega^{per}_{j,n,k}(x) = \begin{cases} \omega_{j,n,k}(x), & \text{if } x \in I_{j,k} \cap [0,1] \\ \omega_{j,n,k}(x+1), & \text{if } x \in [0,1], \text{ and } x \notin I_{j,k}. \end{cases}$$

□

The following result shows that DPWP forms an orthonormal system for $L^2([0,1])$.

Theorem 3.2. *The collection of Daubechies periodized wavelet packets* $\{\omega^{per}_{0,n,k}(x)\}_{n\in\mathbb{N}_0,k\in\mathbb{Z}}$ *is an orthonormal system for* $L^2([0,1])$.

Proof. The details of the proof can be seen in ([32], Section 9.3).

Corollary 3.1. *For each fixed* $j \in \mathbb{Z}$, *the collection of Daubechies periodized wavelet packets* $\{\omega^{per}_{j,n,k}\}_{n\in\mathbb{N}_0,k\in\mathbb{Z}}$ *forms an orthonormal system for* $L^2([0,1])$.

Proof. Proof follows from the Theorem 3.2.

4. Approximation properties of V^{per}_J

The domain of periodized wavelet packets when restricted to $[0,1]$, generate an MRA of $L^2([0,1])$ analogously to that of $L^2(\mathbb{R})$. The significant subspaces involved are defined as

$$V_j^{per} = span\{\omega_{j,0,k}^{per}(x) : x \in [0,1]\}_{k=0}^{2^j-1},$$
$$U_{j,n}^{per} = span\{\omega_{j,n,k}^{per}(x) : x \in [0,1]\}_{n\in\mathbb{N},\ k=0,1,\ldots,2^j-1}.$$

Note that the V_j^{per} are nested similarly as in the case of non-periodic MRA,

$$V_0^{per} \subset V_1^{per} \subset V_2^{per} \subset \cdots \subset L^2([0,1]).$$

So, $\overline{\bigcup_{j=0}^{\infty} V_j^{per}} = L^2([0,1])$. Further, the orthogonality relationship gives

$$L^2([0,1]) = V_{J_1}^{per} \oplus \bigoplus_{j=J_1}^{\infty} \bigoplus_{n=2^j}^{2^{j+1}-1} U_{0,n}^{per}, \text{ for some } J_1 > 0. \tag{4.1}$$

Let $f \in V_J^{per}$ and let $J_1 : 1 \leqslant J_1 \leqslant J$. Then, the periodized wavelet packet expansion is

$$f(x) = \sum_{k=0}^{2^{J_1}-1} c_{J_1,k}\ \omega_{J_1,0,k}^{per}(x) + \sum_{j=J_1}^{J-1} \sum_{n=2^j}^{2^{j+1}-1} \sum_{k=0}^{2^j-1} d_{0,k}^n\ \omega_{0,n,k}^{per}(x),\ x \in [0,1], \tag{4.2}$$

where the coefficients $c_{j,k}$ and $d_{0,k}^n$ are respectively given by

$$c_{j,k} = \int_0^1 f(x)\ \omega_{j,0,k}^{per}(x)\ dx \text{ and } d_{0,k}^n = \int_0^1 f(x)\ \omega_{0,n,k}^{per}\ dx.$$

Let $\omega_n, n \in \mathbb{N}_0$ be wavelet packets. Then the orthogonal projections of $L^2([0,1])$ on V_j^{per} and $U_{0,n}^{per}$ are respectively defined as

$$(P_{V_j^{per}} f)(x) = \sum_{k=-\infty}^{\infty} c_{j,k}\ \omega_{j,0,k}^{per}(x), \tag{4.3}$$

$$(P_{U_{0,n}^{per}} f)(x) = \sum_{k=-\infty}^{\infty} d_{0,k}^n\ \omega_{0,n,k}^{per}(x), \tag{4.4}$$

where

$$c_{j,k} = \int_0^1 f(x)\ \omega_{j,0,k}^{per}(x)\ dx, \qquad d_{0,k}^n = \int_0^1 f(x)\ \omega_{0,n,k}^{per}(x)\ dx$$

and

$$P_{V_J^{per}} f = P_{V_{J_1}^{per}} f + \sum_{j=J_1}^{J-1} \sum_{n=2^j}^{2^{j+1}-1} P_{U_{0,n}^{per}} f,\ J \in \mathbb{Z}.$$

For $f \in L^2([0,1]) \cap C^g(\mathbb{R})$ $(g > 1)$ and $x \in [0,1]$, the approximation error is given by $E_J^{per}(x) = f(x) - (P_{V_J^{per}} f)(x)$.

Now, we give the following result related to the approximation error.

Theorem 4.1. *Let* $f \in L^2([0,1]) \cap C^g(\mathbb{R})$ $(g > 1)$ *be a function and* $\omega_n,\ n \in \mathbb{N}_0$ *be DPWP such that*

(i) $|\omega_n(t)| = O(2^{-gj})$ *for* $n = 2^j, ..., (2^{j+1}-1)$*, where* $j \geqslant 0$,

(ii) $\int_{\mathbb{R}} x^p \ \omega_n(x) \ dx = 0,$ *for* $0 \leqslant p \leqslant g-1$.

Let $J \in \mathbb{Z} : J \geqslant J_1 > 0$, *where* $2^{J_1} \geqslant 2g-1$. *Then*

$$\|E_J^{per}(x)\|_\infty = O(2^{-J(g-1)}).$$

Proof. The periodic wavelet packet expansion for $P_{V_J}^{per}$ is

$$(P_{V_J^{per}} f)(x) = \sum_{k=0}^{2^{J_1}-1} c_{J_1,k} \ \omega_{J_1,0,k}^{per}(x) + \sum_{j=J_1}^{J-1} \sum_{n=2^j}^{2^{j+1}-1} \sum_{k=0}^{2^j-1} d_{0,k}^n \ \omega_{0,n,k}^{per}(x). \tag{4.5}$$

Taking $J \to \infty$, the periodic wavelet packet expansion for $f \in L([0,1])$ is given by

$$f(x) = \sum_{k=0}^{2^{J_1}-1} c_{J_1,k} \ \omega_{J_1,0,k}^{per}(x) + \sum_{j=J_1}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} \sum_{k=0}^{2^j-1} d_{0,k}^n \ \omega_{0,n,k}^{per}(x). \tag{4.6}$$

The approximation error is given by

$$E_J^{per}(x) = f(x) - (P_{V_J^{per}} f)(x), \ x \in [0,1].$$

Therefore, we get

$$E_J^{per}(x) = \sum_{j=J}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} \sum_{k=0}^{2^j-1} d_{0,k}^n \ \omega_{0,n,k}^{per}(x). \tag{4.7}$$

Let $I = [0, 2g-1]$ be the compact support of ω_n. Then, it follows that $\omega_{0,n,k}$ is supported in the interval $I_k = [k, k+2g-1]$ with length $l(I_k) = 2g-1$ and centre $x_k = k + g - \frac{1}{2}$.

Note that

$$d_{0,k}^n = \int_0^1 f(x) \ \overline{\omega_{0,n,k}^{per}(x)} \ dx = \int_{\mathbb{R}} f(x) \ \overline{\omega_{0,n,k}(x)} \ dx. \tag{4.8}$$

Since $f \in C^g(\mathbb{R})$, using Taylor's expansion of f about the point x_k, it follows that

$$\begin{aligned} |d_{0,k}^n| = \Big| \int_{\mathbb{R}} &[f(x_k) + (x-x_k) \ f^{(1)}(x_k) + \cdots \\ &+ \frac{1}{(g-1)!}(x-x_k)^{g-1} \ f^{(g-1)}(x_k) + R_g(x)] \ \omega_{0,n,k}(x) \Big|, \end{aligned}$$

where $R_g(x) = \frac{1}{g!}(x-x_k)^g \ f^{(g)}(\eta)$ for some number η between x_k and x. If $x \in I_k$, then, we have

$$|R_g(x)| \leqslant \frac{1}{g!}\big(g - \frac{1}{2}\big)^g \ \max_{x \in I_k} |f^{(g)}(x)|.$$

Therefore, we compute

$$
\begin{aligned}
|d^n_{0,k}| &= \Big| \int_{I_k} R_g(x)\ \overline{\omega_{0,n,k}(x)} \Big| \\
&\leqslant \frac{1}{g!}\big(g-\frac{1}{2}\big)^g \ \max_{x\in I_k} |f^{(g)}(x)| \int_{I_k} |\omega_{0,n,k}(x)|\ dx \\
&\leqslant \frac{2^{\frac{1}{2}}K}{g!}\big(g-\frac{1}{2}\big)^{g+\frac{1}{2}} \ \max_{x\in I_k} |f^{(g)}(x)| \Big(\int_0^{2g-1} 2^{-2gj}\ dx \Big)^{\frac{1}{2}} \\
&= \frac{2K}{g!}\big(g-\frac{1}{2}\big)^{g+1} \ \max_{x\in I_k} |f^{(g)}(x)|\ 2^{-gj} = M\ 2^{-gj}, \qquad (4.9)
\end{aligned}
$$

where $M = \frac{2K}{g!}\big(g-\frac{1}{2}\big)^{g+1} \ \max\limits_{x\in I_k} |f^{(g)}(x)|$. Using (4.9) in (4.7), we obtain

$$
\|E^{per}_J(x)\|_\infty \leqslant \sum_{j=J}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} \sum_{k=0}^{2^j-1} M\ 2^{-gj} \ \max_{x\in I_k} |\omega^{per}_{0,n,k}(x)|.
$$

Define $C_{\omega^{per}_{0,n,k}} = \max\limits_{x\in I_k} |\omega^{per}_{0,n,k}(x)|$. Then, we compute

$$
\begin{aligned}
\|E^{per}_J(x)\|_\infty &\leqslant C_{\omega^{per}_{0,n,k}}\ M \sum_{j=J}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} \sum_{k=0}^{2^j-1} 2^{-gj} \\
&= C_{\omega^{per}_{0,n,k}}\ M \sum_{j=J}^{\infty} 2^{-(g-2)j} = K'\ 2^{-(g-1)j}, \qquad (4.10)
\end{aligned}
$$

where K' is a constant. Thus, we find that with respect to the resolution J, error E^{per}_J shows an exponential decay. Besides, more is the number of vanishing moments, faster will be the decay. □

5. Compression errors

In this section, using hard thresholding technique, we discuss the compression errors.

The information about a signal f is stored in the form of wavelet packet coefficients $\{\langle f, \omega_{j,n,k}\rangle\}_{j,k\in\mathbb{Z}}$ and this knowledge helps us to reconstruct the signal f. Nevertheless, practically it is not possible to store such an infinite sequence of non-zero numbers and thus it is necessary to chose only finite number of such coefficients. This is primarily done by specifying an independent parameter or threshold $\delta > 0$ such that only those coefficients are retained for which $|\langle f, \omega_{j,n,k}\rangle| \geqslant \delta$. Such coefficients are known as significant wavelet packet coefficients, whereas others which do not satisfy the above inequality are quantized to zero and are known as insignificant wavelet packet coefficients. Thus, threshold value δ separates the insignificant wavelet packet coefficients from the significant ones. One may note that the selection of wavelet packets also plays an essential role as we always look for those wavelet packets which correlates well with the signal under consideration or detection. If there is a large amount of signal information present, one can keep large number of wavelet packet coefficients, as compared to lesser number in case of a noisy signal. The above process is known as hard thresholding. The errors which appeared when small wavelet packet coefficients are repudiated are referred to as compression errors.

Let us define a set of significant wavelet packet coeffficients at level j as

$$S_j^\delta = \{k : 0 \leqslant k \leqslant 2^j - 1 \text{ and } |d_{0,k}^n| > \delta \text{ for } n = 2^j, ..., (2^{j+1} - 1)\}.$$

The set of insignificant wavelet packet coefficients are given by $I_j^\delta = S_j^0 \setminus S_j^\delta$. Thus, δ-truncated wavelet packet expansion for f is given by

$$(P_{V_J^{per}} f)^\delta(x) = \sum_{k=0}^{2^{J_1}-1} c_{J_1,k}\ \omega_{J_1,0,k}^{per}(x) + \sum_{j=J_1}^{J-1} \sum_{n=2^j}^{2^{j+1}-1} \sum_{k \in S_j^\delta} d_{0,k}^n\ \omega_{0,n,k}^{per}(x).$$

Let $n_S(\delta)$ be the number of all significant wavelet packet coefficients, i.e.,

$$n_S(\delta) = \sum_{j=J_1}^{J-1} \sum_{n=2^j}^{2^{j+1}-1} \aleph(S_j^\delta) + 2^{J_1},$$

where $\aleph(S_j^\delta)$ denotes the cardinality of S_j^δ. The last term in the above sum is due to the coefficient of scaling function as they contribute the coarse approximation on which the fine structures are built by wavelet packets. Let us suppose that $n = 2^J$ be the dimension of V_J^{per}. Then, define $n_I(\delta) = n - n_S(\delta)$ to be the number of insignificant wavelet packet coefficients in the expansion. Due to this truncation, an error $E_{\delta,J}^{per}$ has been occured and is given by

$$\begin{aligned} E_{\delta,J}^{per}(x) &= (P_{V_J^{per}} f)(x) - (P_{V_J^{per}} f)^\delta(x) \\ &= \sum_{j=J_1}^{J-1} \sum_{n=2^j}^{2^{j+1}-1} \sum_{k \in I_j^\delta} d_{0,k}^n\ \omega_{0,n,k}^{per}(x) \end{aligned} \tag{5.1}$$

with $n_I(\delta)$ number of terms. This ensures the inequality

$$\|E_{\delta,J}^{per}(x)\|_2 \leqslant \delta\ (n_I(\delta))^{\frac{1}{2}}. \tag{5.2}$$

Now, if we redefine the set of significant wavelet packet coefficients as

$$S_j^\delta = \{k : 0 \leqslant k \leqslant 2^j - 1 \text{ and } |d_{0,k}^n| > \delta\ 2^{-\frac{j}{2}} \text{ for } n = 2^j, ..., (2^{j+1} - 1)\},$$

then $I_j^\delta = S_j^0 \setminus S_j^\delta$. Therefore the scale j can be employed to transmute the threshold value δ.

Hence using (5.1), we finally obtain

$$\begin{aligned} \|E^{per}_{\delta,J}(x)\|_\infty &= \sum_{j=J_1}^{J-1} \sum_{n=2^j}^{2^{j+1}-1} \sum_{k\in I^\delta_j} \max_x(|d^n_{0,k}\ \omega^{per}_{0,n,k}(x)|) \\ &= C'_{\omega^{per}_{0,n,k}} \sum_{j=J_1}^{J-1} \sum_{n=2^j}^{2^{j+1}-1} \sum_{k\in I^\delta_j} |d^n_{0,k}| = C'_{\omega^{per}_{0,n,k}}\ \delta\ n_I(\delta), \end{aligned} \tag{5.3}$$

where $C'_{\omega^{per}_{0,n,k}} = 2^{-\frac{j}{2}}\ C_{\omega^{per}_{0,n,k}}$.

Conclusion

Restrepo et al. [35] studied periodized wavelets by restricting the wavelets on the bounded subsets of $\mathbb{R}$. In the present article, we amalgamated their with that of Daubechies [3, 4] and studied Daubechies periodized wavelet packets and using it obtained approximation of periodic functions. Also, thresholding technique is used to study compression errors. On comparing (5.2) and (5.3), we find that the threshold is scaled in (5.3) which decreases substantially on the increase in the scale resulting in consequence of which the number of wavelet packet coefficients increases at the finer scales. Thus, n_I will be lesser in the latter case. Finally, we have also observed that using wavelet packets instead of just Daubechies wavelet bases, one can expect reduction in the compression errors.

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