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L^p AND HÖLDER ESTIMATES FOR CAUCHY-RIEMANN EQUATIONS ON CONVEX DOMAIN OF FINITE/INFINITE TYPE WITH PIECEWISE SMOOTH BOUNDARY IN $\mathbb{C}^2$

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Abstract. In this paper, we investigate L^p estimates ($1 \leq p \leq +\infty$) and f-Hölder estimates for the Cauchy-Riemann equation in a class of convex domains of finite or infinite type with piecewise smooth boundary in $\mathbb{C}^2$.

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1. Introduction

Let (z_1, z_2) be the complex Euclidean coordinates of $\mathbb{C}^2$ and let $\Omega \subset \mathbb{C}^2$ be a bounded domain. The Cauchy-Riemann complex on $C^1(\Omega)$-functions is defined as follow:

$$\bar{\partial} u = \sum_{j=1}^{2} \frac{\partial u}{\partial \bar{z}_j} d\bar{z}_j,$$

where $\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \sqrt{-1}\frac{\partial}{\partial x_{2+j}}\right)$ with $z_j = x_j + \sqrt{-1}x_{2+j}, j = 1, 2$. One of the most fundamental and important problems in multidimensional complex analysis is to solve the Cauchy-Riemann equation

$$\bar{\partial} u = \varphi$$

for a given $(0,1)$-form $\varphi = \varphi_1 d\bar{z}_1 + \varphi_2 d\bar{z}_2$. In the case when Ω is a smoothly bounded, convex domain, L^p estimates and Hölder estimates of the $\bar{\partial}$-equation were studied by many mathematicians. The book [3] by Chen and Shaw is an excellent reference for this literature. In this paper, we investigate the problem on a class of bounded convex domains with non-smooth boundaries.

For each $j = 1, \dots, N$, let $\Omega_j \subset \mathbb{C}^2$ be a domain with smooth boundary $b\Omega_j$ and $\rho_j : \mathbb{C}^2 \to \mathbb{R}$ is a function of class $C^\infty(\mathbb{C}^2)$. Assume that ρ_j is a defining function of Ω_j in the following sense:

- $\rho_j(z) < 0$ if and only if $z \in \Omega_j$;
- $\{z \in \mathbb{C}^2 : \rho_j(z) = 0\} = b\Omega_j$;

- $|\nabla\rho_j(z)| > 0$ if $z \in b\Omega_j$;
- $\nabla\rho_j \perp b\Omega_j$.

The certain domain in this paper is the transversal intersection of $\Omega_1, \dots, \Omega_N$, that is defined as follows

$$\Omega = \Omega_1 \cap \Omega_2 \dots \cap \Omega_N \tag{1.1}$$

so that $d\rho_{j_1} \wedge \dots \wedge d\rho_{j_l} \neq 0$ on $\bigcap_{k=1}^{l} U_{j_k}$ for $1 \leq j_1 < \dots < j_l \leq N$, where U_j is a neighborhood of $b\Omega_j$.

For $z \in \Omega$, let us define

$$\frac{1}{\rho(z)} = \sum_{j=1}^{N} \frac{1}{\rho_j(z)}. \tag{1.2}$$

Since $\rho \in C^\infty(\Omega)$ and

$$N^{-1} \inf_{1\leq j\leq N}\{-\rho_j\} \leq -\rho \leq \inf_{1\leq j\leq N}\{-\rho_j\},$$

we have $\Omega = \{z \in \mathbb{C}^2 : \rho(z) < 0\}$ and $-\rho(z) \approx \operatorname{dist}(z, b\Omega)$. Here and in what follows, the notations $\lesssim$ and $\gtrsim$ denote inequalities up to a positive constant, and $\approx$ means the combination of $\lesssim$ and $\gtrsim$.

Such domains were firstly considered by M. Range in [13] and then by Berndtsson-Andersson in [1]. The following theorem is the first result of this paper.

Theorem 1.1 (L^p estimates)**.** *Let Ω_j, $j = 1, \dots, N$, be smooth bounded, convex domains and admit the maximal type F at all boundary points for a same function F (see Definition (2.1)). Let Ω be a piecewise smooth domain defined by (1.1) and let ρ be defined by (1.2). Let φ be a $(0,1)$-form in $L^p_{(0,1)}(\Omega)$, for $p \in [1, +\infty]$. Then, there is a function $u \in L^p(\Omega)$ satisfying $\bar\partial u = \varphi$ in the weak sense, and*

$$\|u\|_{L^p(\Omega)} \leq C_p \|\varphi\|_{L^p_{(0,1)}}(\Omega).$$

By $\bar\partial u = \varphi$ in the weak sense, we mean that $u = \lim_{\varepsilon\to 0^+} u_\varepsilon$ in $L^p(\Omega)$ (or f-Hölder spaces in Theorem 1.2), where u_ε is the Berndtsson-Andersson solution of $\bar\partial u_\varepsilon = \varphi_\varepsilon$ with smooth φ_ε.

In the lecture of Range [15] given at Cortona (Italy), he proved that on the following smoothly bounded convex domain

$$\Omega^m = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^{2m} + |z_2|^2 - 1 < 0\}, \tag{1.3}$$

where $m = 1, 2, \dots$, the Cauchy-Riemann equation $\bar\partial u = \varphi$ is solvable. This domain is said to be finite type in the sense of Range (see [14, Definition 1.1]). Moreover, the solution u is Hölder continuous of order $\alpha < \frac{1}{2m}$ whenever φ is a $C^1(\overline{\Omega^m})$ $(0,1)$-form. Moreover, he also showed that on the infinite type smooth boundary convex domain

$$\Omega^\infty = \{(z_1, z_2) \in \mathbb{C}^2 : \exp(1 + 2/s) \cdot \exp\left(\frac{-1}{|z_1|^s}\right) + |z_2|^2 - 1 < 0\},$$

for $0 < s < 1$, the Cauchy-Riemann equation is although solvable. Nevertheless, there is no solution which is Hölder continuous of any positive order. For this motivation, we need a class of corresponding non-standard Hölder spaces, namely f-Hölder spaces on Ω. That is, for f being an increasing function such that $\lim_{t\to+\infty} f(t) = +\infty$,

$$\Lambda^f(\Omega) = \{u \in L^\infty(\Omega) : \|u\|_f := \|u\|_{L^\infty(\Omega)} + \sup_{z,z+h\in\Omega} f(|h|^{-1})|u(z+h) - u(z)| < +\infty\}.$$

It is clear that if $f(t) = t^\alpha$, for $0 < \alpha < 1$, the space $\Lambda^f(\Omega)$ coincides to $\Lambda^\alpha(\Omega)$-the classical Hölder space of order α. The f-Hölder space was introduced in [10, 9] and extended to study tangential Cauchy-Riemann equations in [6, 7].

Theorem 1.2 (f-Hölder estimates)**.** *Let $\Omega_1, \dots, \Omega_N$ and Ω be domains defined in Theorem 1.1. Let φ be a continuous $(0,1)$-form. Then, there is a function $u \in \Lambda^f(\Omega)$ satisfying $\bar\partial u = \varphi$ in the weak sense, and*

$$\|u\|_{\Lambda^f(\Omega)} \lesssim \|\varphi\|_{L^\infty_{(0,1)}}(\Omega),$$

where

$$f(d^{-1}) := \left(\int_0^d \frac{\sqrt{F^*(t)}}{t} dt\right)^{-1},$$

and F^* is the inverse function of F.

The paper is organized as follows. We recall the construction of Berndtsson-Andersson $\bar\partial$-solution and maximal type F in Section 2. Then, we prove Theorem 1.1 in Section 3 and Theorem 1.2 in Section 4.

2. Berndtsson-Andersson solution and Maximal type F

In this section, for every $k = 1, \dots, N$, we assume that Ω_k is a bounded convex domain in $\mathbb{C}^2$ with smooth boundary $b\Omega_k$, and that ρ_k is a defining function for Ω_k. The convexity means

$$\sum_{i,j=1}^4 \frac{\partial^2 \rho_k}{\partial x_i \partial x_j}(\zeta) a_i a_j \geq 0 \quad \text{on } b\Omega_k,$$

for every $a = (a_1, \dots, a_4) \in \mathbb{R}^4$ with $\sum_{j=1}^4 a_j \frac{\partial \rho_k}{\partial x_j}(\zeta) = 0$ on $b\Omega_k$. Let us define the following support function of Ω_k, for $\zeta, z \in \overline{\Omega}_k$:

$$\Phi_k(\zeta, z) = \Phi_{\Omega_k}(\zeta, z) = \sum_{j=1}^2 \frac{\partial \rho_k}{\partial \zeta_j}(\zeta)(\zeta_j - z_j). \tag{2.1}$$

For each $\zeta \in b\Omega_k$, the condition $\Phi_k(\zeta, z) = 0$ characterizes the translate of the complex tangent space $T_\zeta^{\mathbb{C}}(b\Omega_k)$. The convexity of Ω_k implies

$$\Phi_k(\zeta, z) \neq 0, \quad \text{for all } \zeta \in b\Omega_k, z \in \overline{\Omega}_k.$$

Theorem 2.1 (Berndtsson-Andersson solution [1]). *Let $\Omega_1, \ldots, \Omega_N$ and Ω be defined as above. Let $\Delta = \{(\zeta, z) \in \Omega \times \Omega | \zeta = z\}$ be the diagonal of Ω. Then for any $r > 1$, there exists a $(2,1)$-form $K^r(\zeta, z)$ defined on $(\Omega \times \Omega) \setminus \Delta$ such that: for any $\bar{\partial}$-closed, continuous $(0,1)$-form φ in $\overline{\Omega}$, the following*

$$S[\varphi](z) = \int_{\zeta \in \Omega} \varphi(\zeta) \wedge K^r(\zeta, z), \quad z \in \Omega,$$

satisfies

$$\bar{\partial}(S[\varphi])(z) = \varphi(z).$$

Moreover, in [4, page 1421], Cho and Park showed that

$$\begin{aligned} |K^r(\zeta, z)| \lesssim & \frac{\prod_{j=1}^{N} |\rho_j(\zeta)|^r}{|\zeta - z|^3 \prod_{j=1}^{N} |\Phi_j(\zeta, z)|^r} + \frac{1}{|\zeta - z|} \sum_{k=1}^{N} \left[\left(\prod_{j \neq k} \frac{|\rho_j(\zeta)|^r . |\rho_k(\zeta)|^{r-1}}{|\Phi_j(\zeta, z)|^r |\Phi_k(\zeta, z)|^{r+1}} \right.\right. \\ & \left.\left. + \frac{\prod_{j=1}^{N} |\rho_j(\zeta)|^r}{\prod_{j \neq k} |\Phi_j(\zeta, z)|^r |\Phi_k(\zeta, z)|^{r+1}} \right) \right] := K_1^r(\zeta, z) + K_2^r(\zeta, z) + K_3^r(\zeta, z). \end{aligned} \tag{2.2}$$

and they also obtained the following L^1-boundedness.

Theorem 2.2 (L^1-estimate ([4])). *Let φ be a $\bar{\partial}$-closed $(0,1)$-form whose coefficients in $L^1(\Omega)$. Then,*

$$\bar{\partial}(S[\varphi]) = \varphi$$

in the weak sense, and $\|S[\varphi]\|_{L^1(\Omega)} \lesssim \|\varphi\|_{L^1_{(0,1)}(\Omega)}$.

In the present work, we are going to study the case L^∞-estimates, for the solution to the $\bar{\partial}$-equation. Hence, we need the following geometric ingredient.

Definition 2.1. Let $F : [0, \infty) \to [0, \infty)$ be a function such that

(1) F is smooth and increasing;

(2) $F(0) = 0$;

(3) $\int_0^\delta |\ln F(t^2)| dt < \infty$ for some small $\delta > 0$;

(4) $\dfrac{F(t)}{t}$ is non-decreasing.

Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded, convex domain. Then, Ω is called a domain admitting the maximal type F at the boundary point $P \in b\Omega$ if there are positive constants c, c', such that, for all $\zeta \in \bar{\Omega} \cap B(P, c')$ we have

$$\rho(z) \gtrsim F(|z-\zeta|^2),$$

for all $z \in B(\zeta, c)$ with $\Phi_\Omega(\zeta, z) = 0$.

In the case $F(t) = t^m$, Ω is called a convex domain of finite type $2m$ in the sense of Range. The maximal type F was introduced in [6, 7] to study tangential Cauchy-Riemann equations, and the global Lipschitz continuity of the Bergman projection weakly pseudoconvex domains in $\mathbb{C}^2$. Some examples are follows.

- Let Ω be a strongly convex domain with its defining function ρ. Then,

$$\Re\Phi(\zeta, z) \geq \rho(\zeta) - \rho(z) + \lambda_0|\zeta - z|^2,$$

  for $|\zeta - z|$ and $|\rho(\zeta)|$ small, and $\lambda_0 > 0$ (see [3] for details). Hence, when $\zeta \in b\Omega \cap \{|\zeta - z| < c\}$, and $\Phi(\zeta, z) = 0$, we have

$$\rho(z) \gtrsim F(|z-\zeta|^2),$$

  with $F(t) = t$. So, Ω in this case is of maximal type F.
- The complex ellipsoid is

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^{2m_1} + |z_2|^{2m_2} < 1\},$$

  where $m_1, m_2 \in \mathbb{N}$. Then Ω is convex of maximal type F with $F(t) = t^m$, for $m = \max\{m_1, m_2\}$, see [15].
- Assume that Ω denote a bounded domain of the type

$$\Omega = \left\{ z = (z_1, z_2) \in \mathbb{C}^n : \rho(z) = \sum_{j=1}^{2} \rho_j(|z_j|^2) - 1 < 0, \right\}$$

  where all functions ρ_j are assumed to be real-analytic in $[0, a_j]$ such that
  (1) $\rho_j'(t) \geq 0, \rho_j'(t) + 2t\rho_j''(t) \geq 0$ for $0 \leq t \leq a_j$;
  (2) $\rho_j'(0) = \rho_j(0) = 0$ and $\rho_j(a_j) > 1$.
  In [2], J. Bruna and J. del Castillo obtained that there exists a positive integer m such that

$$\Re\Phi(\zeta, z) \gtrsim \rho(\zeta) - \rho(z) + \sum_{k=1}^{2} \frac{\partial^2 \rho}{\partial\zeta_k \partial\bar{\zeta}_k}(\zeta)|z_k - \zeta_k|^2 + |\zeta - z|^{2m},$$

  for $\zeta, z \in \bar{\Omega}$ (see [2, Formula (7)]). Therefore Ω is a smoothly bounded, admissibly decoupled, convex domain admitting an F-type, with $F(t) = t^m$.

- Let

$$\Omega^{\infty} = \{(z_1, z_2) \in \mathbb{C}^2 |\, \rho(z) := \exp(1 + 2/s) \cdot \exp\left(\frac{-1}{|z_1|^s}\right) + |z_2|^2 - 1 < 0\}.$$

Since

$$\Re[\Phi_{\Omega^\infty}(\zeta, z)] \geq \rho(\zeta) - \rho(z) + \exp(1 + 2/s) \exp\left\{\frac{-1}{32|\zeta - z|^{2s}}\right\},$$

for $0 < s < 1/2$, Ω^∞ is convex of the maximal type $F(t) = \exp(\frac{-1}{32.t^s})$, see [19]. Note that Ω_∞ is a domain of infinite type.

The most important property of support functions on convex domains admitting a maximal type F is the following.

Lemma 2.1. *[[6, Lemma 3.3, p. 112]] For each $k = 1, \dots, N$, let Ω_k be a smoothly bounded, convex domain in $\mathbb{C}^2$ of maximal type F at $P \in b\Omega_k$. Then there is a positive constant c_k such that the support function $\Phi_k(\zeta, z)$ satisfies the following estimate*

$$|\Phi_k(\zeta, z)| \gtrsim |\rho_k(\zeta)| + |\rho_k(z)| + |\Im\Phi_k(\zeta, z)| + F(|z - \zeta|^2), \tag{2.3}$$

for every $\zeta \in \bar{\Omega}_k \cap B(P, c)$, and $z \in \overline{\Omega}_k$, $|z - \zeta| < c_k$.

3. Proof of L^p-estimate

By Theorem 2.2 and Riesz-Thorin Interpolation Theorem (see Theorem B.6, Appendix B in [3] for more details), we are only going to prove the L^∞-estimate. Let φ be a $(0, 1)$-form with L^∞-coefficients on $\overline{\Omega}$. Then by Hölder inequality,

$$|S[\varphi](z)| \lesssim \|\varphi\|_{L^\infty_{(0,1)}(\Omega)} \int_\Omega |K^r(\zeta, z)| dV(\zeta),$$

where $dV(.)$ is the Lebesgue measure in $\mathbb{R}^4$. Next, we will estimate the integral of each term in the right hand side of (2.2).

Firstly, by Lemma 2.1, we obtain

$$\int_\Omega K_1^r(\zeta, z) dV(\zeta) \lesssim \int_\Omega \frac{\prod_{j=1}^N |\rho_j(\zeta)|^r}{|\zeta - z|^3 \prod_{j=1}^N |\rho_j(\zeta)|^r} dV(\zeta) \lesssim \int_\Omega \frac{dV(\zeta)}{|z - \zeta|^3} \lesssim 1.$$

Next, for the second term $K_2^r(\zeta, z)$ and $K_3^r(\zeta, z)$, we have

$$\int_\Omega K_2^r(\zeta, z) dV(\zeta) \lesssim \sum_{k=1}^N \int_\Omega \frac{dV(\zeta)}{|\zeta - z||\Phi_k(\zeta, z)|^2} \tag{3.1}$$

and

$$\int_{\Omega} K_3^r(\zeta,z)dV(\zeta) \lesssim \sum_{k=1}^{N}\int_{\Omega}\frac{dV(\zeta)}{|\zeta-z\|\Phi_k(\zeta,z)|}. \tag{3.2}$$

We are going to estimate the right hand side of (3.1) and the process for (3.2) is similar and more simple. To do this, we recall the Henkin coordinates on each Ω_k.

Lemma 3.1. [5, page 608] *There exist positive constants M, a and $\eta \le c$, and, for each z with $dist(z, b\Omega_k) \le a$, there is a smooth local coordinate system $(t_1, t_2, t_3, t_4) = t = t(\zeta, z)$ on the ball $B(z, c)$ such that we have*

$$\begin{cases} t(z,z) = 0, \\ t_1(\zeta) = \rho_k(\zeta) - \rho_k(z), \\ t_2(\zeta) = \Im(\Phi_k(\zeta,z)), \\ |t| < \delta \quad for\ \zeta \in B(z,c), \\ |J_{\mathbb{R}}(t)| \le M \quad and \quad |det J_{\mathbb{R}}(t)| \ge \frac{1}{M}, \end{cases}$$

where $J_{\mathbb{R}}(t)$ is the Jacobian of the transformation at t.

Now, for each fixed k, by Lemma 2.1, we have

$$\begin{aligned}\int_{\Omega}\frac{dV(\zeta)}{|\zeta-z\|\Phi_k(\zeta,z)|^2} &\lesssim \int_{\Omega}\frac{dV(\zeta)}{|\zeta_1-z_1|(|\rho_k(\zeta)|+|\Im\Phi_k(\zeta,z)|+F(|z_1-\zeta_1|^2))^2} \\ &\lesssim \int_{t^2<c^2}\frac{dt_1dt_2dt_3dt_4}{|(t_3,t_4)|(t_1+t_2+F(t_3^2+t_4^2))^2} \\ &\lesssim \int_{t_1^2+t_3^2+t_4^2<c^2}\frac{dt_1dt_3dt_4}{|(t_3,t_4)|(t_1+F(t_3^2+t_4^2))} \\ &\lesssim \int_{t_3^2+t_4^2<c^2}\frac{\ln F(t_3^2+t_4^2)}{|(t_3,t_4)|}dt_3dt_4 \\ &\lesssim \int_0^c \ln F(s^2)ds \quad (<\infty \text{ since the condition on } F) \\ &\quad\text{(using the polar coodinates } t_3 = s\cos\theta, t_4 = s\sin\theta).\end{aligned}$$

This estimate completes the proof of the L^∞-estimate and so Theorem 1.1.

4. Proof of f-Hölder estimates

Before to prove, we recall the General Hardy-Littlewood Lemma for $\Lambda^f(\Omega)$-continuous which was established by Khanh in [10].

Lemma 4.1. *Let Ω be a smoothly bounded domain in $\mathbb{R}^n$ and let ρ be a defining function of Ω. Let $G : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function such that $\frac{G(t)}{t}$ is* *decreasing and* $\int_0^d \frac{G(t)}{t}dt < \infty$ *for* $d > 0$ *small enough. If* $u \in C^1(\Omega)$ *such that*

$$|\nabla u(x)| \lesssim \frac{G(|\rho(x)|)}{|\rho(x)|} \quad \textit{for every } x \in \Omega,$$

then

$$f(|x-y|^{-1})|u(x)-u(y)| < \infty$$

uniformly in $x, y \in \Omega$, $x \neq y$, *and where* $f(d^{-1}) := \left(\int_0^d \frac{G(t)}{t}dt\right)^{-1}$.

Hence, to prove Theorem 1.2, we need to show that

$$\int_\Omega |\nabla_z K^r(\zeta,z)|dV(\zeta) \lesssim \frac{\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|}.$$

It is not difficult to show that the term $\int_\Omega |\nabla_z K^r(\zeta,z)|dV(\zeta)$ is bounded from above by

$$C \times \sum_{k=1}^N \left(\int_\Omega \frac{dV(\zeta)}{|\zeta-z|^4} + \int_\Omega \frac{dV(\zeta)}{|\zeta-z|^3|\Phi_k(\zeta,z)|} + \int_\Omega \frac{dV(\zeta)}{|\zeta-z|^2|\Phi_k(\zeta,z)|^2} + \int_\Omega \frac{dV(\zeta)}{|\zeta-z||\Phi_k(\zeta,z)|^3} \right).$$

Moreover, in these integrals, it is most difficult to estimate the followings

$$\int_\Omega \frac{dV(\zeta)}{|\zeta-z|^2|\Phi_k(\zeta,z)|^2}, \int_\Omega \frac{dV(\zeta)}{|\zeta-z||\Phi_k(\zeta,z)|^3},$$

and the others are similar and bounded from above by $|\ln(-\rho(z))|$. On the other hand, since $|\Phi_k(z,\zeta)| \lesssim |z-\zeta|$,

$$\int_\Omega \frac{dV(\zeta)}{|\zeta-z|^2|\Phi_k(\zeta,z)|^2} \lesssim \int_\Omega \frac{dV(\zeta)}{|\zeta-z||\Phi_k(\zeta,z)|^3}.$$

Now, again, by Lemma 2.1 and the Henkin coordinates, we obtain

$$\begin{aligned}
\int_\Omega \frac{dV(\zeta)}{|\zeta-z||\Phi_k(\zeta,z)|^3} &\lesssim \int_{|t|^2<c^2} \frac{dt_1dt_2dt_3dt_4}{|(t_3,t_4)|(|\rho_k(z)|+t_1+t_2+F(|(t_3,t_4)|^2))^3} \\
&\lesssim \int_{t_3^2+t_4^22<c^2} \frac{dt_3dt_4}{|(t_3,t_4)|(|\rho_k(z)|+F(|(t_3,t_4)|^2))} \\
&\lesssim \int_0^c \frac{ds}{|\rho(z)|+F(s^2)}.
\end{aligned}$$

Applying the technique introduced in [10], the right-hand-side is split into two parts

$$\int_0^c \frac{dr}{|\rho(z)|+F(r^2)} = \underbrace{\int_0^{\sqrt{F^*(|\rho(z)|)}} \frac{dr}{|\rho(z)|+F(r^2)}}_{\textbf{easy part}} + \underbrace{\int_{\sqrt{F^*(|\rho(z)|)}}^c \frac{dr}{|\rho(z)|+F(r^2)}}_{\textbf{diff. part}}.$$

It is clear that the "easy part" is bounded from above by $\frac{\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|}$. For the "diff. part", if $r \geq \sqrt{F^*(|\rho(z)|)}$, by F is increasing,

$$\frac{F(r^2)}{r^2} \geq \frac{F(F^*(|\rho(z)|))}{F^*(|\rho(z)|)} = \frac{|\rho(z)|}{|F^*(|\rho(z)|)|},$$

so we have

$$\frac{F(r^2)}{|\rho(z)|} \geq \frac{r^2}{F^*(|\rho(z)|)}.$$

Therefore,

$$\int_{\sqrt{F^*(|\rho(z)|)}}^{c} \frac{dr}{|\rho(z)| + F(r^2)} \leq \frac{1}{|\rho(z)|} \int_{\sqrt{F^*(|\rho(z)|)}}^{c} \frac{dr}{1 + r^2/F^*(|\rho(z)|)}$$
$$\leq \frac{\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|} \int_1^\infty \frac{dy}{1+y^2} = \frac{\pi}{4} \frac{\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|}.$$

Hence we obtain $\int_\Omega \frac{dV(\zeta)}{|\zeta - z\|\Phi_k(\zeta, z)|^3} \lesssim \frac{\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|}$. Moreover, since $|\rho(z)| \approx \mathrm{dist}(z, b\Omega)$, we have

$$\int_\Omega \frac{dV(\zeta)}{|\zeta - z| \cdot |\Phi(\zeta, z)|^3} \lesssim \frac{\sqrt{F^*(\mathrm{dist}(z, b\Omega))}}{\mathrm{dist}(z, b\Omega)}.$$

Next, we are going to check that $\frac{\sqrt{F^*(\mathrm{dist}(z, b\Omega))}}{\mathrm{dist}(z, b\Omega)}$ satisfies all conditions in General Hardy-Littlewood Lemma. The fact $\frac{\sqrt{F^*(t)}}{t}$ is decreasing is trivial.

For $d > 0$ small enough, by a changing variables, we have

$$\int_0^d \frac{\sqrt{F^*(t)}}{t} dt = \int_0^{\sqrt{F^*(d)}} y(\ln F(y^2))' dy$$
$$= \sqrt{F^*(d)} \ln d - \lim_{t \to 0} t(\ln F(t^2))$$
$$- \underbrace{\int_0^{\sqrt{F^*(d)}} (\ln F(y^2)) dy}_{\text{finite by the hypothesis}}.$$

Since $|\ln(F(t^2))|$ is decreasing when $0 \leq t \leq \delta$, for $\delta > 0$ small enough, so

$$|\ln F(\eta^2)|\eta \leq \int_0^\eta |\ln F(t^2)| dt \leq \int_0^\delta |\ln F(t^2)| dt < \infty$$

uniformly in $0 \leq \eta \leq \delta$. Hence, $\sqrt{F^*(t)}|\ln t| < \infty$ for all $0 \leq t \leq \sqrt{F^*(\delta)}$, and $\lim_{t \to 0} t|\ln F(t^2)| = 0$. These imply

$$\int_0^d \frac{\sqrt{F^*(t)}}{t} dt < \infty.$$

The last inequality completes the proof of Theorem 1.2.

For example, we consider the case

$$\Omega = \Omega^\infty \cap B\left((0,1), \frac{1}{2}\right),$$

for $0 < s < 1/2$, where $B\left((0,1), \frac{1}{2}\right) \subset \mathbb{C}^2$ be the ball with center $(0,1)$ and radius $1/2$. Since $F(t) = \exp\left(\frac{-1}{32t^s}\right)$, a direct calculation implies $f(t) = \frac{1024^s(1-2s)}{2s}(|\ln t|)^{\frac{1}{2s}-1}$.

Then, if φ be a continuous $(0,1)$-form, the Berndtsson-Andersson solution $S[\varphi]$ of the equation $\bar{\partial}u = \varphi$ belongs to $\Lambda^f(\Omega)$.

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