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EXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF BOUNDARY VALUE PROBLEMS WITH p-LAPLACIAN IN BANACH SPACES

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Abstract. Weakly sufficient conditions that guarantee the existence of positive bounded solutions are obtained for the equation: $(\phi_p(u'(t)))' + f(u(t)) = \theta$, $0 < t < 1$, subject to bounded conditions, by a simple application of a recent theoretical result for sum of two operators. The nonlinearity f takes values in a general Banach space.

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1. Introduction

Let E be a Banach space with a norm $\|\cdot\|$ and zero element θ. Let also,

$$\mathcal{P} = \{u \in E : u \geqslant \theta\}.$$

With $\mathcal{P}^*$ we will denote the dual cone of the cone $\mathcal{P}$. Set $I = [0,1]$. Then $(\mathcal{C}(I,E), \|\cdot\|_c)$ is a Banach space with $\|x\|_c = \max_{t\in I} \|x(t)\|$, and

$$Q = \{x \in \mathcal{C}(I,E) : x(t) \geqslant \theta, \quad t \in I\}$$

is a cone of the Banach space $\mathcal{C}(I,E)$. Let $r > 1$ be arbitrarily chosen and fixed and

$$B_r = \{x \in \mathcal{C}(I,E) : \|x\|_c \leqslant r\}.$$

In this article, we investigate the following boundary value problem (BVP for short):

$$\begin{aligned} (\phi_p(u'(t)))' + f(u(t)) &= \theta, \quad 0 < t < 1, \\ u'(0) = u(1) &= \theta, \end{aligned} \tag{1.1}$$

where

(H1): $\phi_p(s) = |s|^{p-2}s$, $p > 1$ $\quad \phi_p^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$,

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(H2): $f \in \mathcal{C}(\mathcal{P}, \mathcal{P})$ and

$$\sup\{\|f(u(t))\| : u \in Q \cap B_r\} \leqslant M < \infty,$$

where $M > 0$ is a given constant such that

$$\frac{M^{q-1}}{q} < 1. \tag{1.2}$$

Our main result is as follows.

Theorem 1.1. *Suppose* (*H*1) *and* (*H*2)*. Then the BVP* (1.1) *has at least one positive bounded solution.*

In this paper, a positive solution u of (1.1) means $u(t) \geqslant \theta,\ t \in (0,1)$.

Set

$$\begin{aligned} f^\beta &= \limsup_{\|u\|\to\beta} \frac{\|f(u)\|}{\phi_p(\|u\|)}, \quad f_\beta = \liminf_{\|u\|\to\beta} \frac{\|f(u)\|}{\phi_p(\|u\|)}, \\ (\psi f)_\beta &= \liminf_{\|u\|\to\beta} \frac{\psi(f(u))}{\phi_p(\|u\|)}, \end{aligned}$$

where $\beta = 0$ or ∞, $\psi \in \mathcal{P}^*$ and $\|\psi\| = 1$, and for $r_1 > 0$,

$$T_{r_1} = \{x \in E : \|x\| \leqslant r_1\}.$$

Suppose that $\delta \in \left(0, \frac{1}{2}\right)$. If $\mathcal{P}$ is a normal cone, f is uniformly continuous and bounded on $\mathcal{P}\bigcap T_{r_1}$ and there exists a positive constant L_{r_1} with $(q-1)M^{q-2}L_{r_1} < 1$ such that

$$\alpha(f(D)) \leqslant L_r \alpha(D), \quad \forall\, D \in \mathcal{P} \cap T_{r_1},$$

and if

$$\phi_q(f^0) < 1 < \frac{1}{2}\,\delta\phi_q\left(\left(\frac{1}{2} - \delta\right)(\psi f)_\infty\right),$$

in [1, Theorem 3.1], it is proved that the BVP (1.1) has at least one non zero positive solution. Here $\alpha(\cdot)$ denotes the Kuratowski measure. Evidently, our main result is better than the result in [1].

The paper is organized as follows. In the next Section, we give some auxiliary results. In Section 3, we prove our main result. In Section 4 we lustrate our main result with some examples.

2. Auxiliary results

Consider the nonlinear equation $Tx+Fx = x$, posed in some closed convex subset of a Banach space, where $(I - T)$ is Lipschitz invertible mapping, in particular T is an expansive operator, and F is a k-set contraction.

A transformation which allows, under certain additional conditions, to derive several existence results for this equation by resorting to the theory of the fixed point index in cones for strict set contractions mappings. Some of these results have been improved in several directions, and they have been applied to obtain existence results of initial and boundary value problems subject to ordinary and partial differential equations (see [1]-[6]).

In what follows, $\mathcal{K}$ will refer to a cone in a Banach space $\mathbb{E}$.
The following Proposition 2.1 will be used to be proved our main result.

Proposition 2.1. [6, 5] *Let Ω be a subset of $\mathcal{K}$ and U be a bounded open subset of $\mathcal{K}$ with $0 \in U$. Assume that the mapping $T : \Omega \subset \mathcal{K} \to \mathbb{E}$ be such that $(I - T)$ is Lipschitz invertible with constant $\gamma > 0$, $S : \overline{U} \to \mathbb{E}$ is a k-set contraction with $0 \leqslant k < \gamma^{-1}$, and $S(\overline{U}) \subset (I - T)(\Omega)$. If*

$$Sx \neq (I - T)(\lambda x) \text{ for all } x \in \partial U \bigcap \Omega,\ \lambda \geqslant 1 \text{ and } \lambda x \in \Omega,$$

then the fixed point index $i_* (T + S, U \bigcap \Omega, \mathcal{K}) = 1$.

If $\gamma \in \mathcal{C}(I, E)$, in ([1, Lemma 2.1]), is proved that the unique solution of the BVP

$$\begin{aligned} (\phi_p (u'(t)))' + \gamma(t) &= \theta, \quad 0 < t < 1, \\ u'(0) = u(1) &= \theta, \end{aligned} \tag{2.1}$$

is

$$u(t) = \int_t^1 \phi_q \left(\int_0^s \gamma(\tau) d\tau \right) ds, \quad t \in I. \tag{2.2}$$

3. Proof of the Main Result

Take $\epsilon > 0$. Set $R = \frac{M^{q-1}}{q}$ and

$$\Omega = \{u \in Q : \|u\|_c \leqslant R\}, \quad U = \{u \in Q : \|u\|_c \leqslant r\}.$$

For $u \in Q$, define the operators

$$Tu(t) = (1 - \epsilon)u(t),$$
$$Su(t) = \epsilon \int_t^1 \phi_q \left(\int_0^s f(u(\tau)) d\tau \right) ds, \quad t \in I.$$

Note that any fixed point $u \in Q$ of the operator $T + S$ is a solution of the BVP (1.1).

(1) For, $u \in \Omega$, we have that

$$\|(I - T)u(t)\| = \epsilon \|u(t)\|, \quad t \in I.$$

Therefore $I - T : \Omega \to \mathcal{C}(I, E)$ is Lipschitz invertible with a constant $\gamma = \frac{1}{\epsilon}$.

(2) For $u \in \overline{U}$, we have

$$\begin{aligned} \|Su(t)\| &= \epsilon \left\| \int_t^1 \phi_q \left(\int_0^s f(u(\tau))d\tau \right) ds \right\| \leqslant \epsilon \int_t^1 \phi_q \left(\int_0^s \|f(u(\tau))\| d\tau \right) ds \\ &\leqslant M^{q-1} \epsilon \int_t^1 s^{q-1} ds \leqslant \epsilon \frac{M^{q-1}}{q}, \quad t \in I, \end{aligned}$$

and

$$\|Su\|_c \leqslant \epsilon \frac{M^{q-1}}{q}.$$

Next,

$$\begin{aligned} \left\| \frac{d}{dt} Su(t) \right\| &= \epsilon \left\| -\phi_q \left(\int_0^t f(u(s))ds \right) \right\| \\ &\leqslant \epsilon \phi_q \left(\int_0^1 \|f(u(s))\| ds \right) \leqslant \epsilon M^{q-1}, \quad t \in [0,1]. \end{aligned}$$

Then, $\|(Su)'\|_c \leqslant \epsilon M^{q-1}$. Hence and the Arzela-Ascoli theorem, we conclude that $S : \overline{U} \to \mathcal{C}(I,E)$ is a completely continuous mapping. Therefore $S : \overline{U} \to \mathcal{C}(I,E)$ is a 0-set contraction.

(3) Let $u \in \overline{U}$ be arbitrarily chosen. Take $v = \frac{Su}{\epsilon}$. We have $v \in Q$ and

$$\|v\|_c = \frac{\|Su\|_c}{\epsilon} \leqslant \frac{M^{q-1}}{q},$$

i.e., $v \in \Omega$. Note that $(I-T)v = Su$. Therefore $S(\overline{U}) \subset (I-T)(\Omega)$.

(4) Assume that there are $u \in \partial U$ and $\lambda \geqslant 1$ so that

$$Su = (I-T)(\lambda u) \quad \text{and} \quad \lambda u \in \Omega.$$

We have $Su = \epsilon \lambda u$, $\|u\|_c = r$, and

$$\epsilon \frac{M^{q-1}}{q} \geqslant \|Su\|_c = \epsilon \lambda \|u\|_c \geqslant \epsilon r,$$

whereupon

$$r \leqslant \frac{M^{q-1}}{q} < 1.$$

This is a contradiction, because $r > 1$.

Hence and Proposition 2.1, it follows that the operator $T + S$ has a fixed point $u \in Q$, which is a bounded solution of the BVP (1.1). This completes the proof of the main result.

4. Examples

Example 1. For $m, k \geqslant 0$, consider the BVP:

$$\begin{aligned} \left(|u'|^3 u'\right)'(t) + (u^m(t) + \ln(u^k(t)+1)) &= 0, \quad 0 < t < 1, \\ u'(0) = 0, \quad u(1) = 0, & \end{aligned} \tag{4.1}$$

Here $E=\mathbb{R}, \mathcal{P}=\mathbb{R}^+, \phi_p(s)=|s|^3 s\,(p=5,\ q=\frac{5}{4})$ and $f(y)=y^m+\ln(y^k+1)$. Clearly, the function f is positive continuous and bounded when y is bounded. Moreover, for some $r>1$, the inequality (1.2) in Assumption $(H2)$ is satisfied for all constants m and k satisfying $r^m+\ln(r^k+1)<(\frac{5}{4})^4$.
Therefore, the problem (4.1) has a bounded positive solution.

Example 2. Consider the following BVP of infinite system of scalar differential equations in the infinite-dimensional Banach space $E=l^\infty=\{u=(u_1,\ldots,u_n,\ldots)\mid \sup_n |u_n|<+\infty\}$ with the sup-norm $\|u\|=\sup_n |u_n|$:

$$
\begin{aligned}
&(|u_n'|u_n')'\,(t)+\tfrac{1}{100}(|\sin u_{n+1}(t)|+5\,u_n^2(t))=0, \quad 0<t<1,\\
&x_n'(0)=0,\ \ x_n(1)=0,\ \ n=1,2,\ldots
\end{aligned}
\tag{4.2}
$$

Let $\mathcal{P}=\{x=(x_n)\in l^\infty \mid x_n\geqslant 0,\ n=1,2,\ldots\}$. It is easy to see that $\mathcal{P}$ is a cone in E. System (4.2) can be regarded as a BVP of the form (1.1) in l^∞ with $\phi_p(s)=|s|s\ (p=3,\ q=\frac{3}{2}),\ u=(u_1,\ldots,u_n,\ldots),\ f=(f_1,\ldots,f_n,\ldots)$,

$$
f_n(u(t))=\frac{1}{100}(|\sin u_{n+1}(t)|+5\,u_n^2(t)), \ \ \text{for } n=1,2,\ldots.
$$

Then $f\in C(\mathcal{P},\mathcal{P})$. Furthermore, for any $r>1$ satisfying $\frac{1}{100}(1+5\,r^2)<\frac{9}{4}$,

$$
\sup\{\|f(u(t))\| : u\in Q\cap B_r\}\leqslant M=\frac{1}{100}(1+5\,r^2)<\infty,
$$

and $\frac{M^{q-1}}{q}<1$. Therefore, the system (4.2) has a bounded positive solution.

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