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DIFFERENTIAL SANDWICH-TYPE RESULTS FOR SYMMETRIC FUNCTIONS ASSOCIATED WITH PASCAL DISTRIBUTION SERIES

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Abstract. In this paper we obtain some applications of the theory of differential subordination, differential superordination, and sandwich-type results for some subclasses of symmetric functions associated with Pascal distribution series.

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1. Introduction

Let $\mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{H}[a, m]$ denote the subclass of functions $f \in \mathcal{H}(\mathbb{U})$ of the form

$$f(z) = a + a_m z^m + a_{m+1} z^{m+1} + \dots,\ z \in \mathbb{U},$$

with $a \in \mathbb{C}$ and $m \in \mathbb{N} := \{1, 2, \dots\}$.

Also, let $\mathcal{A}(m)$ denote the subclass of functions $f \in \mathcal{H}(\mathbb{U})$ of the form

$$f(z) = z + \sum_{k=m+1}^{\infty} a_k z^k,\ z \in \mathbb{U}, \tag{1.1}$$

with $m \in \mathbb{N}$, and let $\mathcal{A} := \mathcal{A}(1)$.

A variable x is said to have the *Pascal distribution* if it takes the values $0, 1, 2, 3, \dots$ with the probabilities

$$(1-q)^r, \quad \frac{qr(1-q)^r}{1!}, \quad \frac{q^2 r(r+1)(1-q)^r}{2!}, \quad \frac{q^3 r(r+1)(r+2)(1-q)^r}{3!}, \dots$$

respectively, where q and r are called the parameters, and thus we have the probability formula

$$P(X = k) = \binom{k+r-1}{r-1} q^k (1-q)^r,\ k \in \mathbb{N}_0.$$

Now, we introduce a power series whose coefficients are probabilities of the *Pascal distribution*, that is

$$\mathrm{Q}_{q,m}^{r}(z) := z + \sum_{n=m+1}^{\infty} \binom{n+r-2}{r-1} q^{n-1}(1-q)^{r} z^{n},\ z \in \mathbb{U},$$
$$(m \in \mathbb{N},\ r \geq 1,\ 0 \leq q \leq 1),$$

and using the ratio test we easily deduce that the radius of convergence of the above power series is at least $\frac{1}{q} \geq 1$, hence $\mathrm{Q}_{q,m}^{r} \in \mathcal{A}(m)$.

Defining the functions

$$\mathrm{M}_{q,\lambda}^{r,m}(z) := (1-\lambda)\mathrm{Q}_{q,m}^{r}(z) + \lambda z \left(\mathrm{Q}_{q,m}^{r}(z)\right)'$$
$$= z + \sum_{n=m+1}^{\infty} \binom{n+r-2}{r-1} \left[1+\lambda(n-1)\right] q^{n-1}(1-q)^{r} z^{n},\ z \in \mathbb{U},$$
$$(m \in \mathbb{N},\ r \geq 1,\ 0 \leq q \leq 1,\ \lambda \geq 0),$$

we introduce the linear operator $\mathcal{N}_{q,\lambda}^{r,m} : \mathcal{A}(m) \to \mathcal{A}(m)$ defined by

$$\mathcal{N}_{q,\lambda}^{r,m} f(z) := \mathrm{M}_{q,\lambda}^{r,m}(z) * f(z)$$
$$= z + \sum_{n=m+1}^{\infty} \binom{n+r-2}{r-1} \left[1+\lambda(n-1)\right] q^{n-1}(1-q)^{r} a_n z^{n},\ z \in \mathbb{U},$$
$$(m \in \mathbb{N},\ r \geq 1,\ 0 \leq q \leq 1,\ \lambda \geq 0),$$

where f is given by (1.1), and the symbol "$*$" stands for the *Hadamard (or convolution) product*.

Remark that, for $m = 1$ the function $\mathrm{M}_{q,\lambda}^{r,m}$ reduces to $\mathrm{N}_{q,\lambda}^{r} := \mathrm{M}_{q,\lambda}^{r,1}$ introduced and studied by El-Deeb et al. [9].

Definition 1.1. For $f, g \in \mathcal{H}(\mathbb{U})$, we say that f *is subordinate to* g, written $f(z) \prec g(z)$, if there exists a *Schwarz function* w, which is analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(w(z))$, $z \in \mathbb{U}$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence (see [12, 7]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $k, h \in \mathcal{H}(\mathbb{U})$, and let $\varphi(r, s; z) : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$.

(i) If k satisfies *the first order differential subordination*

$$\varphi(k(z), zk'(z); z) \prec h(z), \tag{1.2}$$

then k is said to be *a solution of the differential subordination* (1.2). The function q is called *a dominant of the solutions of the differential subordination* (1.2) if $k \prec q(z)$ for all the functions k satisfying (1.2). A dominant $\widetilde{q}$ is said to be *the best dominant of* (1.2) if $\widetilde{q}(z) \prec q(z)$ for all the dominants q.

(ii) If k satisfies *the first order differential superordination*

$$h(z) \prec \varphi(k(z), zk'(z); z), \tag{1.3}$$

then k is called to be *a solution of the differential superordination* (1.3). The function q is called *a subordinant of the solutions of the differential superordination* (1.3) if $q(z) \prec k(z)$ for all the functions k satisfying (1.3). A subordinant $\widetilde{q}$ is said to be *the best subordinant of* (1.3) if $q(z) \prec \widetilde{q}(z)$ for all the subordinants q.

Miller and Mocanu [13] obtained conditions on the functions h, q and φ for which the following implication holds:

$$h(z) \prec \varphi(k(z), zk'(z); z) \Rightarrow q(z) \prec k(z).$$

Using the results of Miller and Mocanu [13], Bulboacă [6] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [5]. Ali et al. [1], have used the results of [7] (see also [2, 3, 8]) to obtain sufficient conditions for normalized analytic functions f to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are univalent functions in $\mathbb{U}$ with $q_1(0) = q_2(0) = 1$.

Sakaguchi [15] introduced a class S_s^* of *functions starlike with respect to symmetric points*, which consists of functions $f \in \mathcal{A}$ satisfying the inequality

$$Re\frac{zf'(z)}{f(z) - f(-z)} > 0, \ z \in \mathbb{U},$$

that represents a subclass of *close-to-convex functions*, and hence univalent in $\mathbb{U}$, and moreover, this class includes the class of *convex functions* and *odd starlike functions with respect to the origin* (see [14, 15]).

Also, Aouf et al. [4] introduced and studied the class $S_{s,n}^*T(1,1)$ of *functions n-starlike with respect to symmetric points*, which consists of functions $f \in \mathcal{A}$ with $a_k \leq 0$ for $k \geq 2$, and satisfying the inequality

$$Re\frac{D^{n+1}f(z)}{D^n f(z) - D^n f(-z)} > 0, \ z \in \mathbb{U},$$

where D^n is the *Sălăgean operator* [16].

The classes defined in [14] and [15] could be generalized by introducing the next class of functions, defined with the aid of the $\mathcal{N}_{q,\lambda}^{r,m}$ operator:

Definition 1.2. A function $f \in \mathcal{A}(m)$ with

$$\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z) \neq 0, \ z \in \dot{\mathbb{U}} := \mathbb{U} \setminus \{0\}, \tag{1.4}$$

is said to be in the class $\mathcal{N}_{q,\lambda}^{r,m}(\gamma,\mu,A,B)$ if it satisfies the subordination condition

$$(1.5)\qquad (1+\gamma)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}$$

$$-\gamma\left(\frac{z\left(\mathcal{N}_{q,\lambda}^{r,m}f(z)\right)'-z\left(\mathcal{N}_{q,\lambda}^{r,m}f(-z)\right)'}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}\prec\frac{1+Az}{1+Bz},$$

$$\left(\gamma\in\mathbb{C},\ 0<\mu<1,\ -1\le B<A\le 1,\ m\in\mathbb{N},\ r\ge 1,\ 0\le q\le 1,\ \lambda\ge 0\right).$$

In this paper we will obtain some sharp differential subordination and superordination results for the functions belonging to the class $\mathcal{N}_{q,\lambda}^{r,m}(\gamma,\mu,A,B)$, in order to try to make a connection between a special subclass of analytic functions whose coefficients are probabilities of the *Pascal distribution*, and the differential subordination theory.

2. Preliminaries

In order to prove our results we shall need the following definition and lemmas.

Definition 2.1. [12, Definition 2.2b., p. 21] Let $\mathcal{Q}$ be the set of all functions f that are analytic and injective on $\overline{\mathbb{U}}\setminus E(f)$, where $E(f):=\left\{\zeta\in\partial\mathbb{U}:\lim\limits_{z\to\zeta}f(z)=\infty\right\}$ and are such that $f'(\zeta)\neq 0$ for $\zeta\in\partial\mathbb{U}\setminus E(f)$.

Lemma 2.1. [12, Theorem 3.1b., p. 71] *Let the function H be convex in $\mathbb{U}$, with $H(0)=a$, and $\lambda\neq 0$ with $\operatorname{Re}\lambda\ge 0$. If $\Phi\in\mathcal{H}[a,m]$ and*

$$(2.1)\qquad \Phi(z)+\frac{z\Phi'(z)}{\lambda}\prec H(z),$$

then

$$\Phi(z)\prec\Psi(z):=\frac{\lambda}{mz^{\frac{\lambda}{m}}}\int_0^z t^{\frac{\lambda}{m}-1}H(t)dt\prec H(z),$$

and the function Ψ is convex, $\Psi\in\mathcal{H}[a,m]$, and is the best dominant of (2.1).

Lemma 2.2. [18, Lemma 2.2., p. 3] *Let q be a univalent in $\mathbb{U}$, with $q(0)=1$. Let $\xi,\varphi\in\mathbb{C}$ with $\varphi\neq 0$, and assume that*

$$Re\left(1+\frac{zq''(z)}{q'(z)}\right)>\max\left\{0;-\operatorname{Re}\frac{\xi}{\varphi}\right\},\ z\in\mathbb{U}.$$

If k is analytic in $\mathbb{U}$ and

$$(2.2)\qquad \xi k(z)+\varphi zk'(z)\prec\xi q(z)+\varphi zq'(z),$$

then $k(z)\prec q(z)$, and q is the best dominant of (2.2).

From [13, Theorem 6, p. 820] we could easily obtain the following lemma:

Lemma 2.3. *Let q be convex in $\mathbb{U}$, and $k \neq 0$ with $Rek \geq 0$. If $g \in \mathcal{H}[q(0),1] \cap \mathcal{Q}$, such that $g(z) + kzg'(z)$ is univalent in $\mathbb{U}$, then*

$$q(z) + kzq'(z) \prec g(z) + kzg'(z), \tag{2.3}$$

implies that $q(z) \prec g(z)$, and q is the best subordinant of (2.3).

Lemma 2.4. [10] *Let F be analytic and convex in $\mathbb{U}$, and $0 \leq \lambda \leq 1$. If $f, g \in \mathcal{A}$, such that $f(z) \prec F(z)$ and $g(z) \prec F(z)$, then*

$$\lambda f(z) + (1-\lambda) g(z) \prec F(z).$$

3. Main results

Unless otherwise mentioned, we shall assume in the reminder of this paper that $\gamma \in \mathbb{C}$, $0 < \mu < 1$, $-1 \leq B < A \leq 1$, $m \in \mathbb{N}$, $r \geq 1$, $0 \leq q \leq 1$, $\lambda \geq 0$, and the powers are understood as principle values.

Theorem 3.1. *If $f \in \mathcal{N}_{q,\lambda}^{r,m}(\gamma, \mu, A, B)$ and $\gamma \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ with $Re\gamma \geq 0$, then*

$$\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)} \right)^{\mu} \prec \Psi(z) := \frac{\mu}{\gamma m} \int_0^1 \frac{1+Azu}{1+Bzu} u^{\frac{\mu}{\gamma m}-1} du \prec \frac{1+Az}{1+Bz},$$

and Ψ is convex, $\Psi \in \mathcal{H}[1,m]$, and is the best dominant.

Proof. If we define a function h by

$$h(z) = \left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)} \right)^{\mu}, \ z \in \mathbb{U}, \tag{3.1}$$

from (1.4) it follows that h is an analytic function in $\mathbb{U}$, with $h(0) = 1$. Differentiating (3.1) with respect to z, we obtain that

$$\begin{aligned} &(1+\gamma) \left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)} \right)^{\mu} \\ &-\gamma \left(\frac{z \left(\mathcal{N}_{q,\lambda}^{r,m} f(z)\right)' - z \left(\mathcal{N}_{q,\lambda}^{r,m} f(-z)\right)'}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)} \right) \left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)} \right)^{\mu} \\ &= h(z) + \frac{\gamma}{\mu} z h'(z) \prec \frac{1+Az}{1+Bz}. \end{aligned} \tag{3.2}$$

Since

$$\mathcal{N}_{q,\lambda}^{r,m} f(z) = z + \sum_{n=m+1}^{\infty} \alpha_n z^n, \quad \text{and} \quad \mathcal{N}_{q,\lambda}^{r,m} f(-z) = -z + \sum_{n=m+1}^{\infty} \alpha_n (-1)^n z^n,$$

where

$$\alpha_n = \binom{n+r-2}{r-1} \left[1 + \lambda(n-1)\right] q^{n-1} (1-q)^r a_n, \ n \geq m+1,$$

we have

$$U(z) := \frac{2z}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)} = \frac{2z}{2z + \sum\limits_{n=m+1}^{\infty} \alpha_n \left[1 + (-1)^{n+1}\right] z^n} = \frac{1}{1 + \sum\limits_{k=m}^{\infty} \beta_k z^k},$$

with

$$\beta_k = \frac{\alpha_{k+1}\left[1 + (-1)^k\right]}{2},\ k \geq m.$$

Moreover,

$$U(z) = \frac{1}{1 + \sum\limits_{k=m}^{\infty} \beta_k z^k} = 1 + \sum_{j=1}^{\infty} \gamma_j z^j,\ z \in \mathbb{U},$$

with unknowns γ_j, $j \geq 1$, we have

$$1 = \left(1 + \beta_m z^m + \beta_{m+1} z^{m+1} + \dots\right) \left(1 + \gamma_1 z + \gamma_2 z^2 + \dots + \gamma_m z^m + \gamma_{m+1} z^{m+1} + \dots\right),$$

and equating the corresponding coefficients it follows that

$$\gamma_1 = \gamma_2 = \dots = \gamma_{m-1} = 0, \quad \gamma_m = -\beta_m, \quad \gamma_{m+1} = -\beta_{m+1}, \dots,$$

hence

$$U(z) = 1 + \sum_{j=m}^{\infty} \gamma_j z^j \in \mathcal{H}[1, m].$$

According to (3.1), we have

$$h = U^{\mu}, \quad \text{with} \quad U \in \mathcal{H}[1, m],$$

and using the binomial power expansion formula we get

$$h = U^{\mu} \in \mathcal{H}[1, m].$$

Now, from the subordination (3.2), using Lemma 2.1 for $\lambda = \dfrac{\mu}{\gamma}$ we obtain our result. $\square$

Remark 3.1. The above theorem shows that

$$\mathcal{N}_{q,\lambda}^{r,m}(\gamma, \mu, A, B) \subset \mathcal{N}_{q,\lambda}^{r,m}(0, \mu, A, B),$$

for all $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma \geq 0$.

Moreover, the next inclusion result for the classes $\mathcal{N}_{q,\lambda}^{r,m}(\gamma, \mu, A, B)$ holds:

Theorem 3.2. *If $\gamma_1, \gamma_2 \in \mathbb{R}$ such that $0 \leq \gamma_1 \leq \gamma_2$, and $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$, then*

$$\mathcal{N}_{q,\lambda}^{r,m}(\gamma_2, \mu, A_2, B_2) \subset \mathcal{N}_{q,\lambda}^{r,m}(\gamma_1, \mu, A_1, B_1). \tag{3.3}$$

Proof. If $f \in \mathcal{N}_{q,\lambda}^{r,m}(\gamma_2, \mu, A_2, B_2)$, since $-1 \le B_1 \le B_2 < A_2 \le A_1 \le 1$, it is easy to check that

$$(1+\gamma_2)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}$$

$$-\gamma_2\left(\frac{z\left(\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)\right)'}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}$$

$$\prec \frac{1+A_2 z}{1+B_2 z} \prec \frac{1+A_1 z}{1+B_1 z}, \tag{3.4}$$

that is $f \in \mathcal{N}_{q,\lambda}^{r,m}(\gamma_1, \mu, A_1, B_1)$, hence the assertion (3.3) holds for $\gamma_1 = \gamma_2$.

If $0 \le \gamma_1 < \gamma_2$, from Remark 3.1 and (3.4) it follows $f \in \mathcal{N}_{q,\lambda}^{r,m}(0, \mu, A_1, B_1)$, that is

$$\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu} \prec \frac{1+A_1 z}{1+B_1 z}. \tag{3.5}$$

A simple computation shows that

$$(1+\gamma_1)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}$$

$$-\gamma_1\left(\frac{z\left(\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)\right)'}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}$$

$$= \left(1-\frac{\gamma_1}{\gamma_2}\right)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu} + \frac{\gamma_1}{\gamma_2}\left[(1+\gamma_2)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}\right.$$

$$\left. -\gamma_2\left(\frac{z\left(\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)\right)'}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}\right],\ z \in \mathbb{U}. \tag{3.6}$$

Moreover,

$$0 \le \frac{\gamma_1}{\gamma_2} < 1,$$

and the function $\dfrac{1+A_1 z}{1+B_1 z}$, with $-1 \le B_1 < A_1 \le 1$, is analytic and convex in $\mathbb{U}$. According to (3.6), using the subordinations (3.4) and (3.5), from Lemma 2.4 we deduce that

$$(1+\gamma_1)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}$$

$$-\gamma_1\left(\frac{z\left(\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)\right)'}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu} \prec \frac{1+A_1 z}{1+B_1 z},$$

that is $f \in \mathcal{N}_{q,\lambda}^{r,m}(\gamma_1, \mu, A_1, B_1)$. $\square$

Theorem 3.3. *Suppose that q is univalent in $\mathbb{U}$, with $q(0) = 1$, and let $\gamma \in \mathbb{C}^*$ such that*

$$Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0; -Re\frac{\mu}{\gamma}\right\},\ z \in \mathbb{U}. \tag{3.7}$$

If $f \in \mathcal{A}(m)$ such that (1.4) *holds, and satisfies the subordination*

$$(1+\gamma)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z) - \mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}$$
$$-\gamma\left(\frac{z\left(\mathcal{N}_{q,\lambda}^{r,m}f(z) - \mathcal{N}_{q,\lambda}^{r,m}f(-z)\right)'}{\mathcal{N}_{q,\lambda}^{r,m}f(z) - \mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z) - \mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}$$
$$\prec q(z) + \frac{\gamma}{\mu}zq'(z), \tag{3.8}$$

then

$$\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z) - \mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu} \prec q(z),$$

and q is the best dominant of (3.8).

Proof. Since $f \in \mathcal{A}(m)$ such that (1.4) holds, it follows that the function h defined by (3.1) is analytic in $\mathbb{U}$, and $h(0) = 1$. Like in the proof of Theorem 3.1, differentiating (3.1) with respect to z, we obtain that (3.8) is equivalent to

$$h(z) + \frac{\gamma}{\mu}zh'(z) \prec q(z) + \frac{\gamma}{\mu}zq'(z).$$

Using Lemma 2.2 for $\xi := 1$ and $\varphi := \dfrac{\gamma}{\mu}$, we get that the above subordination implies $h(z) \prec q(z)$, and q is the best dominant of (3.8). $\square$

For the special case $q(z) = \dfrac{1+Az}{1+Bz}$, with $-1 \le B < A \le 1$, Theorem 3.3 reduces to the following corollary:

Corollary 3.1. *Let $\gamma \in \mathbb{C}^*$ and $-1 \le B < A \le 1$, such that*

$$\max\left\{-1; -\frac{1+\operatorname{Re}\frac{\mu}{\gamma}}{1-\operatorname{Re}\frac{\mu}{\gamma}}\right\} \le B \le 0, \quad or \quad 0 \le B \le \min\left\{1; \frac{1+\operatorname{Re}\frac{\mu}{\gamma}}{1-\operatorname{Re}\frac{\mu}{\gamma}}\right\}. \tag{3.9}$$

If $f \in \mathcal{A}(m)$ such that (1.4) *holds, and satisfies the subordination*

$$(1+\gamma)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z) - \mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}$$
$$-\gamma\left(\frac{z\left(\mathcal{N}_{q,\lambda}^{r,m}f(z) - \mathcal{N}_{q,\lambda}^{r,m}f(-z)\right)'}{\mathcal{N}_{q,\lambda}^{r,m}f(z) - \mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z) - \mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}$$
$$\prec \frac{1+Az}{1+Bz} + \frac{\gamma}{\mu}\frac{(A-B)z}{(1+Bz)^2}, \tag{3.10}$$

then

$$\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}\prec\frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant of* (3.10).

Proof. For $q(z)=\frac{1+Az}{1+Bz}$, the condition (3.7) reduces to

$$Re\frac{1-Bz}{1+Bz}>\max\left\{0;-Re\frac{\mu}{\gamma}\right\},\ z\in\mathbb{U}. \tag{3.11}$$

Since

$$\inf\left\{Re\frac{1-Bz}{1+Bz}:z\in\mathbb{U}\right\}=\begin{cases}\frac{1+B}{1-B}, & \text{if}\quad -1\le B\le 0,\\ \frac{1-B}{1+B}, & \text{if}\quad 0\le B<1,\end{cases}$$

we easily check that (3.11) holds if and only if the assumption (3.9) is satisfied, whenever $-1\le B<1$. □

Theorem 3.4. *Let q be convex in $\mathbb{U}$, with $q(0)=1$, and $\gamma\in\mathbb{C}^*$, with $Re\gamma\ge 0$. Also, let $f\in\mathcal{A}(m)$ such that*

$$\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}\in\mathcal{H}[q(0),1]\cap\mathcal{Q}, \tag{3.12}$$

and assume that the function

$$(1+\gamma)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}$$

$$-\gamma\left(\frac{z\left(\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)\right)'}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}\ \text{is univalent in}\ \mathbb{U}. \tag{3.13}$$

If

$$q(z)+\frac{\gamma}{\mu}zq'(z)\prec(1+\gamma)\left(\frac{2z}{D^nf(z)-D^nf(-z)}\right)^{\mu}$$

$$-\gamma\left(\frac{D^{n+1}f(z)+D^{n+1}f(-z)}{D^nf(z)-D^nf(-z)}\right)\left(\frac{2z}{D^nf(z)-D^nf(-z)}\right)^{\mu}, \tag{3.14}$$

then

$$q(z)\prec\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu},$$

and q is the best subordinant of (3.14).

Proof. Letting the function h defined by (3.1), then $h \in \mathcal{H}[q(0), m]$, and from (3.12) we have that $h \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}$. Like in the proof of Theorem 3.1, differentiating (3.1) with respect to z, we obtain that

$$q(z) + \frac{\gamma}{\mu} z q'(z) \prec h(z) + \frac{\gamma}{\mu} z h'(z).$$

Now, according to Lemma 2.3 for $k := \dfrac{\gamma}{\mu}$ we obtain the desired result. □

Taking $q(z) = \dfrac{1 + Az}{1 + Bz}$, with $-1 \leq B < A \leq 1$, in Theorem 3.4 we obtain the following corollary:

Corollary 3.2. *Let $\gamma \in \mathbb{C}^*$, with $Re\gamma \geq 0$, and $-1 \leq B < A \leq 1$. If $f \in \mathcal{A}(m)$ such that the assumption* (3.12) *and* (3.13) *hold, and satisfies the subordination*

$$\frac{1 + Az}{1 + Bz} + \frac{\gamma}{\mu} \frac{(A - B)z}{(1 + Bz)^2} \prec (1 + \gamma) \left(\frac{2z}{\mathcal{N}^{r,m}_{q,\lambda} f(z) - \mathcal{N}^{r,m}_{q,\lambda} f(-z)} \right)^{\mu}$$

$$-\gamma \left(\frac{z \left(\mathcal{N}^{r,m}_{q,\lambda} f(z) - \mathcal{N}^{r,m}_{q,\lambda} f(-z) \right)'}{\mathcal{N}^{r,m}_{q,\lambda} f(z) - \mathcal{N}^{r,m}_{q,\lambda} f(-z)} \right) \left(\frac{2z}{\mathcal{N}^{r,m}_{q,\lambda} f(z) - \mathcal{N}^{r,m}_{q,\lambda} f(-z)} \right)^{\mu}, \tag{3.15}$$

then

$$\frac{1 + Az}{1 + Bz} \prec \left(\frac{2z}{\mathcal{N}^{r,m}_{q,\lambda} f(z) - \mathcal{N}^{r,m}_{q,\lambda} f(-z)} \right)^{\mu},$$

and $\dfrac{1 + Az}{1 + Bz}$ *is the best subordinant of* (3.15).

Combining Theorem 3.3 and Theorem 3.4 we obtain the following sandwich-type theorem:

Theorem 3.5. *Let q_1 and q_2 be two convex functions in $\mathbb{U}$, with $q_1(0) = q_2(0) = 1$, and let $\gamma \in \mathbb{C}^*$, with $Re\gamma \geq 0$. If $f \in \mathcal{A}(m)$ such that the assumption* (3.12) *and* (3.13) *hold, then*

$$q_1(z) + \frac{\gamma}{\mu} z q_1'(z) \prec (1 + \gamma) \left(\frac{2z}{\mathcal{N}^{r,m}_{q,\lambda} f(z) - \mathcal{N}^{r,m}_{q,\lambda} f(-z)} \right)^{\mu}$$

$$-\gamma \left(\frac{z \left(\mathcal{N}^{r,m}_{q,\lambda} f(z) - \mathcal{N}^{r,m}_{q,\lambda} f(-z) \right)'}{\mathcal{N}^{r,m}_{q,\lambda} f(z) - \mathcal{N}^{r,m}_{q,\lambda} f(-z)} \right) \left(\frac{2z}{\mathcal{N}^{r,m}_{q,\lambda} f(z) - \mathcal{N}^{r,m}_{q,\lambda} f(-z)} \right)^{\mu} \prec q_2(z) + \frac{\gamma}{\mu} z q_2'(z), \tag{3.16}$$

implies that

$$q_1(z) \prec \left(\frac{2z}{\mathcal{N}^{r,m}_{q,\lambda} f(z) - \mathcal{N}^{r,m}_{q,\lambda} f(-z)} \right)^{\mu} \prec q_2(z),$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant of (3.16).

Theorem 3.6. *If* $f \in \mathcal{N}_{q,\lambda}^{r,m}(0,\mu,1-2\rho,-1)$, *with* $0 \le \rho < 1$, *then* $f \in \mathcal{N}_{q,\lambda}^{r,m}(\gamma,\mu,1-2\rho,-1)$ *for* $|z| < R$, *where*

$$R = \left(\sqrt{\frac{|\gamma|^2 m^2}{\mu^2} + 1} - \frac{|\gamma| m}{\mu}\right)^{\frac{1}{m}}. \tag{3.17}$$

Proof. For $f \in \mathcal{N}_{q,\lambda}^{r,m}(0,\mu,1-2\rho,-1)$, with $0 \le \rho < 1$, let define the function h by

$$\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)}\right)^{\mu} = (1-\rho)h(z) + \rho,\ z \in \mathbb{U}. \tag{3.18}$$

Hence, the function h is analytic in $\mathbb{U}$, with $h(0) = 1$, and since $f \in \mathcal{N}_{q,\lambda}^{r,m}(0,\mu,1-2\rho,-1)$ is equivalent to,

$$\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)}\right)^{\mu} \prec \frac{1+(1-2\rho)z}{1-z},$$

it follows that Reh(z) > 0, $z \in \mathbb{U}$.

Like in the proof of Theorem 3.1, since $f \in \mathcal{N}_{q,\lambda}^{r,m}(0,\mu,1-2\rho,-1)$, with $0 \le \rho < 1$, we deduce that

$$\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)}\right)^{\mu} \in \mathcal{H}[1,m],$$

and from the relation (3.18) we get $h \in \mathcal{H}[1,m]$. Therefore, the following estimate holds

$$|zh'(z)| \le \frac{2mr^m \mathrm{Reh(z)}}{1-r^{2m}},\ |z| = r < 1,$$

that represents the result of Shah [17] (the inequality (6), p. 240, for $\alpha = 0$), which generalize Lemma 2 of [11].

A simple computation shows that

$$\frac{1}{1-\rho}\left\{(1+\gamma)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)}\right)^{\mu}\right.$$
$$\left.-\gamma\left(\frac{z\left(\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)\right)'}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)}\right)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)}\right)^{\mu} - \rho\right\}$$
$$= h(z) + \frac{\gamma}{\mu} zh'(z),\ z \in \mathbb{U},$$

hence, we obtain

$$\mathrm{Re}\left\{\frac{1}{1-\rho}\left[(1+\gamma)\left(\frac{2\mathrm{z}}{\mathcal{N}_{\mathrm{q},\lambda}^{\mathrm{r,m}}\mathrm{f}(\mathrm{z})-\mathcal{N}_{\mathrm{q},\lambda}^{\mathrm{r,m}}\mathrm{f}(-\mathrm{z})}\right)^{\mu}\right.\right.$$

$$\left.\left.-\gamma\left(\frac{z\left(\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)\right)'}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}-\rho\right]\right\}$$

$$\geq \mathrm{Reh(z)}\left[1-\frac{2\left|\gamma\right|\mathrm{mr^m}}{\mu\left(1-\mathrm{r^{2m}}\right)}\right],\ |\mathrm{z}|=\mathrm{r}<1, \tag{3.19}$$

and the right-hand side of (3.19) is positive provided that $r<R$, where R is given by (3.17). □

Theorem 3.7. *Let $f\in\mathcal{N}_{q,\lambda}^{r,m}(\gamma,\mu,A,B)$, let $\gamma\in\mathbb{C}^{*}$ with $\mathrm{Re}\gamma\geq 0$, and $-1\leq B<A\leq 1$.*

1. Then,

$$\frac{\mu}{\gamma m}\int_0^1\frac{1-Au}{1-Bu}u^{\frac{\mu}{\gamma m}-1}du<\mathrm{Re}\left(\frac{2\mathrm{z}}{\mathcal{N}_{\mathrm{q},\lambda}^{\mathrm{r,m}}\mathrm{f}(\mathrm{z})-\mathcal{N}_{\mathrm{q},\lambda}^{\mathrm{r,m}}\mathrm{f}(-\mathrm{z})}\right)^{\mu}$$

$$<\frac{\mu}{\gamma m}\int_0^1\frac{1+Au}{1+Bu}u^{\frac{\mu}{\gamma m}-1}du,\ z\in\mathbb{U}. \tag{3.20}$$

2. For $|z|=r<1$, we have

$$2r\left(\frac{\mu}{\gamma m}\int_0^1\frac{1+Aur}{1+Bur}u^{\frac{\mu}{\gamma m}-1}du\right)^{-\frac{1}{\mu}}<\left|\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)\right|$$

$$<2r\left(\frac{\mu}{\gamma m}\int_0^1\frac{1-Aur}{1-Bur}u^{\frac{\mu}{\gamma m}-1}du\right)^{-\frac{1}{\mu}}. \tag{3.21}$$

All these inequalities are the best possible.

Proof. From the assumptions, using Theorem 3.1 we obtain that

$$\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}\prec\Psi(z):=\frac{\mu}{\gamma m}\int_0^1\frac{1+Azu}{1+Bzu}u^{\frac{\mu}{\gamma m}-1}du, \tag{3.22}$$

and the convex function $\Psi\in\mathcal{H}[1,m]$ is the best dominant. Therefore,

$$\mathrm{Re}\left(\frac{2\mathrm{z}}{\mathcal{N}_{\mathrm{q},\lambda}^{\mathrm{r,m}}\mathrm{f}(\mathrm{z})-\mathcal{N}_{\mathrm{q},\lambda}^{\mathrm{r,m}}\mathrm{f}(-\mathrm{z})}\right)^{\mu}<\sup_{z\in\mathbb{U}}\mathrm{Re}\left(\frac{\mu}{\gamma\mathrm{m}}\int_0^1\frac{1+\mathrm{Azu}}{1+\mathrm{Bzu}}\mathrm{u}^{\frac{\mu}{\gamma\mathrm{m}}-1}\mathrm{du}\right)$$

$$=\frac{\mu}{\gamma m}\int_0^1\sup_{z\in\mathbb{U}}\mathrm{Re}\left(\frac{1+\mathrm{Azu}}{1+\mathrm{Bzu}}\right)\mathrm{u}^{\frac{\mu}{\gamma\mathrm{m}}-1}\mathrm{du}=\frac{\mu}{\gamma\mathrm{m}}\int_0^1\frac{1+\mathrm{Au}}{1+\mathrm{Bu}}\mathrm{u}^{\frac{\mu}{\gamma\mathrm{m}}-1}\mathrm{du},\ \mathrm{z}\in\mathbb{U},$$

and

$$\mathrm{Re}\left(\frac{2\mathrm{z}}{\mathcal{N}_{\mathrm{q},\lambda}^{\mathrm{r,m}}\mathrm{f(z)}-\mathcal{N}_{\mathrm{q},\lambda}^{\mathrm{r,m}}\mathrm{f(-z)}}\right)^{\mu}>\inf_{\mathrm{z}\in\mathbb{U}}\mathrm{Re}\left(\frac{\mu}{\gamma\mathrm{m}}\int\limits_{0}^{1}\frac{1-\mathrm{Azu}}{1-\mathrm{Bzu}}\mathrm{u}^{\frac{\mu}{\gamma\mathrm{m}}-1}\mathrm{du}\right)$$

$$=\frac{\mu}{\gamma m}\int\limits_{0}^{1}\inf_{z\in\mathbb{U}}\mathrm{Re}\left(\frac{1-\mathrm{Azu}}{1-\mathrm{Bzu}}\right)\mathrm{u}^{\frac{\mu}{\gamma\mathrm{m}}-1}\mathrm{du}=\frac{\mu}{\gamma\mathrm{m}}\int\limits_{0}^{1}\frac{1-\mathrm{Au}}{1-\mathrm{Bu}}\mathrm{u}^{\frac{\mu}{\gamma\mathrm{m}}-1}\mathrm{du},\ \mathrm{z}\in\mathbb{U}.$$

Also, since

$$\left|\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right|^{\mu}<\sup_{z\in\mathbb{U}}\left|\frac{\mu}{\gamma m}\int\limits_{0}^{1}\frac{1+Azu}{1+Bzu}u^{\frac{\mu}{\gamma m}-1}du\right|$$

$$=\frac{\mu}{\gamma m}\int\limits_{0}^{1}\sup_{z\in\mathbb{U}}\left|\frac{1+Azu}{1+Bzu}\right|u^{\frac{\mu}{\gamma m}-1}du=\frac{\mu}{\gamma m}\int\limits_{0}^{1}\frac{1+Aur}{1+Bur}u^{\frac{\mu}{\gamma m}-1}du,\ |z|=r<1,$$

we get

$$\left|\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)\right|>2r\left(\frac{\mu}{\gamma m}\int\limits_{0}^{1}\frac{1+Aur}{1+Bur}u^{\frac{\mu}{\gamma m}-1}du\right)^{-\frac{1}{\mu}},$$

while

$$\left|\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right|^{\mu}>\inf_{z\in\mathbb{U}}\left|\frac{\mu}{\gamma m}\int\limits_{0}^{1}\frac{1-Azu}{1-Bzu}u^{\frac{\mu}{\gamma m}-1}du\right|$$

$$=\frac{\mu}{\gamma m}\int\limits_{0}^{1}\inf_{z\in\mathbb{U}}\left|\frac{1-Azu}{1-Bzu}\right|u^{\frac{\mu}{\gamma m}-1}du=\frac{\mu}{\gamma m}\int\limits_{0}^{1}\frac{1-Aur}{1-Bur}u^{\frac{\mu}{\gamma m}-1}du,\ |z|=r<1,$$

implies

$$\left|\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)\right|<2r\left(\frac{\mu}{\gamma m}\int\limits_{0}^{1}\frac{1-Aur}{1-Bur}u^{\frac{\mu}{\gamma m}-1}du\right)^{-\frac{1}{\mu}}.$$

The inequalities of (3.20) and (3.21) are the best possible because the subordination (3.22) is sharp. □

Concluding, all the above results give us information about subordination and superordination properties, inclusion results, radius problem, and sharp estimations for the classes $\mathcal{N}_{q,\lambda}^{r,m}(\gamma,\mu,A,B)$, together general sharp subordination and superordination for the operator $\mathcal{N}_{q,\lambda}^{r,m}$. For special choices of the parameters $\gamma\in\mathbb{C}$, $0<\mu<1$, $-1\le B<A\le 1$, $m\in\mathbb{N}$, $r\ge 1$, $0\le q\le 1$, and $\lambda\ge 0$ we may obtain several simple applications connected with the above mentioned classes and operator.

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