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NON-EXISTENCE OF GLOBAL SOLUTIONS FOR A FRACTIONAL INTEGRO-DIFFERENTIAL PROBLEM WITH A CONVOLUTION KERNEL

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Abstract. In this paper, we investigate the nonexistence of nontrivial global solutions for a fractional integro-differential problem in the space of absolutely continuous functions. We provide criteria under which no nontrivial global solutions exist. It is shown that a dissipation of order between zero and one or even a (frictional) dissipation of order one does not help providing global nontrivial solutions. The test function method is used with several derived estimations. Examples with numerical computations are given to illustrate the results.

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1. Introduction

We consider the following fractional integro-differential inequality

$$u'(t)+\left({}^{C}D_{0+}^{\alpha}u\right)(t)\geq\int_0^t g(t-s)f\left(u\left(s\right)\right)ds,\quad t>0,\ 0\leq\alpha<2, \tag{1.1}$$

subject to

$$u(0)=u_0,\ \text{ when } 0\leq\alpha<1, \tag{1.2}$$

or,

$$u(0)=u_0,\ u'(0)=u_1,\ \text{ when } 1\leq\alpha<2, \tag{1.3}$$

where ${}^{C}D_{0+}^{\alpha}$ is the Caputo fractional derivative of order α and $u_0,u_1\in\mathbb{R}$ are given initial data.

This initial value problem is a generalization of many interesting initial value problems. When the kernel g represents the Dirac delta function, $f\left(u\right)=u^p\left(t\right)$, $p>1$ and $\alpha=0$, the equality in (1.1) represents the Bernoulli differential equation

$$u'(t)+u(t)=u^p\left(t\right),\ t>0,\ p>1. \tag{1.4}$$

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Equation (1.4) with $u(0) = u_0$ has the solution

$$u(t) = \left(\left(u_0^{1-p} - 1\right) e^{(p-1)t} + 1\right)^{\frac{1}{1-p}},$$

that blows up in the finite time

$$T_b = \frac{1}{1-p} \ln \left(1 - u_0^{1-p}\right)$$

if and only if $u_0 > 1$, (see [4]).

The solution of the nonlinear Volterra integro-differential equation

$$u'(t) = -c + \int_0^t u^p(s) ds, \tag{1.5}$$

is given by

$$u\left(t\right) = \left(\frac{1-p}{2}\sqrt{\frac{2}{p+1}}t + u_0^{\frac{1-p}{2}}\right)^{\frac{2}{1-p}},$$

and it blows up in the finite time

$$T_b = \frac{2}{p-1}\sqrt{\frac{p+1}{2}}u_0^{\frac{1-p}{2}},$$

when $c = \sqrt{\frac{2}{p+1}u_0^{p+1}}$ and $u_0 > 0$.

When $\alpha = 0$, $u_0 \geq 0$ and the kernel $g(t)$ is positive, locally integrable and $\lim\limits_{t\to\infty} \int_0^t g(s)ds = \infty$, it has been shown in [11] that the solution of

$$u'(t) + u(t) = \int_0^t g(t-s) f\left(u(s)\right) ds, \quad t > 0, \tag{1.6}$$

blows up in finite time if and only if for some $\beta > 0$,

$$\int_\nu^\infty \left(\frac{s}{f\left(s\right)}\right)^{\frac{1}{\beta}} \frac{ds}{s} < \infty, \text{ for any } \nu > 0. \tag{1.7}$$

It has been assumed that $f(t)$ is nonnegative, continuous and nondecreasing for $t > 0$, $f \equiv 0$ for $t \leq 0$, and $\lim\limits_{t\to\infty} \frac{f(t)}{t} = \infty$. Obviously, when $f(u\left(s\right)) = |u(s)|^p$ in (1.6), the condition (1.7) is fulfilled if $p > 1$.

By choosing $g(t)$ to be the Dirac delta function and $f\left(u\right) = |u(t)|^p, p > 1$ in the equality in (1.1), we obtain

$$u'(t) + \left({}^C D_{0+}^\alpha u\right)(t) = |u(t)|^p, \quad t > 0, \quad p > 1. \tag{1.8}$$

As proven in [12], the solution of the system

(1.9)
$$\begin{cases} u_t + \sum\limits_{i=1}^{n} a_i \left(D_{0^+}^{\alpha_i} u\right)(t) = \int_0^t \frac{(t-s)^{-\gamma_1}}{\Gamma(1-\gamma_1)} f_1(u(s),v(s))ds, \quad t>0,\ 0<\alpha_i,\gamma_1<1, \\ v_t + \sum\limits_{i=1}^{n} a_i \left(D_{0^+}^{\beta_i} v\right)(t) = \int_0^t \frac{(t-s)^{-\gamma_1}}{\Gamma(1-\gamma_2)} f_2(u(s),v(s))ds, \quad t>0,\ 0<\beta_i,\gamma_2<1, \\ u(0)=u_0,\ v(0)=v_0\ ,\ \ 0<u_0,v_0\in\mathbb{R}, \end{cases}$$

blows up in finite time for the continuously differentiable functions f_1 and f_2 satisfying the growth conditions:

$$f_1(u,v) \geq a\,|v|^p \quad \text{and} \quad f_2(u,v) \geq b\,|u|^q\,, \quad a,b>0 \ \text{ for all } \ u,v\in\mathbb{R}.$$

Although, the authors in [12], treated a system rather than an equation, the kernel there is a special case of ours, that is $k(t-s) = \frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)}$.

The present authors studied, in [3], the nonexistence of nontrivial global solutions for the fractional integro-differential problem

$$\begin{cases} \left(D_{0^+}^{\alpha} u\right)(t) + \left(D_{0^+}^{\beta} u\right)(t) \geq \int_0^t h(t-s)\,|u(s)|^p\,ds, \quad t>0,\ p>1, \\ \left(I^{1-\alpha}u\right)(0^+) = b,\ b\in\mathbb{R}, \end{cases} \tag{1.10}$$

where $D_{0^+}^{\alpha}$ and $D_{0^+}^{\beta}$ are the Riemann-Liouville fractional derivatives of orders α and β, respectively, $0\leq\beta<\alpha\leq1$ and h is a nonnegative function different from zero almost everywhere. It has been shown that if $\left(t^{-\alpha p'} + t^{-\beta p'}\right) h^{1-p'}(t) \in L^1_{loc}[0,\infty)$ and

$$\lim_{T\to\infty} T^{1-p'} \left(\int_0^T t^{-\alpha p'} h^{1-p'}(t)dt + \int_0^T t^{-\beta p'} h^{1-p'}(t)dt \right) = 0,$$

where $p' = \frac{p}{p-1}$, then, the problem (1.10) has no nontrivial global solution when $b\geq0$.

In this paper, we prove nonexistence of nontrivial global solutions for Problems (1.1) − (1.2) and (1.1) − (1.3) under some conditions on the functions g and f . The test function method introduced in [13] is adopted to the fractional case and used here, see also [5, 9, 10, 15].

It is well known that lower order derivatives usually represent damping terms and therefore help stabilizing the system in addition to the existence of solutions for all time. On the contrary, polynomial sources destabilize the system and they can even force solutions to blow up in finite time. In fact they are sometimes called blowing up terms. When they are both present in the system we will have a competition between these two terms. When $0<\alpha\leq1$, the fractional derivative acts as a damping term, while when $1<\alpha\leq2$, it is the first derivative which plays this role.

Many results on the existence of solutions for fractional differential equations are available in the literature, (see e.g. [1, 2, 8]). The most important recent results on fractional differential equations with Caputo fractional derivatives are surveyed in [1]. The study of the nonexistence of solutions for differential equations is as important as the study of the existence of solutions. It is particularly capital for the nonlinear differential equations where solutions cannot be found explicitly. We refer the reader to [5, 6, 12, 9, 10, 15] and the references therein.

The rest of this paper is structured as follows. Section 2 is devoted to the required notions and notations from fractional calculus that will be used throughout this paper. Also, we present the test function and some of its properties we use. The statements and proofs of our results are presented in Section 3. In the last section, we provide some examples of special types of kernels with the numerical treatment at various values of the parameters.

2. Preliminaries

In this section, we begin with some fractional-order operators relevant to our study and recall some of their properties. We introduce our selected test function with some of its characteristics.

The Riemann-Liouville left-sided and right-sided fractional derivatives of order $\alpha \geq 0$, are defined by

$$\left(D_{a^+}^{\alpha} u\right)(t) = D^n \left(I_{a^+}^{n-\alpha} u\right)(t), \tag{2.1}$$

$$\left(D_{b^-}^{\alpha} u\right)(t) = (-1)^n D^n \left(I_{b^-}^{n-\alpha} u\right)(t), \tag{2.2}$$

respectively, where $D^n = \frac{d^n}{dt^n}$, $n = [\alpha] + 1$ and $[\alpha]$ is the integral part of α. I_{a+}^{α} and $I_{b^-}^{\alpha}$ are the Riemann-Liouville left-sided and right-sided fractional integrals of order $\alpha > 0$ defined by

$$\left(I_{a+}^{\alpha} u\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s) ds, \ t > a,$$

$$\left(I_{b^-}^{\alpha} u\right)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} u(s) ds, \quad t < b,$$

respectively, provided the right-hand sides exist. The function Γ is the Euler Gamma function. We define $I_{a+}^{0} u = I_{b^-}^{0} u = u$. In particular, when $\alpha = m \in \mathbb{N}_0$, it follows from the definitions that

$$D_{a^+}^{m} u = D^m u, \ D_{b^-}^{\alpha} u = (-1)^m D^m u.$$

The Caputo left-sided and right-sided fractional derivatives of order $\alpha \geq 0$, are defined by

$$\left({}^{C}D_{a+}^{\alpha}u\right)(t) = \left(D_{a+}^{\alpha}\left(u(s)-\sum_{i=0}^{n-1}\frac{u^{(i)}(a)}{i!}(s-a)^{i}\right)\right)(t),$$

$$\left({}^{C}D_{b-}^{\alpha}u\right)(t) = \left(D_{b-}^{\alpha}\left(u(s)-\sum_{i=0}^{n-1}\frac{u^{(i)}(b)}{i!}(b-s)^{i}\right)\right)(t),$$

respectively, where $n=[\alpha]+1$ for $\alpha \notin \mathbb{N}_0$ and $n=\alpha$ for $\alpha \in \mathbb{N}_0$.

In particular, when $\alpha = n \in \mathbb{N}_0$, it follows from the definitions that

$${}^{C}D_{a+}^{0}u = {}^{C}D_{b-}^{0}u = u,\ {}^{C}D_{a+}^{n}u = D^{n}u,\ {}^{C}D_{b-}^{n}u = (-1)^{n}D^{n}u.$$

Notice that if $u^{(i)}(a)=0$ for all $i=0,1,...,n-1$, then ${}^{C}D_{a+}^{\alpha}u = D_{a+}^{\alpha}u$, and if $u^{(i)}(b)=0$ for all $i=0,1,...,n-1$, then ${}^{C}D_{b-}^{\alpha}u = D_{b-}^{\alpha}u$. For more details about fractional operators, we refer to the books [7, 14].

The space of absolutely continuous functions on $[a,b]$ is denoted by $AC[a,b]$. In general, for $n \in \mathbb{N}$,

$$AC^{n}[a,b] = \left\{u : [a,b] \to \mathbb{R} \text{ such that } D^{n-1}u \in AC[a,b]\right\}.$$

If $u \in AC^{n}[a,b]$, then ${}^{C}D_{a+}^{\alpha}u$ and ${}^{C}D_{b-}^{\alpha}u$ exist almost everywhere on the interval $[a,b]$ and

$$\left({}^{C}D_{a+}^{\alpha}u\right)(t) = \left(I_{a+}^{n-\alpha}D^{n}u\right)(t), \tag{2.3}$$

$$\left({}^{C}D_{b-}^{\alpha}u\right)(t) = (-1)^{n}\left(I_{b-}^{n-\alpha}D^{n}u\right)(t). \tag{2.4}$$

Lemma 2.1. [7] *If $\alpha \geq 0$, $\beta > 0$, then*

$$\left(I_{b-}^{\alpha}(b-s)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(b-t)^{\beta+\alpha-1},$$

$$\left(D_{b-}^{\alpha}(b-s)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(b-t)^{\beta-\alpha-1}.$$

Lemma 2.2. [14] *Let $\alpha \geq 0$, $p \geq 1, q \geq 1$ and $\frac{1}{p}+\frac{1}{q} \leq 1+\alpha$ ($p \neq 1$ and $q \neq 1$ in the case when $\frac{1}{p}+\frac{1}{q} = 1+\alpha$). If $f \in L^{p}(a,b)$ and $g \in L^{q}(a,b)$, then*

$$\int_a^b f(t)\left(I_{a+}^{\alpha}g\right)(t)\,dt = \int_a^b g(t)\left(I_{b-}^{\alpha}f\right)(t)\,dt.$$

Lemma 2.3. *Let $\alpha \geq 0$ and $n=[\alpha]+1$ for $\alpha \notin \mathbb{N}_0$ and $n=\alpha$ for $\alpha \in \mathbb{N}_0$. For $f \in C[a,b]$ and $g, I_{b-}^{n-\alpha}f \in AC^{n}[a,b]$, we have*

$$\int_a^b f(t)\left({}^{C}D_{a+}^{\alpha}g\right)(t)\,dt = \int_a^b g(t)\left(D_{b-}^{\alpha}f\right)(t)\,dt + \sum_{i=0}^{n-1}\left[\left(D_{b-}^{\alpha+i-n}f\right)(t)\left(D^{n-1-i}g\right)(t)\right]_a^b.$$

Proof. Since $g \in AC^n[a,b]$, then we have from the definition (2.3),

$$\int_a^b f(t)\left({}^C D_{a+}^{\alpha} g\right)(t)\,dt = \int_a^b f(t)\left(I_{a^+}^{n-\alpha} D^n g\right)(t)dt.$$

Because $f \in L^{m_1}(a,b)$ for any $m_1 \geq 1$ and $D^n g \in L^1(a,b)$, we deduce from Lemma 2.2,

$$\int_a^b f(t)\left(I_{a^+}^{n-\alpha} D^n g\right)(t)dt = \int_a^b D^n g(t)\left(I_{b^-}^{n-\alpha} f\right)(t)dt.$$

As $I_{b-}^{n-\alpha} f \in AC^n[a,b]$ and $D^{n-1} g \in AC[a,b]$, then integrating by parts n times yields

$$\int_a^b f(t)\left({}^C D_{a+}^{\alpha} g\right)(t)\,dt = \sum_{i=0}^{n-1}\left[\left(D_{b^-}^{\alpha+i-n} f\right)(t)\left(D^{n-1-i} g\right)(t)\right]_a^b +$$

$$+(-1)^n \int_a^b g(t) D^n\left(I_{b^-}^{n-\alpha} f\right)(t)\,dt.$$

Owing to (2.2), the proof is complete. □

In this paper, we use the following test function

$$\phi(t) := \begin{cases} \left(1-\frac{t}{T}\right)^{\theta}, & 0 \leq t \leq T, \\ 0, & t > T. \end{cases} \tag{2.5}$$

The function ϕ has the following properties.

Lemma 2.4. *Let ϕ be the function defined in (2.5), then for $\theta > nr-1$, $r > 1$, $n = 0, 1, 2, ...,$ we have*

$$\int_0^T \phi^{1-r}(t)\left|D^n \phi(t)\right|^r dt = C_{n,r} T^{1-nr},\ T > 0,$$

where

$$C_{n,r} = \frac{\Gamma^r(\theta+1)}{(\theta-nr+1)\,\Gamma^r(\theta-n+1)}.$$

Proof. Since

$$\begin{aligned} D^n \phi(t) &= (-1)^n \theta(\theta-1)(\theta-2)\ldots(\theta-n+1) T^{-\theta}(T-t)^{\theta-n} \\ &= \frac{(-1)^n \Gamma(\theta+1)}{\Gamma(\theta-n+1)} T^{-\theta}(T-t)^{\theta-n}, \end{aligned}$$

it follows that

$$\begin{aligned} \int_0^T \phi^{1-r}(t)\left|D^n \phi(t)\right|^r dt &= \left(\frac{\Gamma(\theta+1)}{\Gamma(\theta-n+1)}\right)^r T^{-\theta} \int_0^T (T-t)^{\theta-nr} dt \\ &= C_{n,r} T^{1-nr}. \end{aligned}$$

□

Lemma 2.5. *Let $\alpha \geq 0$ and ϕ be as in (2.5) with $\theta > \alpha - 1$, then we have for all $0 \leq t \leq T$,*

$$\left(D_{T^-}^{\alpha}\phi\right)(t) = \frac{\Gamma(\theta+1)}{\Gamma(\theta-\alpha+1)} T^{-\theta}\left(T-t\right)^{\theta-\alpha}, \tag{2.6}$$

$$\int_0^T t^m \left(D_{T^-}^{\alpha}\phi\right)(t)\,dt = \xi_{m,\theta} T^{m+1-\alpha}, \quad m = 0,1,2,...,n-1, \;\; n = [\alpha]+1, \tag{2.7}$$

where $\xi_{m,\theta} = \frac{(-1)^m m!\Gamma(\theta+1)}{\Gamma(\theta-\alpha+m+2)}$.

Proof. We have from Lemma 2.1,

$$\left(D_{T^-}^{\alpha}\phi\right)(t) = \left(D_{T^-}^{\alpha} T^{-\theta}\left(T-s\right)^{\theta}\right)(t) = \frac{\Gamma(\theta+1)}{\Gamma(\theta-\alpha+1)} T^{-\theta}\left(T-t\right)^{\theta-\alpha}.$$

An integration m times by parts yields

$$\begin{aligned} \int_0^T t^m \left(D_{T^-}^{\alpha}\phi\right)(t)\,dt &= \sum_{i=0}^{m-1}\left[(-1)^i \frac{m!}{(m-i)!} t^{m-i}\left(I_{T^-}^{i+1} D_{T^-}^{\alpha}\phi\right)(t)\right]_0^T \\ &\quad + (-1)^m m! \int_0^T \left(I_{T^-}^{m} D_{T^-}^{\alpha}\phi\right)(t)\,dt. \end{aligned} \tag{2.8}$$

Using (2.6) and Lemma 2.1, we find

$$\left(I_{T^-}^{i+1} D_{T^-}^{\alpha}\phi\right)(t) = \frac{\Gamma(\theta+1)}{\Gamma\left(\theta-\alpha+i+2\right)} T^{-\theta}\left(T-t\right)^{\theta-\alpha+i+1},$$

$$\left(I_{T^-}^{m} D_{T^-}^{\alpha}\phi\right)(t) = \frac{\Gamma(\theta+1)}{\Gamma\left(\theta-\alpha+m+1\right)} T^{-\theta}\left(T-t\right)^{\theta-\alpha+m}.$$

Therefore

$$\left[t^{m-i}\left(I_{T^-}^{i+1} D_{T^-}^{\alpha}\phi\right)(t)\right]_0^T = 0 \;\; \text{for all } i = 0,1,2,...,m-1, \tag{2.9}$$

and

$$\int_0^T \left(I_{T^-}^{m} D_{T^-}^{\alpha}\phi\right)(t)\,dt = \frac{\Gamma(\theta+1)}{\Gamma\left(\theta-\alpha+m+2\right)} T^{m-\alpha+1}. \tag{2.10}$$

Now, by substituting (2.9) and (2.10) in (2.8) we obtain (2.7). □

Lemma 2.6. *Let $\alpha \geq 0$, $n = [\alpha]+1$ and ϕ be as in (2.5) with $\theta > \alpha - 1$, then*

$$\left(I_{T^-}^{n-\alpha}\phi\right)(t) = \frac{\Gamma(\theta+1)}{\Gamma(\theta+n-\alpha+1)} T^{-\theta}\left(T-t\right)^{\theta+n-\alpha},$$

for all $0 \leq t \leq T$. Moreover, $I_{T^-}^{n-\alpha}\phi \in AC^n[0,T]$.

Proof. From an application of Lemma 2.1, we deduce that

$$\left(I_{T^-}^{n-\alpha}\phi\right)(t) = \left(I_{T^-}^{n-\alpha}\left(T^{-\theta}\left(T-s\right)^{\theta}\right)\right)(t) = \frac{\Gamma(\theta+1)}{\Gamma(\theta+n-\alpha+1)} T^{-\theta}\left(T-t\right)^{\theta+n-\alpha}.$$

It is clear that $I_{T^-}^{n-\alpha}\phi$ is in the space $AC^n[0,T]$ for $\theta > \alpha - 1$. □

Lemma 2.7. *Let $\alpha \geq 0$ and ϕ be as in (2.5) with $\theta > \max\{0, \alpha - 1\}$. Suppose that $g \in AC^n[0,T]$, $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}_0$ and $n = \alpha$ for $\alpha \in \mathbb{N}_0$. Then*

$$\int_0^T \phi(t) \left({}^C D_{0+}^{\alpha} g\right)(t)\, dt = \int_0^T g(t) \left(D_{T^-}^{\alpha} \phi\right)(t)\, dt - \sum_{i=0}^{n-1} \bar{\xi}_{i,\alpha} T^{n-\alpha-i} \left(D^{n-1-i} g\right)(0),$$

where $\bar{\xi}_{i,\alpha} = \frac{\Gamma(\theta+1)}{\Gamma(\theta+1-\alpha-i+n)}$.

Proof. As a consequence of (2.6) in Lemma 2.5, we get for $i = 0, 1, 2, ..., n-1$,

$$\left(D_{T^-}^{\alpha+i-n} \phi\right)(0) = \frac{\Gamma(\theta+1)}{\Gamma(\theta+1-\alpha-i+n)} T^{n-\alpha-i},$$

$$\left(D_{T^-}^{\alpha+i-n} \phi\right)(T) = 0.$$

Since $\phi \in C[0,T]$ for $\theta > 0$ and $I_{T^-}^{n-\alpha} \phi \in AC^n[0,T]$, then the conclusion follows in the light of Lemma 2.3. $\square$

3. The main results

In this section we prove the nonexistence of a nontrivial global solution for the initial value problems $(1.1) - (1.2)$ and $(1.1) - (1.3)$.

Definition 3.1. By a nontrivial global solution of Problem $(1.1) - (1.2)$ or Problem $(1.1) - (1.3)$, we mean a nonzero function u defined on $[0, \infty)$ such that $u \in AC[0,T]$ or $u \in AC^2[0,T]$ for all $T > 0$, for which the inequality (1.1) holds for all $t > 0$, and satisfying (1.2) or (1.3), respectively.

Firstly, we need to prove the following auxiliary lemma.

Lemma 3.1. *Let $\beta \geq 0$, $n = [\beta] + 1$ and $r > 1$. Let ϕ be as in (2.5) with $\theta > nr - 1$. Suppose that g is a nonnegative function that is different from zero almost everywhere and $t^{r(n-\beta-1)} g^{1-r}(t) \in L^1_{loc}[0, +\infty)$. Then, for any $T > 0$*

$$\int_0^T \left(D_{T^-}^{\beta} \phi\right)^r(t) \left(\int_t^T g(s-t) \phi(s)\, ds\right)^{1-r} dt \leq \hat{C}_{\beta,r} T^{1-nr} \int_0^T t^{r(n-\beta-1)} g^{1-r}(t) dt,$$

where $\hat{C}_{\beta,r} = \frac{C_{n,r}}{\Gamma^r(n-\beta)}$, $C_{n,r}$ is given in Lemma 2.4.

Proof. Since $\phi^{(i)}(T) = 0$ for all $i = 0, 1, ..., n-1$, then $D_{T^-}^{\beta} \phi = {}^C D_{T^-}^{\beta} \phi$. Also, since $\phi \in AC^n[0,T]$ for $\theta > n - 1$, then ${}^C D_{T^-}^{\beta} \phi = (-1)^n I_{T^-}^{n-\beta} D^n \phi$ and

$$\left(D_{T^-}^{\beta} \phi\right)(t) \leq \left(I_{T^-}^{n-\beta} |D^n \phi|\right)(t) = \frac{1}{\Gamma(n-\beta)} \int_t^T (s-t)^{n-\beta-1} |(D^n \phi)(s)|\, ds$$

$$= \frac{1}{\Gamma(n-\beta)} \int_t^T (s-t)^{n-\beta-1} g^{\frac{1}{r'}}(s-t) \phi^{\frac{1}{r'}}(s)\, g^{-\frac{1}{r'}}(s-t) \phi^{-\frac{1}{r'}}(s)\, |(D^n \phi)(s)|\, ds.$$

Using Hölder inequality with $\frac{1}{r}+\frac{1}{r'}=1$, we find

$$\begin{aligned}\left(D_{T^-}^{\beta}\phi\right)(t) &\leq \frac{1}{\Gamma(n-\beta)}\left(\int_t^T g(s-t)\phi(s)\,ds\right)^{\frac{1}{r'}} \\ &\quad\times\left(\int_t^T (s-t)^{(n-\beta-1)r} g^{-\frac{r}{r'}}(s-t)\phi^{-\frac{r}{r'}}(s)\left|\left(D^n\phi\right)(s)\right|^r ds\right)^{\frac{1}{r}}.\end{aligned}$$

Therefore

$$\begin{aligned}&\int_0^T \left(D_{T^-}^{\beta}\phi\right)^r(t)\left(\int_t^T g(s-t)\phi(s)\,ds\right)^{1-r} dt \\ \leq\ & b_1\int_0^T\int_t^T (s-t)^{r(n-\beta-1)} g^{-\frac{r}{r'}}(s-t)\phi^{-\frac{r}{r'}}(s)\left|\left(D^n\phi\right)(s)\right|^r dsdt,\ b_1=\frac{1}{\Gamma^r(n-\beta)} \\ =\ & b_1\int_0^T\int_0^s (s-t)^{r(n-\beta-1)} g^{1-r}(s-t)\phi^{1-r}(s)\left|\left(D^n\phi\right)(s)\right|^r dtds \\ =\ & b_1\int_0^T \phi^{1-r}(s)\left|\left(D^n\phi\right)(s)\right|^r\left(\int_0^s (s-t)^{r(n-\beta-1)} g^{1-r}(s-t)dt\right)ds.\end{aligned}$$

Let $\tau=s-t$, then the following uniform bound is obtained

$$\int_0^s \tau^{r(n-\beta-1)}g^{1-r}(\tau)d\tau\leq\int_0^T \tau^{r(n-\beta-1)}g^{1-r}(\tau)d\tau,$$

and the result follows from Lemma 2.4. □

Now, we are able to prove the nonexistence of solutions for the problem (1.1) − (1.2) when $0\leq\alpha<1$.

Theorem 3.1. *Let $0\leq\alpha<1$ and f be $C^1(\mathbb{R},\mathbb{R})$ function satisfies*

$$f(x)\geq a|x|^p \ \text{ for all } x\in\mathbb{R} \text{ for some positive constant } a \text{ and } p>1.$$

Suppose that g is a nonnegative function different from zero almost everywhere with $g^{1-p'}$, $t^{-\alpha p'}g^{1-p'}(t)\in L^1_{loc}[0,+\infty)$ and

$$\lim_{T\to\infty} T^{1-p'}\left(T^{-p'}\int_0^T g^{1-p'}(t)dt+\int_0^T t^{-\alpha p'}g^{1-p'}(t)dt\right)=0, \tag{3.1}$$

where $p'=\frac{p}{p-1}$. Then the problem (1.1) − (1.2) does not admit any global nontrivial solution when $u_0\geq 0$.

Proof. Assume, on the contrary, that a solution $u\in AC[0,T]$ exists for all $T>0$. Multiplying both sides of (1.1) by the test function ϕ defined in (2.5) with $\theta>2p'-1$

and integrating, we obtain

(3.2)
$$\int_0^T \phi(t)\, u'(t)dt + \int_0^T \phi(t) \left({}^C D_{0+}^{\alpha} u\right)(t)dt \geq \int_0^T \phi(t) \left(\int_0^t g(t-s) f(u(s))\, ds \right) dt.$$

By Lemma 2.7 we have,

$$\int_0^T \phi(t) \left({}^C D_{0+}^{\alpha} u\right)(t)\, dt = \int_0^T u(t) \left(D_{T-}^{\alpha} \phi\right)(t)\, dt - u_0 \bar{\xi}_{0,\alpha} T^{1-\alpha}, \tag{3.3}$$

$$\int_0^T \phi(t)\, u'(t)dt = -\int_0^T u(t) \phi'(t)\, dt - u_0, \tag{3.4}$$

where $\bar{\xi}_{0,\alpha} = \frac{\Gamma(\theta+1)}{\Gamma(\theta-\alpha+2)}$.

Substituting (3.3) and (3.4) in (3.2) yields

$$W + u_0 \left(1 + \bar{\xi}_{0,\alpha} T^{1-\alpha}\right) \leq \int_0^T u(-\phi')\, dt + \int_0^T u D_{T-}^{\alpha} \phi dt \tag{3.5}$$

where

$$W := \int_0^T \phi(t) \left(\int_0^t g(t-s) f(u(s))\, ds \right) dt. \tag{3.6}$$

To have a bound for the integral$W$, we rewrite it as

$$W = \int_0^T f(u(s)) \left(\int_s^T g(t-s) \phi(t)\, dt \right) ds = \int_0^T f(u(s))\, G(s) ds,$$

where

$$G(s) := \int_s^T g(t-s) \phi(t)\, dt, \quad 0 \leq s < t \leq T.$$

Applying Hölder inequality with $\frac{1}{p} + \frac{1}{p'} = 1$ for the two integrals in right hand side of (3.5), we obtain

$$\int_0^T u(-\phi')\, dt \leq \left(\int_0^T |u|^p\, G dt \right)^{\frac{1}{p}} \left(\int_0^T G^{-\frac{p'}{p}} (-\phi')^{p'}\, dt \right)^{\frac{1}{p'}} \leq W^{\frac{1}{p}} U^{\frac{1}{p'}},$$

$$\int_0^T u D_{T-}^{\alpha} \phi dt \leq \left(\int_0^T |u|^p\, G dt \right)^{\frac{1}{p}} \left(\int_0^T G^{-\frac{p'}{p}} \left(D_{T-}^{\alpha} \phi\right)^{p'} dt \right)^{\frac{1}{p'}} \leq W^{\frac{1}{p}} V^{\frac{1}{p'}},$$

where

$$U := a^{-\frac{p'}{p}} \int_0^T G^{-\frac{p'}{p}} (-\phi')^{p'}\, dt \quad \text{and} \quad V := a^{-\frac{p'}{p}} \int_0^T G^{-\frac{p'}{p}} \left(D_{T-}^{\alpha} \phi\right)^{p'} dt. \tag{3.7}$$

Therefore (3.5) can be rewritten as

$$W + u_0 \left(1 + \bar{\xi}_{0,\alpha} T^{1-\alpha}\right) \leq W^{\frac{1}{p}} \left(U^{\frac{1}{p'}} + V^{\frac{1}{p'}} \right). \tag{3.8}$$

From the positivity of W, u_0 and $\bar{\xi}_{0,\alpha}$, we get from (3.8)

$$W \leq W^{\frac{1}{p}} \left(U^{\frac{1}{p'}} + V^{\frac{1}{p'}} \right),$$

which implies that

$$W \leq 2^{p'-1}(U+V). \tag{3.9}$$

Now, we estimate the integral U defined in (3.7),

$$\begin{aligned} U &= a^{-\frac{p'}{p}} \int_0^T G^{-\frac{p'}{p}}(t)\left(-\phi'(t)\right)^{p'} dt \\ &= a^{-\frac{p'}{p}} \int_0^T \left(\int_t^T g(s-t)\phi(s)\, ds \right)^{1-p'} \left(D^1_{T^-}\phi(t)\right)^{p'} dt \\ &\leq a^{-\frac{p'}{p}} \hat{C}_{1,p'} T^{1-2p'} \int_0^T g^{1-p'}(t)dt, \ \text{(Lemma 3.1 with } \beta = 1). \end{aligned} \tag{3.10}$$

Similarly, we see that

$$\begin{aligned} V &= a^{-\frac{p'}{p}} \int_0^T G^{-\frac{p'}{p}}(t)\left(D^{\alpha}_{T^-}\phi\right)^{p'}(t)\, dt \\ &= a^{-\frac{p'}{p}} \int_0^T \left(\int_t^T g(s-t)\phi(s)\, ds \right)^{1-p'} \left(D^{\alpha}_{T^-}\phi\right)^{p'}(t)\, dt \\ &\leq a^{-\frac{p'}{p}} \hat{C}_{\alpha,p'} T^{1-p'} \int_0^T t^{-\alpha p'} g^{1-p'}(t)dt, \end{aligned} \tag{3.11}$$

(Lemma 3.1 with $0 \leq \beta = \alpha < 1$).

Substituting (3.10) and (3.11) in (3.9) we end up with

$$W \leq M\left(T^{1-2p'} \int_0^T g^{1-p'}(t)dt + T^{1-p'} \int_0^T t^{-\alpha p'} g^{1-p'}(t)dt \right), \tag{3.12}$$

where $M = 2^{p'-1} \max\left\{ a^{-\frac{p'}{p}} \hat{C}_{1,p'}, a^{-\frac{p'}{p}} \hat{C}_{\alpha,p'} \right\}$. Eventually, we deduce from Condition (3.1) that $u \equiv 0$ and the proof is complete. □

The following result is a corollary of Theorem 3.1.

Corollary 3.1. *Let α and f be as in the assumptions of Theorem 3.1. Suppose that, for any $T > 0$, there exist positive constants k_1, k_2,*

$$\omega_1 < \frac{p+1}{p-1} \ \text{and} \ \omega_2 < \frac{1}{p-1} \tag{3.13}$$

such that

$$\int_0^T g^{1-p'}(t)dt \leq k_1 T^{\omega_1}, \ \text{and} \ \int_0^T t^{-\alpha p'} g^{1-p'}(t)dt \leq k_2 T^{\omega_2}, \tag{3.14}$$

where $p' = \frac{p}{p-1}$ and g is a nonnegative function that is different from zero almost everywhere with $g^{1-p'}$, $t^{-\alpha p'} g^{1-p'}(t) \in L^1_{loc}[0, +\infty)$. Then the problem (1.1) – (1.2) has no nontrivial global solution when $u_0 \geq 0$.

Proof. It suffices to show that the assumptions (3.13) and (3.14) imply that (3.1) is fulfilled. We deduce from the hypothesis (3.14), that

$$0 \leq T^{1-p'}\left(T^{-p'}\int_0^T g^{1-p'}(t)dt + \int_0^T t^{-\alpha p'} g^{1-p'}(t)dt\right) \leq k_1 T^{1-2p'+\omega_1} + k_2 T^{1-p'+\omega_2}.$$

We find from (3.13) that $1-2p'+\omega_1 < 0,\ 1-p'+\omega_2 < 0$ and consequently (3.1) follows. □

The following corollary considers an important type of kernels appear in numerous applications.

Corollary 3.2. *Let α and f be as in the assumptions of Theorem 3.1. Suppose that $g(t) \geq bt^{-\eta}$, $t>0$, for some constant $b>0$, where $1-p(1-\alpha) < \eta < 2+p(\alpha-1)$. Then the problem (1.1) – (1.2) has no nontrivial global solution when $u_0 \geq 0$.*

Proof. It suffices to show that Hypothesis (3.1) is satisfied with the function g. Indeed, since $g(t) \geq bt^{-\eta}$; $b,\ \eta > 0$, then $g^{1-p'}(t) \leq b^{1-p'} t^{\eta(p'-1)}$ and

$$\int_0^T g^{1-p'}(t)dt \leq b^{1-p'}\int_0^T t^{\eta(p'-1)}dt = \frac{b^{1-p'}}{\eta(p'-1)+1}T^{\eta(p'-1)+1}$$

$$\begin{aligned}\int_0^T t^{-\alpha p'} g^{1-p'}(t)dt &\leq b^{1-p'}\int_0^T t^{\eta(p'-1)-\alpha p'}dt \\ &= \frac{b^{1-p'}}{\eta(p'-1)-\alpha p'+1}T^{\eta(p'-1)-\alpha p'+1}.\end{aligned}$$

Therefore

$$\begin{aligned}&T^{1-p'}\left(T^{-p'}\int_0^T g^{1-p'}(t)dt + \int_0^T t^{-\alpha p'} g^{1-p'}(t)dt\right) \\ &\leq \frac{b^{1-p'}}{\eta(p'-1)+1}T^{\sigma_1} + \frac{b^{1-p'}}{\eta(p'-1)-\alpha p'+1}T^{\sigma_2},\end{aligned}$$

where

$$\sigma_1 = 2-\eta+p'(\eta-2),\ \sigma_2 = 2-\eta+p'(\eta-\alpha-1).$$

It follows from $1-p(1-\alpha) < \eta < 2+p(\alpha-1)$ that both σ_1 and σ_2 are negative and so (3.1) is satisfied. □

Remark 3.1. Corollary 3.2 can be considered also as a consequence of Corollary 3.1 with

$$\begin{aligned}k_1 &= \frac{b^{1-p'}}{\eta(p'-1)+1},\ k_2 = \frac{b^{1-p'}}{\eta(p'-1)-\alpha p'+1}, \\ \omega_1 &= \eta(p'-1)+1 = \frac{p+\eta-1}{p-1}, \\ \omega_2 &= \eta(p'-1)-\alpha p'+1 = \frac{p(1-\alpha)+\eta-1}{p-1},\quad 0 \leq \alpha < 1.\end{aligned}$$

It is clear from $1-p(1-\alpha)<\eta<2-p(1-\alpha)$ that $0<\omega_1<\frac{p+1}{p-1}$, $0<\omega_2<\frac{1}{p-1}$.

The next theorem deals with the case $1\leq\alpha<2$.

Theorem 3.2. *Let $1\leq\alpha<2$ and f be as in the assumptions of Theorem 3.1. Assume that g is a nonnegative function that is different from zero almost everywhere with $g^{1-p'}$, $t^{(1-\alpha)p'}g^{1-p'}(t)\in L^1_{loc}[0,+\infty)$. Suppose that*

$$\lim_{T\to\infty}T^{1-2p'}\left(\int_0^T g^{1-p'}(t)dt+\int_0^T t^{(1-\alpha)p'}g^{1-p'}(t)dt\right)=0, \tag{3.15}$$

where $p'=\frac{p}{p-1}$. Then (1.1) *subject to* (1.3) *has no nontrivial global solution when $u_0,u_1\geq 0$.*

Proof. Assume, on the contrary, that a solution $u\in AC^2[0,T]$ exists for all $T>0$. Then as in the proof of Theorem 3.1, we have

$$W+u_0\left(1+\bar{\xi}_{1,\alpha}T^{1-\alpha}\right)+u_1\bar{\xi}_{0,\alpha}T^{2-\alpha}\leq W^{\frac{1}{p}}\left(U^{\frac{1}{p'}}+V^{\frac{1}{p'}}\right),$$

where W, U and V are as in (3.6) and (3.7). Accordingly, for $1\leq\alpha<2$, from Lemma 3.1 with $1\leq\beta=\alpha<2$, we obtain the following estimates

$$\begin{aligned}
U &\leq a^{-\frac{p'}{p}}\hat{C}_{1,p'}T^{1-2p'}\int_0^T g^{1-p'}(t)dt,\\
V &= \int_0^T\left(\int_t^T g(s-t)\phi(s)\,ds\right)^{1-p'}\left(D^\alpha_{T-}\phi\right)^{p'}(t)\,dt\\
&\leq a^{-\frac{p'}{p}}\hat{C}_{\alpha,p'}T^{1-2p'}\int_0^T t^{(1-\alpha)p'}g^{1-p'}(t)dt,\quad \text{(Lemma 3.1 with } 1\leq\beta=\alpha<2).
\end{aligned}$$

By the assumptions (3.15), we get $u\equiv 0$ and the proof is complete. □

Applying Theorem 3.2 for kernels of the type $g(t)\geq bt^{-\eta}$, we obtain the following result.

Corollary 3.3. *Let α and f be as in the assumptions of Theorem 3.2. Suppose that $g(t)\geq bt^{-\eta}$, $t>0$, for some constant $b>0$, where $1-p(2-\alpha)<\eta<2$. Then* (1.1) *subject to* (1.3) *has no nontrivial global solution when $u_0,u_1\geq 0$.*

Proof. The hypotheses of Theorem 3.2 are satisfied with the given kernel g. In fact,

$$\int_0^T g^{1-p'}(t)dt\leq b^{1-p'}\int_0^T t^{\eta(p'-1)}dt=b_2T^{\eta_1},$$

$$\int_0^T t^{(1-\alpha)p'}g^{1-p'}(t)dt\leq b^{1-p'}\int_0^T t^{\eta(p'-1)+(1-\alpha)p'}dt=b_3T^{\eta_2},$$

where

$$b_2 = \frac{b^{1-p'}}{\eta(p'-1)+1},\ b_3 = \frac{b^{1-p'}}{\eta(p'-1)+(1-\alpha)p'+1},$$

$$\eta_1 = \eta(p'-1)+1 = \frac{p+\eta-1}{p-1} > 0,$$

$$\eta_2 = \eta(p'-1)+(1-\alpha)p'+1 = \frac{p(2-\alpha)+\eta-1}{p-1} > 0.$$

It is easy to check that $\eta_1, \eta_2 > 0$ and $1-2p'+\eta_1, 1-2p'+\eta_2 < 0$. □

Remark 3.2. The same results of Theorem 3.2 and Corollary 3.3 can be obtained with more relaxed conditions on the initial data. It is enough to have $a_0u_0+a_1u_1 \geq 0$, for some positive constants a_0 and a_1, instead of $u_0 \geq 0$ and $u_1 \geq 0$. Indeed, a_0 and a_1 can be given in terms of the constants $T, \bar{\xi}_{0,\alpha}$ and $\bar{\xi}_{1,\alpha}$.

4. Applications

In this section, we provide a special case of the kernel $g(t)$ in Corollaries 3.2 and 3.3, when the source term is the Riemann-Liouville fractional integral of a power of the state. We show here by computing the solutions numerically that the solutions can not exist globally.

Example 4.1. *Consider the fractional integro-differential inequality*

$$u'(t) + \left({}^C D^{\alpha}_{0+}u\right)(t) \geq \left(I^{\beta}_{0+}|u(s)|^p\right)(t), \quad t>0,\ \beta>0,\ p>1, \tag{4.1}$$

subject to (1.2) *when* $0 \leq \alpha < 1$,*or,* (1.3) *when* $1 \leq \alpha < 2$. *The problem consists of* (4.1) *subject to* (1.2) *is a special case of* (1.1) – (1.2) *when*

$$g(t) = t^{\beta-1}, \quad 0 < \beta < p(1-\alpha),\ 0 \leq \alpha < 1.$$

Therefore, we deduce from Corollary 3.2, when $g(t) = t^{-\eta}$, $\eta = 1-\beta$, *that Problem* (4.1) *has no nontrivial global solutions when* $u_0 \geq 0$. *Similarly,* (4.1) *subject to* (1.3) *has no nontrivial global solution when* $u_0, u_1 \geq 0$. *This result is a special case of Corollary 3.3 when*

$$g(t) = t^{-\eta},\ \eta = 1-\beta, \quad 0 < \beta < p(2-\alpha),\ 1 \leq \alpha < 2.$$

For treating the two problems in Example 4.1 numerically, we consider the case of equality and write

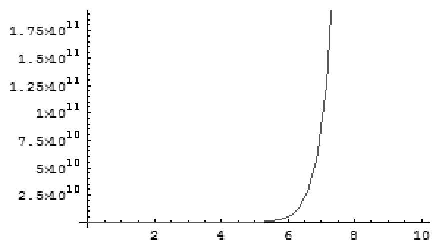
$$u(t) = u_0 - \int_0^t \left({}^C D^{\alpha}_{0+}u\right)(s)ds + \left(I^{\beta+1}_{0+}|u(s)|^p\right)(t).$$

Using the iterative schemes

$$u^{(n)}(t) = u_0 - \int_0^t D^{\alpha}_{0+} \left(u^{(n-1)}(\tau) - u_0 \right)(s) ds + \left(I^{\beta+1}_{0^+} \left| u^{(n-1)}(s) \right|^p \right)(t),$$

$$u^{(n)}(t) = u_0 - \int_0^t \left(D^{\alpha}_{0+} u^{(n-1)}(\tau) - u_0 - u_1\tau \right)(s) ds + \left(I^{\beta+1}_{0^+} \left| u^{(n-1)}(s) \right|^p \right)(t),$$

$n = 1, 2, ...$ with $u^{(0)}(t) = u_0$, for (4.1) subject to (1.2) and (1.3), respectively, the curves of the fourth iteration $u^{(4)}$ show, for different values of the parameters, that the solutions can not be extended for all t.



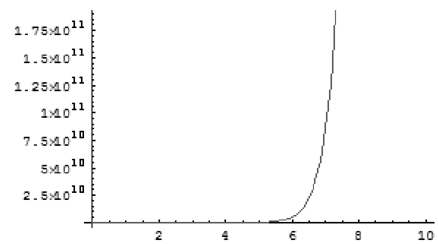


Figure 1: $\alpha = \beta = \frac{1}{2}, p = 2, u_0 = 1$. Figure 2: $\alpha = \frac{3}{2}, \beta = \frac{1}{2}, p = 2, u_0 = u_1 = 1$.

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