Известия НАН Армении, Математика, том 56, н. 3, 2021, стр. 61 – 78.

ON THE FREDHOLM PROPERTY OF SEMIELLIPTIC OPERATORS IN ANISOTROPIC WEIGHTED SPACES IN $\mathbb{R}^n$

A. TUMANYAN

Russian-Armenian University, Mentor, a Siemens Business[1]
E-mail: *ani.tumanyan92@gmail.com*

Abstract. We study the Fredholm property of semielliptic operators in anisotropic weighted spaces in $\mathbb{R}^n$. In this paper necessary conditions are obtained for fulfillment of a priori estimates for such operators. Necessary and sufficient conditions are obtained for the Fredholm property of semielliptic operators with variable coefficients that have certain rate at infinity.

MSC2010 numbers: 35H30; 47A53.

Keywords: Fredholm property; semielliptic operator; a priori estimate; anisotropic weighted space.

1. Introduction, basic notions and definitions

This paper studies the Fredholm property of semielliptic operators with variable coefficients in anisotropic weighted Sobolev spaces in $\mathbb{R}^n$. The class of semielliptic operators is a special subclass of hypoelliptic operators which contains elliptic, parabolic, $2b$–parabolic operators, etc. (see [1]). The analysis of the Fredholm property of semielliptic operators in Sobolev spaces in $\mathbb{R}^n$ has certain difficulties related to the facts that Fredholm theorems for compact manifolds cannot always be used in this case and characteristic polynomials of semielliptic operators are not homogeneous as in elliptic case. The Fredholm property of such operators has been a subject of interest for many authors.

The Fredholm property of elliptic operators in special weighted spaces is studied in the works of L.A. Bagirov [2], R.B. Lockhart, R.C. McOwen [3, 4], E. Schrohe [5] and others.

L.A. Bagirov [6], G.A. Karapetyan, A.A. Darbinyan [7] and A.A. Darbinyan, A.G. Tumanyan [8, 9] studied the Fredholm property of semielliptic operators in anisotropic weighted spaces. In G.V. Demidenko's works [10, 11] the isomorphism properties are obtained on the special scale of weighted spaces for quasi-homogenous semielliptic operator with constant coefficients.

[1]This work is supported in part by Science Committee of Ministry of Education and Science of Armenia and Russian Foundation of Basic Research under Thematic Program no. 18RF-004.

In this work necessary and sufficient conditions are obtained for the Fredholm property of semielliptic operators with special variable coefficients acting in anisotropic Sobolev spaces with certain weight functions. The classes of considered operators and the weight functions are extended compared to the ones from the works [8, 9].

Definition 1.1. *A bounded linear operator A, acting from a Banach space X to a Banach space Y, is called an n–normal operator, if the following conditions hold:*

(1) *the image of operator A is closed* $\left(\operatorname{Im}(A) = \overline{\operatorname{Im}(A)}\right)$*;*
(2) *the kernel of operator A is finite dimensional* $(\dim \operatorname{Ker}(A) < \infty)$.

An operator A is called a Fredholm operator if conditions 1-2 hold and

(3) *the cokernel of operator A is finite dimensional*
$(\dim \operatorname{coker}(A) = \dim Y/\operatorname{Im}(A) < \infty)$.

The difference between the dimension of the kernel and the cokernel of operator A is called index of the operator:

$$\operatorname{ind}(A) = \dim \operatorname{Ker}(A) - \dim \operatorname{coker}(A).$$

Definition 1.2. *For a bounded linear operator A, acting from a Banach space X to a Banach space Y, bounded linear operator $R_1 : Y \to X$ and $R_2 : Y \to X$ are called respectively left and right regularizers if the following holds: $R_1 A = I_X + T_1, AR_2 = I_Y + T_2$, where I_X, I_Y– identity operators, $T_1 : X \to X$ and $T_2 : Y \to Y$ are compact operators.*

Definition 1.3. *For a bounded linear operator A, acting from a Banach space X to a Banach space Y, bounded linear operator $R : Y \to X$ is called a regularizer for operator A, if it is left and right regularizer.*

Let $n \in \mathbb{N}$ and $\mathbb{R}^n$ be Euclidean n-dimensional space, $\mathbb{Z}_+^n$, $\mathbb{N}^n$ be the sets of n-dimensional multiindices and multiindices with natural components respectively.

Consider the differential form

$$P(x, \mathbb{D}) = \sum_{(\alpha:\nu)\le s} a_\alpha(x) D^\alpha, \tag{1.1}$$

where $s \in \mathbb{N}, \alpha \in \mathbb{Z}_+^n, \nu \in \mathbb{N}^n, (\alpha : \nu) = \frac{\alpha_1}{\nu_1} + \cdots + \frac{\alpha_n}{\nu_n}, D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}, D_j = i^{-1}\frac{\partial}{\partial x_j}$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n, a_\alpha(x) \in C(\mathbb{R}^n)$.

Denote

$$P_s(x, \mathbb{D}) = \sum_{(\alpha:\nu)=s} a_\alpha(x) D^\alpha \tag{1.2}$$

the principal part of $P(x, \mathbb{D})$, and

$$P_s\left(x,\xi\right)=\sum_{(\alpha:\nu)=s}a_\alpha\left(x\right)\xi^\alpha \tag{1.3}$$

the symbol of $P_s\left(x,\mathbb{D}\right)$.

Definition 1.4. *The differential form $P\left(x,\mathbb{D}\right)$ is called semielliptic at point $x_0\in\mathbb{R}^n$, if the following is satisfied:*

$$P_s\left(x_0,\xi\right)\neq 0,\forall\xi\in\mathbb{R}^n,|\xi|\neq 0.$$

Definition 1.5. *The differential form $P\left(x,\mathbb{D}\right)$ is called semielliptic in $\mathbb{R}^n$, if $P\left(x,\mathbb{D}\right)$ is semielliptic at each point $x\in\mathbb{R}^n$.*

For $\xi\in\mathbb{R}^n$ denote by

$$|\xi|_\nu=\left(\sum_{i=1}^n\xi_i^{2\nu_i}\right)^{1/2}.$$

Definition 1.6. *The differential form $P\left(x,\mathbb{D}\right)$ is called uniformly semielliptic in $\mathbb{R}^n$, if there exists a constant $C>0$ such that:*

$$\left|P_s\left(x,\xi\right)\right|\geq C\left|\xi\right|_\nu^s,\forall x\in\mathbb{R}^n,\forall\xi\in\mathbb{R}^n.$$

For $k\in\mathbb{R},\nu\in\mathbb{N}^n$ denote by $H^{k,\nu}(\mathbb{R}^n)$ the space

$$H^{k,\nu}\left(\mathbb{R}^n\right):=\left\{u\in S^{'}:\widehat{u}-\text{function},\left\|u\right\|_{k,\nu}=\left(\int\left|\widehat{u}\left(\xi\right)\right|^2\left(1+|\xi|_\nu\right)^{2k}d\xi\right)^{\frac{1}{2}}<\infty\right\},$$

S' is the set of tempered distributions, $\widehat{u}$ is the Fourier transform of function u.

For $r\in\mathbb{Z}_+,\nu\in\mathbb{N}^n$ denote

$$C^{r,\nu}\left(\mathbb{R}^n\right):=\left\{a:D^\beta a(x)\in C(\mathbb{R}^n),\ \sup_{x\in\mathbb{R}^n}\left|D^\beta a(x)\right|<\infty,\forall\beta\in\mathbb{Z}_+^n\ s.t.\ (\beta:\nu)\leq r\right\},$$

$$Q:=\left\{g\in C\left(\mathbb{R}^n\right):g(x)>0,\forall x\in\mathbb{R}^n\right\},$$

$$Q^{r,\nu}:=\left\{g\in Q:D^\beta g(x)\in C(\mathbb{R}^n)\text{ and }\frac{1}{g(x)}\rightrightarrows 0,\max_{y,|x-y|\leq 1}\frac{|g(x)-g(y)|}{g(y)}\rightrightarrows 0,\right.$$
$$\left.\frac{|D^\beta g(x)|}{g(x)^{1+(\beta:\nu)}}\rightrightarrows 0\text{ when }|x|\to\infty,\forall\beta\in\mathbb{Z}_+^n,0<(\beta:\nu)\leq r\right\}.$$

Let $\nu_{max}=\max\limits_{1\leq i\leq n}\nu_i$. The examples of weight functions from $Q^{r,\nu}$ include polynomial functions as well as special exponential functions, for example:

$$\left(1+|x|_\nu\right)^l,l>0,\exp\left(1+|x|_\nu\right)^\sigma,0<\sigma<\frac{1}{\nu_{max}}.$$

For $k \in \mathbb{Z}_+, \nu \in \mathbb{N}^n$, $q \in Q$ and domain $\Omega \subset \mathbb{R}^n$ denote by $H_q^{k,\nu}(\mathbb{R}^n)$ and $H_q^{k,\nu}(\Omega)$ respectively the spaces of measurable functions $\{u\}$ with norms

$$\|u\|_{k,\nu,q} := \|u\|_{H_q^{k,\nu}(\mathbb{R}^n)} := \sum_{(\alpha:\nu)\leq k} \|D^\alpha u \cdot q^{k-(\alpha:\nu)}\|_{L_2(\mathbb{R}^n)} < \infty,$$
$$\|u\|_{H_q^{k,\nu}(\Omega)} := \sum_{(\alpha:\nu)\leq k} \|D^\alpha u \cdot q^{k-(\alpha:\nu)}\|_{L_2(\Omega)} < \infty.$$

Let $k \in \mathbb{N}, k \geq s, q \in Q$ and the coefficients of differential expression $P(x, \mathbb{D})$ of the form (1.1) satisfy the following conditions:

(1.4)
$$|D^\beta a_\alpha(x)| \leq C_{\alpha,\beta}\; q(x)^{s-(\alpha:\nu)+(\beta:\nu)} \quad \left(\forall \alpha, \beta \in \mathbb{Z}_+^n \; (\alpha:\nu) \leq s, (\beta:\nu) \leq k-s\right).$$

Then $P(x, \mathbb{D})$ generates a bounded linear operator, acting from $H_q^{k,\nu}(\mathbb{R}^n)$ to $H_q^{k-s,\nu}(\mathbb{R}^n)$.

In the paper [8] the fulfillment of special a priori estimate and the Fredholm property of semielliptic operators are studied in anisotropic Sobolev spaces. The following theorem is proved:

Theorem 1.1. *Let the differential form $P(x, \mathbb{D})$ with some constant $C > 0$ satisfies the following estimate:*

$$\|u\|_{k,\nu,q} \leq C \left(\|Pu\|_{k-s,\nu,q} + \|u\|_{L_2(\mathbb{R}^n)}\right) \;, \forall u \in H_q^{k,\nu}\left(\mathbb{R}^n\right). \tag{1.5}$$

Then $P\left(x, \mathbb{D}\right)$ is uniformly semielliptic in $\mathbb{R}^n$.

It is easy to check that in the case $q \equiv 1$ inverse statement is true with some smoothness conditions on the coefficients of the principal part of the differential form. In this paper it is proved that under the special conditions on the weight function and coefficients of the differential form $P(x, \mathbb{D})$ uniform semiellipticity in $\mathbb{R}^n$ does not imply the fulfillment of a priori estimate of the form (1.5) and stronger conditions are necessary for it. The results related to a priori estimates are further used to establish necessary conditions for the Fredholm property of the considered class of operators.

In this work necessary and sufficient conditions are obtained for the Fredholm property of semielliptic operators with special variable coefficients acting in anisotropic spaces $H_q^{k,\nu}(\mathbb{R}^n)$.

2. Main results

Let $k, s \in \mathbb{N}, k \geq s$. Consider the differential form

$$P(x, \mathbb{D}) = \sum_{(\alpha:\nu)\leq s} a_\alpha q(x)^{s-(\alpha:\nu)} D^\alpha, \tag{2.1}$$

where a_α – some constant numbers, $q \in Q^{k-s,\nu}$ and denotations from (1.1) are used.

For $N > 0$ and $x_0 \in \mathbb{R}^n$ denote

$$K_N(x_0) := \{x \in \mathbb{R}^n : |x - x_0| \leq N\}, \quad K_N := K_N(0).$$

Theorem 2.1. *Let $P(x, \mathbb{D})$ be the differential form (2.1) and $k \in \mathbb{N}, k \geq s, q \in Q^{k-s,\nu}$. Let the differential form $P(x, \mathbb{D})$ with some constant $\kappa > 0$ satisfies the following estimate:*

$$\|u\|_{k,\nu,q} \leq \kappa \left(\|Pu\|_{k-s,\nu,q} + \|u\|_{L_2(\mathbb{R}^n)}\right), \quad u \in H_q^{k,\nu}(\mathbb{R}^n). \tag{2.2}$$

Then there exists a constant $\delta > 0$ such that

$$\left| \sum_{(\alpha:\nu)\leq s} a_\alpha \lambda^{s-(\alpha:\nu)} \xi^\alpha \right| \geq \delta(\lambda + |\xi|_\nu)^s, \quad \xi \in \mathbb{R}^n, \lambda > 0.$$

Proof. Let $M > 0$, $x_M \in \mathbb{R}^n \backslash K_M$, $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp} \varphi \subset K_1(x_M)$, $\|\varphi\|_{L_2(\mathbb{R}^n)} = 1$ and $\xi \in \mathbb{R}^n$. Consider the function $\tilde{u}(x) = e^{i(q(x_M)^{\frac{1}{\nu}} \xi, x)} \varphi(x)$.

Since $\lim\limits_{|x|\to\infty} \max\limits_{|x-y|\leq 1} \frac{|q(x)-q(y)|}{q(y)} = 0$, then for any $r > 0$ the following inequality is fulfilled

$$|q(x)^r - q(x_M)^r| \leq \varepsilon_r(M) q(x_M)^r, \quad x \in K_1(x_M), \tag{2.3}$$

where $\varepsilon_r(M) \to 0$ when $M \to \infty$.

Using (2.3) and the fact that $\operatorname{supp} \widetilde{u} \subset K_1(x_M)$ it is easy to see that there exists a function $\varepsilon(M)$ such that $\varepsilon(M) \to 0$ when $M \to \infty$ and the following inequalities hold:

$$\|\tilde{u}\|_{k,\nu,q} \geq (1 - \varepsilon(M)) \|\tilde{u}\|_{k,\nu,q(x_M)}, \tag{2.4}$$

$$\|P\tilde{u}\|_{k-s,\nu,q} \leq (1 + \varepsilon(M)) \|P\tilde{u}\|_{k-s,\nu,q(x_M)}. \tag{2.5}$$

Taking into consideration the definition of function $\tilde{u}$ one can check that for any $\alpha \in \mathbb{Z}_+^n, (\alpha : \nu) \leq k$ with some constant $C_1 = C_1(\varphi) > 0$ the following holds

$$\|D^\alpha \tilde{u}\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-(\alpha:\nu)} \geq |\xi^\alpha| \|\varphi\|_{L_2(\mathbb{R}^n)} q(x_M)^k - C_1(1 + |\xi|_\nu)^k q(x_M)^{k - \frac{1}{\nu_{max}}}.$$

Using previous inequality and the fact that $\|\varphi\|_{L_2(\mathbb{R}^n)} = 1$ we get that with some constant $C_2 = C_2(\varphi) > 0$ the following holds

$$\|\tilde{u}\|_{k,\nu,q(x_M)} \geq \sum_{(\alpha:\nu)\leq k} |\xi^\alpha| \, q(x_M)^k - C_2(1 + |\xi|_\nu)^k q(x_M)^{k - \frac{1}{\nu_{max}}}. \tag{2.6}$$

For $\beta \in \mathbb{Z}_+^n, (\beta : \nu) \leq k - s$ with some constant $C_3 = C_3(P) > 0$ we have the following estimates

(2.7)
$$\left\|D^\beta(P(x,\mathbb{D})\tilde{u})\right\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-s-(\beta:\nu)} \le \left\|D^\beta(P(x_M,\mathbb{D})\tilde{u})\right\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-s-(\beta:\nu)}$$
$$+ \left\|D^\beta\left((P(x,\mathbb{D}) - P(x_M,\mathbb{D}))\tilde{u}\right)\right\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-s-(\beta:\nu)}$$
$$\le \left\| \sum_{(\alpha:\nu)\le s} a_\alpha q(x_M)^{s-(\alpha:\nu)} D^{\alpha+\beta}\tilde{u} \right\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-s-(\beta:\nu)}$$
$$+ C_3 \sum_{(\alpha:\nu)\le s} \left\|D^\beta\left((q(x)^{s-(\alpha:\nu)} - q(x_M)^{s-(\alpha:\nu)})D^\alpha\tilde{u}\right)\right\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-s-(\beta:\nu)}.$$

Taking into account $q \in Q^{k-s,\nu}$, inequality (2.3), the definition of function $\tilde{u}$ and the fact that $\operatorname{supp}\widetilde{u} \subset K_1(x_M)$, then for all $\alpha, \beta \in \mathbb{Z}_+$ such that $(\alpha:\nu) \le s, (\beta:\nu) \le k-s$ with some constants $C_4 > 0, C_5 = C_5(\varphi) > 0$ we get the following estimate

$$\left\|D^\beta\left(\left(q(x)^{s-(\alpha:\nu)} - q(x_M)^{s-(\alpha:\nu)}\right) D^\alpha\widetilde{u}\right)\right\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-s-(\beta:\nu)} \tag{2.8}$$
$$\le \left\|\left(q(x)^{s-(\alpha:\nu)} - q(x_M)^{s-(\alpha:\nu)}\right) D^{\beta+\alpha}\widetilde{u}\right\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-s-(\beta:\nu)}$$
$$+ C_4 \sum_{0\le\gamma<\beta} \left\|D^{\beta-\gamma}\left(q(x)^{s-(\alpha:\nu)}\right) D^{\gamma+\alpha}\widetilde{u}\right\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-s-(\beta:\nu)}$$
$$\le \tau(M)(1+|\xi|_\nu)^k q(x_M)^k + C_5(1+|\xi|_\nu)^k q(x_M)^{k-\frac{1}{\nu_{max}}},$$

where $\tau(M)$ is such a function that $\tau(M) \to 0$ when $M \to \infty$.

Similarly, using the definition of function $\tilde{u}$, with some constant $C_6 = C_6(P,\varphi) > 0$ we can get

$$\left\| \sum_{(\alpha:\nu)\le s} a_\alpha q(x_M)^{s-(\alpha:\nu)} D^{\alpha+\beta}\widetilde{u} \right\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-s-(\beta:\nu)} \tag{2.9}$$
$$\le \left| \sum_{(\alpha:\nu)\le s} a_\alpha \xi^\alpha \right| \left|\xi^\beta\right| q(x_M)^k + C_6(1+|\xi|_\nu)^k q(x_M)^{k-\frac{1}{\nu_{max}}}.$$

Then from (2.7), (2.8) and (2.9), with some constant $C_7 = C_7(P,\varphi) > 0$ we get

$$\left\|D^\beta(P(x,\mathbb{D})\tilde{u})\right\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-s-(\beta:\nu)} \tag{2.10}$$
$$\le \left| \sum_{(\alpha:\nu)\le s} a_\alpha \xi^\alpha \right| \left|\xi^\beta\right| q(x_M)^k + \omega(M)(1+|\xi|_\nu)^k q(x_M)^k + C_7(1+|\xi|_\nu)^k q(x_M)^{k-\frac{1}{\nu_{max}}},$$

where $\omega(M)$ is such a function that $\omega(M) \to 0$ when $M \to \infty$.

Therefore, with some constant $C_8 = C_8(P, \varphi) > 0$ the following holds

$$\begin{aligned} (2.11) \quad \|P\tilde{u}\|_{k-s,\nu,q(x_M)} \leq & \sum_{(\beta:\nu)\leq k-s} \left|\xi^\beta\right| \left| \sum_{(\alpha:\nu)\leq s} a_\alpha \xi^\alpha \right| q(x_M)^k \\ & + C_8(1+|\xi|_\nu)^k q(x_M)^{k-\frac{1}{\nu_{max}}} + \tilde{\omega}(M)(1+|\xi|_\nu)^k q(x_M)^k, \end{aligned}$$

where $\tilde{\omega}(M) \to 0$ when $M \to \infty$.

From (2.2), according to inequalities (2.4)–(2.6), (2.11) and the definition of the function $\tilde{u}$ we get

$$\begin{aligned} &(1-\varepsilon(M)) \left(\sum_{(\alpha:\nu)\leq k} |\xi^\alpha|\, q(x_M)^k - C_2(1+|\xi|_\nu)^k q(x_M)^{k-\frac{1}{\nu_{max}}} \right) \\ &\quad \leq \kappa \left((1+\varepsilon(M)) \left(\sum_{(\beta:\nu)\leq k-s} \left|\xi^\beta\right| \left| \sum_{(\alpha:\nu)\leq s} a_\alpha \xi^\alpha \right| q(x_M)^k \right.\right. \\ &\qquad \left.\left. + C_8(1+|\xi|_\nu)^k q(x_M)^{k-\frac{1}{\nu_{max}}} + \tilde{\omega}(M)(1+|\xi|_\nu)^k q(x_M)^k \right) + 1 \right). \end{aligned}$$

From the last inequality, according to the facts that $\frac{1}{q(x)} \rightrightarrows 0$ when $|x| \to \infty$ and $\varepsilon(M) \to 0, \tilde{\omega}(M) \to 0$ when $M \to \infty$, dividing by $(q(x_M))^k$ and tending $M \to \infty$ we get

$$\sum_{(\beta:\nu)\leq k} \left|\xi^\beta\right| \leq \kappa \sum_{(\beta:\nu)\leq k-s} \left|\xi^\beta\right| \left| \sum_{(\alpha:\nu)\leq s} a_\alpha \xi^\alpha \right|.$$

Since $k, s \in \mathbb{N}, k \geq s, \nu \in \mathbb{N}^n$, then there exist the constants $\delta_1, \delta_2 > 0$ such that

$$(2.12) \qquad \sum_{(\alpha:\nu)\leq k} |\xi^\alpha| \geq \delta_1(1+|\xi|_\nu)^k, \quad \sum_{(\alpha:\nu)\leq k-s} |\xi^\alpha| \leq \delta_2(1+|\xi|_\nu)^{k-s}, \quad \xi \in \mathbb{R}^n.$$

Then with some constant $\delta = \frac{\delta_1}{\kappa\delta_2} > 0$ we get

$$\left| \sum_{(\alpha:\nu)\leq s} a_\alpha \xi^\alpha \right| \geq \delta(1+|\xi|_\nu)^s, \quad \xi \in \mathbb{R}^n.$$

Let $\lambda > 0$. By substituting $\xi \in \mathbb{R}^n$ in the last inequality with $\frac{\xi}{\lambda^{\frac{1}{\nu}}} = \left(\frac{\xi_1}{\lambda^{\frac{1}{\nu_1}}}, \ldots, \frac{\xi_n}{\lambda^{\frac{1}{\nu_n}}} \right)$, it is easy to get the following estimate:

$$\left| \sum_{(\alpha:\nu)\leq s} a_\alpha \lambda^{s-(\alpha:\nu)} \xi^\alpha \right| \geq \delta(\lambda+|\xi|_\nu)^s, \quad \xi \in \mathbb{R}^n,\ \lambda > 0.$$

$\square$

Let $k, s \in \mathbb{N},\ k \geq s$. Consider the differential form $P(x, \mathbb{D})$ (see (1.1)), which is expressed in the following way:

$$P(x,\mathbb{D}) = \sum_{(\alpha:\nu)\leq s} a_\alpha(x) D^\alpha = \sum_{(\alpha:\nu)\leq s} \Big(a^0_\alpha(x) q(x)^{s-(\alpha:\nu)} + b_\alpha(x) \Big) D^\alpha, \tag{2.13}$$

where $a_\alpha(x) = a^0_\alpha(x) q(x)^{s-(\alpha:\nu)} + b_\alpha(x)$, $a^0_\alpha(x) \in C^{k-s,\nu}(\mathbb{R}^n), q \in Q^{k-s,\nu}$ and

$$D^\beta(b_\alpha(x)) = o(q(x)^{s-(\alpha:\nu)+(\beta:\nu)}),\ \text{when } |x| \to \infty, \quad (\alpha:\nu) \leq s,\ (\beta:\nu) \leq k-s.$$

Denote

$$L(x,\mathbb{D}) = \sum_{(\alpha:\nu)\leq s} b_\alpha(x) D^\alpha. \tag{2.14}$$

Theorem 2.2. *Let $k, s \in \mathbb{N}, k \geq s$, $q \in Q^{k-s,\nu}$ and $P(x, \mathbb{D})$ be the differential form (2.13) with the coefficients that satisfy $\lim\limits_{|x|\to\infty} \max\limits_{|x-y|\leq 1} |a^0_\alpha(x) - a^0_\alpha(y)| = 0$ for $\alpha \in \mathbb{Z}^n_+$, $(\alpha:\nu) \leq s$. Let there exists a constant $\kappa > 0$ such that:*

$$\|u\|_{k,\nu,q} \leq \kappa \left(\|Pu\|_{k-s,\nu,q} + \|u\|_{L_2(\mathbb{R}^n)} \right), \forall u \in H^{k,\nu}_q(\mathbb{R}^n). \tag{2.15}$$

Then there exist constants $\delta > 0$ and $M > 0$ such that

$$\left| \sum_{(\alpha:\nu)\leq s} a^0_\alpha(x) \lambda^{s-(\alpha:\nu)} \xi^\alpha \right| \geq \delta(\lambda + |\xi|_\nu)^s, \forall \xi \in \mathbb{R}^n, \lambda > 0, |x| \geq M.$$

Proof. Let $M > 0, x_M \in \mathbb{R}^n \backslash K_M, \varphi \in C^\infty_0(\mathbb{R}^n), \operatorname{supp} \varphi \subset K_1(x_M)$, $\|\varphi\|_{L_2(\mathbb{R}^n)} = 1$ and $\xi \in \mathbb{R}^n$. Consider the function $\tilde{u}(x) = e^{i(q(x_M)^{\frac{1}{\nu}} \xi, x)} \varphi(x)$.

Similar to the proof of Theorem 2.1 it is easy to check, that there exists a function $\varepsilon(M)$ such that $\varepsilon(M) \to 0$ when $M \to \infty$ and the following inequalities hold:

$$\|\tilde{u}\|_{k,\nu,q} \geq (1 - \varepsilon(M)) \|\tilde{u}\|_{k,\nu,q(x_M)}, \tag{2.16}$$

$$\|P\tilde{u}\|_{k-s,\nu,q} \leq (1 + \varepsilon(M)) \|P\tilde{u}\|_{k-s,\nu,q(x_M)}. \tag{2.17}$$

Taking into account the definition of the function $\tilde{u}$ one can check that with some constant $C_1 = C_1(\varphi) > 0$ the following holds:

$$\|\tilde{u}\|_{k,\nu,q(x_M)} \geq \sum_{(\alpha:\nu)\leq k} |\xi^\alpha|\, q(x_M)^k - C_1 (1 + |\xi|_\nu)^k q(x_M)^{k - \frac{1}{\nu_{max}}}. \tag{2.18}$$

For any $\beta \in \mathbb{Z}_+^n, (\beta:\nu) \leq k-s$

$$\left\|D^\beta\left(P(x,\mathbb{D})\widetilde{u}\right)\right\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-s-(\beta:\nu)} \leq \tag{2.19}$$

$$\leq \left\|\sum_{(\alpha:\nu)\leq s} a_\alpha^0(x_M)q(x_M)^{s-(\alpha:\nu)}D^{\alpha+\beta}\widetilde{u}\right\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-s-(\beta:\nu)}$$

$$+\sum_{(\alpha:\nu)\leq s}\left\|D^\beta\left(\left[a_\alpha^0(x)q(x)^{s-(\alpha:\nu)}-a_\alpha^0(x_M)q(x_M)^{s-(\alpha:\nu)}\right]D^\alpha\widetilde{u}\right)\right\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-s-(\beta:\nu)}$$

$$+\sum_{(\alpha:\nu)\leq s}\left\|D^\beta(b_\alpha(x)D^\alpha\widetilde{u})\right\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-s-(\beta:\nu)}.$$

Since $a_\alpha^0(x) \in C^{k-s,\nu}(\mathbb{R}^n)$, $\lim\limits_{|x|\to\infty}\max\limits_{|x-y|\leq 1}\left|a_\alpha^0(x)-a_\alpha^0(y)\right| = 0$ and $q \in Q^{k-s,\nu}$, it is easy to check that for $\beta \in \mathbb{Z}_+^n$, $0 < (\beta:\nu) \leq k-s$ there exist functions $\tilde{\varepsilon}_1(M), \tilde{\varepsilon}_2(M)$ such that $\tilde{\varepsilon}_1(M), \tilde{\varepsilon}_2(M) \to 0$ when $M \to \infty$ and some constant $C_2 = C_2(P) > 0$ that the following inequalities hold

$$\left|a_\alpha^0(x)q(x)^{s-(\alpha:\nu)} - a_\alpha^0(x_M)q(x_M)^{s-(\alpha:\nu)}\right| \tag{2.20}$$

$$\leq \left|a_\alpha^0(x)-a_\alpha^0(x_M)\right| q(x)^{s-(\alpha:\nu)} + \left|a_\alpha^0(x_M)\left(q(x)^{s-(\alpha:\nu)}-q(x_M)^{s-(\alpha:\nu)}\right)\right|$$

$$\leq \tilde{\varepsilon}_1(M)q(x_M)^{s-(\alpha:\nu)}, \forall x \in K_1(x_M),$$

$$\left|D^\beta\left(a_\alpha^0(x)q(x)^{s-(\alpha:\nu)}\right)\right| \leq \tilde{\varepsilon}_2(M)q(x_M)^{s-(\alpha:\nu)+(\beta:\nu)} \tag{2.21}$$

$$+ C_2 q(x_M)^{s-(\alpha:\nu)+(\beta:\nu)-\frac{1}{\nu_{max}}}, \forall x \in K_1(x_M).$$

Taking into account that $D^\beta(b_\alpha(x)) = o(q(x)^{s-(\alpha:\nu)+(\beta:\nu)})$ when $|x| \to \infty$ and the definition of function $\tilde{u}$, for multiindices $\alpha, \beta \in \mathbb{Z}_+^n$ such that $(\alpha:\nu) \leq s, (\beta:\nu) \leq k-s$ with some constants $C_3 > 0, C_4 = C_4(P,\varphi) > 0$ we get the following estimate

$$\left\|D^\beta\left(b_\alpha(x)D^\alpha\widetilde{u}\right)\right\|_{L_2(\mathbb{R}^n)} \leq C_3\sum_{0\leq\gamma\leq\beta}\left\|D^\gamma(b_\alpha(x))D^{\beta-\gamma+\alpha}\widetilde{u}\right\|_{L_2(\mathbb{R}^n)} \tag{2.22}$$

$$\leq \delta(M)(1+|\xi|_\nu)^{(\alpha:\nu)+(\beta:\nu)}q(x_M)^{s+(\beta:\nu)} + C_4(1+|\xi|_\nu)^{(\alpha:\nu)+(\beta:\nu)}q(x_M)^{s+(\beta:\nu)-\frac{1}{\nu_{max}}},$$

where $\delta(M)$ is such a function that $\delta(M) \to 0$ when $M \to \infty$. Then from the estimates (2.19)–(2.22) with some constant $C_5 = C_5(\varphi, P) > 0$ we get

$$\|P\tilde{u}\|_{k-s,\nu,q(x_M)} \leq \sum_{(\beta:\nu)\leq k-s}\left|\xi^\beta\right|\left|\sum_{(\alpha:\nu)\leq s}a_\alpha^0(x_M)\xi^\alpha\right| q(x_M)^k \tag{2.23}$$

$$+ C_5(1+|\xi|_\nu)^k q(x_M)^{k-\frac{1}{\nu_{max}}} + \tilde{\omega}(M)(1+|\xi|_\nu)^k q(x_M)^k,$$

where $\tilde{\omega}(M)$ is such a function that $\tilde{\omega}(M) \to 0$ when $M \to \infty$.

From the estimate (2.15), according to (2.16)–(2.18), (2.23) and the definition of the function $\tilde{u}$, we get

$$(1-\varepsilon(M))\left(\sum_{(\beta:\nu)\le k}\left|\xi^\beta\right|q(x_M)^k - C_1(1+|\xi|_\nu)^k\right)q(x_M)^{k-\frac{1}{\nu_{max}}}$$
$$\le \kappa\left((1+\varepsilon(M))\left(\sum_{(\beta:\nu)\le k-s}\left|\xi^\beta\right|\left|\sum_{(\alpha:\nu)\le s}a^0(x_M)\xi^\alpha\right|q(x_M)^k\right.\right.$$
$$\left.\left.+C_5(1+|\xi|_\nu)^k q(x_M)^{k-\frac{1}{\nu_{max}}}+\tilde{\omega}(M)(1+|\xi|_\nu)^k q(x_M)^k\right)+1\right).$$

From the last inequality, taking into account that $\frac{1}{q(x)}\rightrightarrows 0$ when $|x|\to\infty$, $\tilde{\omega}(M)\to 0, \varepsilon(M)\to 0$ when $M\to\infty$, dividing by $(q(x_M))^k$, we get

$$\sum_{(\beta:\nu)\le k}\left|\xi^\beta\right| - \tau(M)(1+|\xi|_\nu)^k \le \kappa\sum_{(\beta:\nu)\le k-s}\left|\xi^\beta\right|\left|\sum_{(\alpha:\nu)\le s}a^0_\alpha(x_M)\xi^\alpha\right|,$$

where $\tau(M)$ is such a function that $\tau(M)\to 0$ when $M\to\infty$.

From the last estimate, using inequalities (2.12), we get

$$\left|\sum_{(\alpha:\nu)\le s}a^0_\alpha(x_M)\xi^\alpha\right| \ge \frac{\delta_1}{\kappa\delta_2}(1+|\xi|_\nu)^s - \frac{\tau(M)}{\kappa\delta_2}(1+|\xi|_\nu)^s. \tag{2.24}$$

Since $\tau(M)\to 0$ when $M\to\infty$, then there exists $M_0=M_0(P,\varphi,\delta_1,\delta_2,\kappa)>0$ such that for any $M\ge M_0$ with some constant $\delta=\delta(\kappa,\delta_1,\delta_2)>0$ the following is true

$$\left|\sum_{(\alpha:\nu)\le s}a^0_\alpha(x)\xi^\alpha\right| \ge \delta(1+|\xi|_\nu)^s, \forall\xi\in\mathbb{R}^n, |x|\ge M.$$

Similarly to the proof of Theorem 2.1 from the last inequality it is easy to get the following

$$\left|\sum_{(\alpha:\nu)\le s}a^0_\alpha(x)\lambda^{s-(\alpha:\nu)}\xi^\alpha\right| \ge \delta(\lambda+|\xi|_\nu)^s, \forall\xi\in\mathbb{R}^n, \lambda>0, |x|\ge M.$$

□

Theorem 2.3. (see [14], theorem 7.1). *Let E, F and E_0 be Banach spaces such that E is compactly embedded in E_0. Let A be a bounded linear operator acting from E to F. Operator $A: E\to F$ is an $n-$normal if and only if there exists a constant $C>0$ such that*

$$\|x\|_E \le C\left(\|Ax\|_F + \|x\|_{E_0}\right), \quad x\in E.$$

Applying the previous theorem for operator $P(x,\mathbb{D})$, acting from $H^{k,\nu}_q(\mathbb{R}^n)$ to $H^{k-s,\nu}_q(\mathbb{R}^n)$, we get

Theorem 2.4. *Let $P(x,\mathbb{D})$ be differential form* (1.1)*. Then operator $P(x,\mathbb{D})$, acting from $H_q^{k,\nu}(\mathbb{R}^n)$ to $H_q^{k-s,\nu}(\mathbb{R}^n)$, is an $n-$normal if and only if there exist constants $\kappa>0$ and $R>0$ such that the following holds*

$$\|u\|_{k,\nu,q} \leq \kappa \left(\|Pu\|_{k-s,\nu,q} + \|u\|_{L_2(K_R)}\right), \quad u \in H_q^{k,\nu}(\mathbb{R}^n).$$

Corollary 2.1. *Let $k,s\in\mathbb{N}, k\geq s$, $q\in Q^{k-s,\nu}$ and $P(x,\mathbb{D})$ be the differential form* (2.13) *with the coefficients that satisfy $\lim\limits_{|x|\to\infty}\max\limits_{|x-y|\leq 1}|a_\alpha^0(x)-a_\alpha^0(y)|=0$ for $\alpha\in\mathbb{Z}_+^n$, $(\alpha:\nu)\leq s$. Let operator $P(x,\mathbb{D})$, acting from $H_q^{k,\nu}(\mathbb{R}^n)$ to $H_q^{k-s,\nu}(\mathbb{R}^n)$, be a Fredholm operator. Then there exist constants $\delta>0$ and $M>0$ such that*

$$\left|\sum_{(\alpha:\nu)\leq s} a_\alpha^0(x)\lambda^{s-(\alpha:\nu)}\xi^\alpha\right| \geq \delta(\lambda+|\xi|_\nu)^s, \quad \xi\in\mathbb{R}^n,\ \lambda>0,\ |x|\geq M.$$

Proof. Since operator $P(x,\mathbb{D})$, acting from $H_q^{k,\nu}(\mathbb{R}^n)$ to $H_q^{k-s,\nu}(\mathbb{R}^n)$, is a Fredholm operator, then it is an $n-$normal operator. From Theorem 2.4 we get that there exist such constants $\kappa>0$ and $R>0$ that the following estimate holds

$$\|u\|_{k,\nu,q} \leq \kappa \left(\|Pu\|_{k-s,\nu,q} + \|u\|_{L_2(K_R)}\right) \leq \kappa \left(\|Pu\|_{k-s,\nu,q} + \|u\|_{L_2(\mathbb{R}^n)}\right),\ u \in H_q^{k,\nu}(\mathbb{R}^n).$$

From last estimate and the conditions on the coefficients of $P(x,\mathbb{D})$ using Theorem 2.2 we obtain that there exist constants $\delta>0$ and $M>0$ such that

$$\left|\sum_{(\alpha:\nu)\leq s} a_\alpha^0(x)\lambda^{s-(\alpha:\nu)}\xi^\alpha\right| \geq \delta(\lambda+|\xi|_\nu)^s, \quad \xi\in\mathbb{R}^n,\ \lambda>0,\ |x|\geq M.$$

□

Theorem 8.5.14 from [12] can be formulated in the following equivalent way:

Theorem 2.5. *Let A be a bounded linear operator acting from a Banach space X to a Banach space Y. Then the following holds:*

(1) *if operator A has left regularizer, then kernel of operator A in X is finite dimensional;*
(2) *if operator A has right regularizer, then the image of operator A is closed in Y and cokernel is finite dimensional;*
(3) *operator A has left and right regularizers if and only if A is a Fredholm operator.*

It is easy to check that the following proposition holds:

Proposition 2.1. *Let $k,s\in\mathbb{N},k\geq s,q\in Q^{k-s,\nu}$, $P(x,\mathbb{D})$ be the differential expression of the form* (1.1) *with the coefficients that satisfy conditions* (1.4) *and $\varphi\in C_0^\infty(\mathbb{R}^n)$. Then operator*

$$Tu := P(u\varphi) - \varphi Pu, \quad u \in H_q^{k,\nu}(\mathbb{R}^n)$$

is a compact operator acting from $H_q^{k,\nu}(\mathbb{R}^n)$ *to* $H_q^{k-s,\nu}(\mathbb{R}^n)$.

Theorem 2.6. *Let* $k, s \in \mathbb{N}, k \geq s, q \in Q^{k-s,\nu}$ *and the differential form* $P(x, \mathbb{D})$ *(see (2.13)) be semielliptic in* $\mathbb{R}^n$ *with the coefficients that satisfy*

$$\lim_{|x|\to\infty} \max_{|x-y|\leq 1} |a_\alpha^0(x) - a_\alpha^0(y)| = 0, \quad \alpha \in \mathbb{Z}_+^n, (\alpha:\nu) \leq s.$$

Then the operator $P(x, \mathbb{D}) : H_q^{k,\nu}(\mathbb{R}^n) \to H_q^{k-s,\nu}(\mathbb{R}^n)$ *is a Fredholm operator if and only if there exist constants* $\delta > 0$ *and* $M > 0$ *such that*

$$\left| \sum_{(\alpha:\nu)\leq s} a_\alpha^0(x) \lambda^{s-(\alpha:\nu)} \xi^\alpha \right| \geq \delta(\lambda + |\xi|_\nu)^s, \quad \xi \in \mathbb{R}^n,\ \lambda > 0,\ |x| \geq M. \tag{2.25}$$

Proof. Let's first prove sufficient part.

Let $\delta_0 > 0, \varphi(x) \in C_0^\infty(\mathbb{R}^n)$ be such that $0 \leq \varphi(x) \leq 1$ for all $x \in \mathbb{R}^n$ and $\varphi(x) = 1$ for $x \in K_{\frac{\delta_0}{2}}$, $\varphi(x) = 0$ for $|x| \geq \delta_0$ and $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\psi \subset K_{2\delta_0}$ and $\psi(x) = 1$ for $x \in K_{\delta_0}$. Let $\omega > 0$ be such that $\omega\sqrt{n} < \delta_0$. Let's denote $\{z_m\}_{m=0}^\infty$ points on the lattice in $\mathbb{R}^n$ with a side equals to ω.

Denote

$$\varphi_m(x) := \varphi(x - z_m) \left(\sum_{l=0}^\infty \varphi(x - z_l) \right)^{-1}, \quad \psi_m(x) := \psi(x - z_m), \quad m \in \mathbb{Z}_+.$$

Then $\{\varphi_m\}_{m=0}^\infty$ is a partition of unity that satisfies the following condition:

(i) $\max_{x,y\in\operatorname{supp}\varphi_m} |x - y| < \delta_0$,

(ii) there exists $r \in \mathbb{N}$ such that for any number i there are no more than r functions $\varphi_j(x)$ such that $\operatorname{supp}\varphi_i \cap \operatorname{supp}\varphi_j \neq \varnothing$;

(iii) for any $\alpha \in \mathbb{Z}_+^n$ there exists some constant $C_\alpha > 0$ such that $|D^\alpha \varphi_m(x)| \leq C_\alpha$, ny $x \in \mathbb{R}^n,\ m \in \mathbb{Z}_+$.

Denote $W_m = \operatorname{supp}\varphi_m,\ m \in \mathbb{Z}_+$. Let $x_m \in W_m$ and $m_0 \in \mathbb{N}$. For $m \leq m_0$ denote

$$P^m(x, \mathbb{D}) := \sum_{(\alpha:\nu)\leq s} \left(\psi_m(x) \left(a_\alpha(x) - a_\alpha(x_m)\right) + a_\alpha(x_m)\right) D^\alpha.$$

For $m > m_0$ denote

$$P^m(x, \mathbb{D}) :=$$
$$\sum_{(\alpha:\nu)\leq s} \left(\psi_m(x) \left(a_\alpha^0(x)q(x)^{s-(\alpha:\nu)} - a_\alpha^0(x_m)q(x_m)^{s-(\alpha:\nu)}\right) + a_\alpha^0(x_m)q(x_m)^{s-(\alpha:\nu)}\right) D^\alpha.$$

Since $q \in Q^{k-s,\nu}$ and $\lim_{m\to\infty} \max_{|x-x_m|\leq 1} |a_\alpha^0(x) - a_\alpha^0(x_m)| = 0$, according to Theorem 2.2 from [7], we can choose m_0 big enough such that for $m > m_0$ operator $P^m : H_q^{k,\nu}(\mathbb{R}^n) \to H_q^{k-s,\nu}(\mathbb{R}^n)$ has the inverse operator $R^m : H_q^{k-s,\nu}(\mathbb{R}^n) \to H_q^{k,\nu}(\mathbb{R}^n)$.

For $m \le m_0$ consider

$$R_0^m := F^{-1} \frac{|\xi|_\nu^s}{(1+|\xi|_\nu^s) P_s^m(x_m,\xi)} F.$$

Since $P(x,\mathbb{D})$ is semielliptic in $\mathbb{R}^n$, then using Lemma 4.3 from work [13] we get that for a small enough δ_0 from condition (i) the following holds

(2.26)
$$R_0^m P^m(x,\mathbb{D}) = R_0^m P^m(x_m,\mathbb{D}) + R_0^m \left(P^m(x,\mathbb{D}) - P^m(x_m,\mathbb{D})\right) = I + T_1^m + T_2^m,$$

where $T_1^m : H^{k,\nu}(\mathbb{R}^n) \to H^{k+\sigma,\nu}(\mathbb{R}^n)$ with some number $\sigma = \sigma(\nu) > 0$ and operator $T_2^m : H^{k,\nu}(\mathbb{R}^n) \to H^{k,\nu}(\mathbb{R}^n)$ satisfies $\|T_2^m\| < 1$.

For $m \le m_0$ let $R^m := (I + T_2^m)^{-1} R_0^m$. From (2.26) we have

$$R^m P^m(x,\mathbb{D}) = I + T^m, \tag{2.27}$$

where $T^m : H^{k,\nu}(\mathbb{R}^n) \to H^{k+\sigma,\nu}(\mathbb{R}^n)$ with some number $\sigma = \sigma(\nu) > 0$. Denote

$$Rf := \sum_{l=0}^{\infty} \psi_l R^l(\varphi_l f),\, f \in H_q^{k-s,\nu}(\mathbb{R}^n).$$

Since (2.25) holds one can check that the norms of operators R^l, acting from $H_q^{k-s,\nu}(\mathbb{R}^n)$ to $H_q^{k,\nu}(\mathbb{R}^n)$, are uniformly bounded. From this fact, taking into account that $\frac{1}{q(x)} \rightrightarrows 0$ when $|x| \to \infty$ and properties (i)–(iii) of the functions $\{\varphi_m\}_{m=0}^\infty, \{\psi_m\}_{m=0}^\infty$, it is easy to check that R is a bounded linear operator, acting from $H_q^{k-s,\nu}(\mathbb{R}^n)$ to $H_q^{k,\nu}(\mathbb{R}^n)$.

For $P(x,\mathbb{D})$ and $RP(x,\mathbb{D})$, taking into account (2.13), (2.14) and definitions of the functions $\{\varphi_m\}_{m=0}^\infty, \{\psi_m\}_{m=0}^\infty$, we have the following representations

$$P(x,\mathbb{D})u = \sum_{m=0}^{\infty} \varphi_m P(x,\mathbb{D})(\psi_m u)$$
$$= \sum_{m=0}^{m_0} \varphi_m P^m(x,\mathbb{D})(\psi_m u) + \sum_{m=m_0+1}^{\infty} \varphi_m P^m(x,\mathbb{D})(\psi_m u) + \sum_{m=m_0+1}^{\infty} \varphi_m L(x,\mathbb{D})(\psi_m u),$$

(2.28)
$$RP(x,\mathbb{D})u = \sum_{l=0}^{m_0} \sum_{m=0}^{m_0} \psi_l R^l \left(\varphi_l \varphi_m P^m(\psi_m u)\right) + \sum_{l=0}^{m_0} \sum_{m=m_0+1}^{\infty} \psi_l R^l \left(\varphi_l \varphi_m P^m(\psi_m u)\right)$$
$$+ \sum_{l=m_0+1}^{\infty} \sum_{m=0}^{m_0} \psi_l R^l \left(\varphi_l \varphi_m P^m(\psi_m u)\right) + \sum_{l=m_0+1}^{\infty} \sum_{m=m_0+1}^{\infty} \psi_l R^l \left(\varphi_l \varphi_m P^m(\psi_m u)\right)$$
$$+ \sum_{l=0}^{m_0} \sum_{m=m_0+1}^{\infty} \psi_l R^l \left(\varphi_l \varphi_m L(\psi_m u)\right) + \sum_{l=m_0+1}^{\infty} \sum_{m=m_0+1}^{\infty} \psi_l R^l \left(\varphi_l \varphi_m L(\psi_m u)\right),$$

where $u \in H_q^{k,\nu}(\mathbb{R}^n)$.

For $m, l \in \mathbb{Z}_+$ such that $l \le m_0$ and $m \le m_0$, based on the definitions of $P^m(x, \mathbb{D})$ and the functions $\{\varphi_m\}_{m=0}^{\infty}$, $\{\psi_m\}_{m=0}^{\infty}$, the following holds:

$$\varphi_l \varphi_m P^m(x, \mathbb{D})\,(\psi_m u) = \varphi_l \varphi_m P(x, \mathbb{D})\,(\psi_m u) = \varphi_l \varphi_m P^l(x, \mathbb{D})\,(\psi_m u)\,.$$

From the last equality, using (2.27) and the fact that $\varphi_m(x)\psi_m(x) = \varphi_m(x)$ for all $x \in \mathbb{R}^n$ and $m \in \mathbb{Z}_+$, we get

$$\begin{aligned}
&\sum_{l=0}^{m_0}\sum_{m=0}^{m_0} \psi_l R^l\left(\varphi_l\varphi_m P^m(\psi_m u)\right) = \sum_{l=0}^{m_0}\sum_{m=0}^{m_0} \psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m u)\right)\\
&= \sum_{l=0}^{m_0}\sum_{m=0}^{m_0} \psi_l R^l P^l(\varphi_l\varphi_m\psi_m u) + \sum_{l=0}^{m_0}\sum_{m=0}^{m_0} \psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m u) - P^l(\varphi_l\varphi_m\psi_m u)\right)\\
&\qquad = \sum_{l=0}^{m_0}\sum_{m=0}^{m_0} \varphi_l\varphi_m u + \sum_{l=0}^{m_0}\sum_{m=0}^{m_0} \psi_l T^l(\varphi_l\varphi_m u)\\
&\qquad\qquad + \sum_{l=0}^{m_0}\sum_{m=0}^{m_0} \psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m u) - P^l(\varphi_l\varphi_m\psi_m u)\right),
\end{aligned}$$

where $u \in H_q^{k,\nu}(\mathbb{R}^n)$. Consider

$$T_1 := \sum_{l=0}^{m_0}\sum_{m=0}^{m_0} \psi_l T^l(\varphi_l\varphi_m\cdot) + \sum_{l=0}^{m_0}\sum_{m=0}^{m_0} \psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m\cdot) - P^l(\varphi_l\varphi_m\psi_m\cdot)\right).$$

Using Proposition 2.1 we get that $\psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m\cdot) - P^l(\varphi_l\varphi_m\psi_m\cdot)\right)$, acting from $H_q^{k,\nu}(\mathbb{R}^n)$ to $H_q^{k,\nu}(\mathbb{R}^n)$, is a compact operator. Similarly, since $T^l : H^{k,\nu}(\mathbb{R}^n) \to H^{k+\sigma,\nu}(\mathbb{R}^n)$ with some $\sigma > 0$, it is easy to check that operator $\psi_l T^l(\varphi_l\varphi_m\cdot)$, acting from $H_q^{k,\nu}(\mathbb{R}^n)$ to $H_q^{k,\nu}(\mathbb{R}^n)$ is a compact operator. As the finite sum of compact operators T_1 is a compact operator, acting from $H_q^{k,\nu}(\mathbb{R}^n)$ to $H_q^{k,\nu}(\mathbb{R}^n)$. So we get

$$\sum_{l=0}^{m_0}\sum_{m=0}^{m_0} \psi_l R^l\left(\varphi_l\varphi_m P^m(\psi_m u)\right) = \sum_{l=0}^{m_0}\sum_{m=0}^{m_0} \varphi_l\varphi_m u + T_1 u, u \in H_q^{k,\nu}(\mathbb{R}^n),$$

where $T_1 : H_q^{k,\nu}(\mathbb{R}^n) \to H_q^{k,\nu}(\mathbb{R}^n)$ is a compact operator.

For $m, l \in \mathbb{Z}_+$ such that $l \le m_0$ and $m > m_0$, based on the definitions of $P^m(x, \mathbb{D})$, $L(x, \mathbb{D})$ and the functions $\{\varphi_m\}_{m=0}^{\infty}$, $\{\psi_m\}_{m=0}^{\infty}$, the following holds:

$$\begin{aligned}
\varphi_l\varphi_m P^m(x, \mathbb{D})\,(\psi_m u) &= \varphi_l\varphi_m\left(P(x, \mathbb{D}) - L(x, \mathbb{D})\right)(\psi_m u)\\
&= \varphi_l\varphi_m P^l(x, \mathbb{D})\,(\psi_m u) - \varphi_l\varphi_m L(x, \mathbb{D})\,(\psi_m u)\,.
\end{aligned}$$

From the last equality we get

$$\begin{aligned}
\sum_{l=0}^{m_0}\sum_{m=m_0+1}^{\infty} \psi_l R^l\left(\varphi_l\varphi_m P^m(\psi_m u)\right) &= \sum_{l=0}^{m_0}\sum_{m=m_0+1}^{\infty} \psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m u)\right)\\
&- \sum_{l=0}^{m_0}\sum_{m=m_0+1}^{\infty} \psi_l R^l\left(\varphi_l\varphi_m L(x, \mathbb{D})\,(\psi_m u)\right), u \in H_q^{k,\nu}(\mathbb{R}^n).
\end{aligned} \tag{2.29}$$

Now consider

$$\sum_{l=0}^{m_0}\sum_{m=m_0+1}^{\infty}\psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m\cdot)\right).$$

Using (2.27) and the properties of the functions $\{\varphi_m\}_{m=0}^{\infty}$, $\{\psi_m\}_{m=0}^{\infty}$ we can check that the following holds:

$$\sum_{l=0}^{m_0}\sum_{m=m_0+1}^{\infty}\psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m u)\right) = \sum_{l=0}^{m_0}\sum_{m=m_0+1}^{\infty}\varphi_l\varphi_m u + T_2 u, u\in H_q^{k,\nu}(\mathbb{R}^n)$$

where

$$T_2 := \sum_{l=0}^{m_0}\sum_{m=m_0+1}^{\infty}\psi_l T^l(\varphi_l\varphi_m\cdot) + \sum_{l=0}^{m_0}\sum_{m=m_0+1}^{\infty}\psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m\cdot) - P^l(\varphi_l\varphi_m\psi_m\cdot)\right)$$

$$= \sum_{l=0}^{m_0}\sum_{m=m_0+1}^{m_1}\psi_l T^l(\varphi_l\varphi_m\cdot) + \sum_{l=0}^{m_0}\sum_{m=m_0+1}^{m_1}\psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m\cdot) - P^l(\varphi_l\varphi_m\psi_m\cdot)\right),$$

where

$$m_1 := \max_{m>m_0}\{m : \operatorname{supp}\varphi_m \bigcap \left(\bigcup_{l=0}^{m_0}\operatorname{supp}\varphi_l\right) \neq \varnothing\}.$$

Since T_2 contains the finite number of terms for which $\varphi_l\varphi_m \neq 0$, similarly as for operator T_1, we can show that T_2 is a compact operator, acting from $H_q^{k,\nu}(\mathbb{R}^n)$ to $H_q^{k,\nu}(\mathbb{R}^n)$.

For $m, l \in \mathbb{Z}_+$ such that $l > m_0$ and $m \le m_0$, based on the definitions of $P^m(x,\mathbb{D})$, $L(x,\mathbb{D})$ and the functions $\{\varphi_m\}_{m=0}^{\infty}$, $\{\psi_m\}_{m=0}^{\infty}$, the following holds:

$$\varphi_l\varphi_m P^m(x,\mathbb{D})\,(\psi_m u) = \varphi_l\varphi_m P^l(x,\mathbb{D})\,(\psi_m u) + \varphi_l\varphi_m L(x,\mathbb{D})\,(\psi_m u)\,.$$

Analogously, from the last equality and the fact that for $l > m_0$ operators $R^l : H_q^{k-s,\nu}(\mathbb{R}^n) \to H_q^{k,\nu}(\mathbb{R}^n)$ are the inverse operators of $P^l : H_q^{k,\nu}(\mathbb{R}^n) \to H_q^{k-s,\nu}(\mathbb{R}^n)$ we get

$$\sum_{l=m_0+1}^{\infty}\sum_{m=0}^{m_0}\psi_l R^l\left(\varphi_l\varphi_m P^m(\psi_m u)\right) = \sum_{l=m_0+1}^{\infty}\sum_{m=0}^{m_0}\psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m u)\right) + \sum_{l=m_0+1}^{\infty}\sum_{m=0}^{m_0}\psi_l R^l\left(\varphi_l\varphi_m L(x,\mathbb{D})(\psi_m u)\right), \tag{2.30}$$

$$\sum_{l=m_0+1}^{\infty}\sum_{m=0}^{m_0}\psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m u)\right) = \sum_{l=m_0+1}^{\infty}\sum_{m=0}^{m_0}\varphi_l\varphi_m u + T_3 u,\ u\in H_q^{k,\nu}(\mathbb{R}^n) \tag{2.31}$$

where

$$T_3 := \sum_{l=m_0+1}^{\infty}\sum_{m=0}^{m_0}\psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m\cdot) - P^l(\varphi_l\varphi_m\psi_m\cdot)\right) =$$

$$\sum_{l=m_0+1}^{m_1}\sum_{m=0}^{m_0}\psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m\cdot) - P^l(\varphi_l\varphi_m\psi_m\cdot)\right),$$

$$m_1 := \max_{l>m_0}\{l : \operatorname{supp}\varphi_l \cap \left(\bigcup_{j=0}^{m_0} \operatorname{supp}\varphi_j\right) \neq \varnothing\}.$$

As T_3 contains the finite number of terms for which $\varphi_l\varphi_m \neq 0$, taking into account Proposition 2.1, we get that operator T_3 is a compact operator, acting from $H_q^{k,\nu}(\mathbb{R}^n)$ to $H_q^{k,\nu}(\mathbb{R}^n)$.

For $l > m_0$ and $m > m_0$, based on the definitions of $P^m(x,\mathbb{D})$ and the functions $\{\varphi_m\}_{m=0}^{\infty}$, $\{\psi_m\}_{m=0}^{\infty}$, we have:

$$\varphi_l\varphi_m P^m(x,\mathbb{D})\,(\psi_m u) = \varphi_l\varphi_m P^l(x,\mathbb{D})\,(\psi_m u)\,.$$

From the last equality and the fact that for $m > m_0$ operators $R^m : H_q^{k-s,\nu}(\mathbb{R}^n) \to H_q^{k,\nu}(\mathbb{R}^n)$ are the inverse operators of $P^m : H_q^{k,\nu}(\mathbb{R}^n) \to H_q^{k-s,\nu}(\mathbb{R}^n)$ we get

$$\sum_{l=m_0+1}^{\infty}\sum_{m=m_0+1}^{\infty}\psi_l R^l\left(\varphi_l\varphi_m P^m(\psi_m u)\right) = \sum_{l=m_0+1}^{\infty}\sum_{m=m_0+1}^{\infty}\psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m u)\right)$$
$$= \sum_{l=m_0+1}^{\infty}\sum_{m=m_0+1}^{\infty}\varphi_l\varphi_m u + \sum_{l=m_0+1}^{\infty}\sum_{m=m_0+1}^{\infty}\psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m u) - P^l(\varphi_l\varphi_m\psi_m u)\right),$$

where $u \in H_q^{k,\nu}(\mathbb{R}^n)$.

Taking into account (2.13), the definitions of $P^l(x,\mathbb{D})$ and the properties of functions $\{\varphi_m\}_{m=0}^{\infty}, \{\psi_m\}_{m=0}^{\infty}$, for $l > m_0$ and $m > m_0$ with some constant $C_1 > 0$ we get

$$\|\varphi_l\varphi_m P^l(\psi_m u) - P^l(\varphi_l\varphi_m\psi_m u)\|_{k-s,\nu,q}$$
$$\leq C_1\left\|\sum_{(\alpha:\nu)\leq s}\ \sum_{\beta+\gamma=\alpha,|\gamma|>0} a_\alpha^0(x)D^\beta(\psi_m u)D^\gamma(\varphi_l\varphi_m)q(x)^{s-(\alpha:\nu)}\right\|_{k-s,\nu,q}$$
$$\leq C_1\left\|\sum_{(\alpha:\nu)\leq s}\ \sum_{\beta+\gamma=\alpha,|\gamma|>0} a_\alpha^0(x)D^\gamma(\varphi_l\varphi_m)\frac{1}{q(x)^{(\gamma:\nu)}}D^\beta(\psi_m u)q(x)^{s-(\beta:\nu)}\right\|_{k-s,\nu,q}.$$

From the last inequality, taking into account that $\frac{1}{q(x)} \rightrightarrows 0$ when $|x| \to \infty$, properties (i)–(iii) of the functions $\{\varphi_m\}_{m=0}^{\infty}, \{\psi_m\}_{m=0}^{\infty}$ and the conditions on the coefficients $\{a_\alpha^0(x)\}$ (see (2.13)) we get

$$\|\varphi_l\varphi_m P^l(\psi_m u) - P^l(\varphi_l\varphi_m\psi_m u)\|_{k-s,\nu,q} \leq \omega(m_0)\|u\|_{H_q^{k,\nu}(W_l\cap W_m)}. \tag{2.32}$$

where $\omega(m_0)$ is such a function that $\omega(m_0) \to 0$ when $m_0 \to \infty$.

Since (2.25) holds the norms of operators R^l, acting from $H_q^{k-s,\nu}(\mathbb{R}^n)$ to $H_q^{k,\nu}(\mathbb{R}^n)$, are uniformly bounded. Using this fact, inequality (2.32), the properties (i)–(iii) of the functions $\{\varphi_m\}_{m=0}^{\infty}, \{\psi_m\}_{m=0}^{\infty}$, it is easy to check that for a big enough m_0

operator

$$T_4 := \sum_{l=m_0+1}^{\infty} \sum_{m=m_0+1}^{\infty} \psi_l R^l \left[\varphi_l \varphi_m P^l(\psi_m \cdot) - P^l(\varphi_l \varphi_m \psi_m \cdot)\right],$$

acting from $H_q^{k,\nu}(\mathbb{R}^n)$ to $H_q^{k,\nu}(\mathbb{R}^n)$, satisfies $\|T_4\| < \frac{1}{2}$.

Similarly for remained terms from (2.28), (2.29) and (2.30), taking into account that $D^\beta(b_\alpha(x)) = o(q(x)^{s-(\alpha:\nu)+(\beta:\nu)})$ when $|x| \to \infty$, $(\alpha:\nu) \le s, (\beta:\nu) \le k-s$ (see (2.13), (2.14)), for a big enough m_0 we get that the operator

$$T_5 := \sum_{l=0}^{\infty} \sum_{m=m_0+1}^{\infty} \psi_l R^l \left(\varphi_l \varphi_m L(\psi_m \cdot)\right) - \sum_{l=0}^{m_0} \sum_{m=m_0+1}^{\infty} \psi_l R^l \left(\varphi_l \varphi_m L(\psi_m \cdot)\right) + \sum_{l=m_0+1}^{\infty} \sum_{m=0}^{m_0} \psi_l R^l \left(\varphi_l \varphi_m L(\psi_m \cdot)\right),$$

acting from $H_q^{k,\nu}(\mathbb{R}^n)$ to $H_q^{k,\nu}(\mathbb{R}^n)$, has a norm that satisfies $\|T_5\| < \frac{1}{2}$.

Denote

$$T^{'} := T_1 + T_2 + T_3, T^{''} := T_4 + T_5.$$

From the representation (2.28) we get

$$RPu = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \varphi_l \varphi_m u + T_1 u + T_2 u + T_3 u + T_4 u + T_5 u = u + T^{'} u + T^{''} u,$$

where $u \in H_q^{k,\nu}(\mathbb{R}^n)$, $T^{'} : H_q^{k,\nu}(\mathbb{R}^n) \to H_q^{k,\nu}(\mathbb{R}^n)$ is a compact operator and for operator $T^{''} : H_q^{k,\nu}(\mathbb{R}^n) \to H_q^{k,\nu}(\mathbb{R}^n)$ we have $\|T^{''}\| < 1$.

Therefore

$$\left(I + T^{''}\right)^{-1} RP = I + \left(I + T^{''}\right)^{-1} T^{'},$$

where $T := \left(I + T^{''}\right)^{-1} T^{'} : H_q^{k,\nu}(\mathbb{R}^n) \to H_q^{k,\nu}(\mathbb{R}^n)$ is a compact operator. So we get that operator $\left(I + T^{''}\right)^{-1} R : H_q^{k-s,\nu}(\mathbb{R}^n) \to H_q^{k,\nu}(\mathbb{R}^n)$ is a left regularizer.

Analogously we can construct a right regularizer.

Since right and left regularizers exist, applying Theorem 2.5, we obtain the Fredholm property of operator $P(x,\mathbb{D}) : H_q^{k,\nu}(\mathbb{R}^n) \to H_q^{k-s,\nu}(\mathbb{R}^n)$.

Necessity of condition (2.25) for the Fredholm property of $P(x,\mathbb{D}) : H_q^{k,\nu}(\mathbb{R}^n) \to H_q^{k-s,\nu}(\mathbb{R}^n)$ follows from Corollary 2.1. □

Список литературы

[1] L. Hyormander, Linear Partial Differential Operators, Springer-Verlag Berlin Heidelberg (1969).

[2] L. A. Bagirov, "Elliptic operators in unbounded domain", Sbornik: Mathematics, **86** (128):1(9), 122 – 139 (1971).

[3] R. C. McOwen, "Behavior of the Laplacian on weighted Sobolev spaces", Comm. Pure Appl. Math., **32**, 783 – 795 (1979).

[4] R. B. Lockhart, R. C. McOwen, "On elliptic systems in $\mathbb{R}^n$", Acta Math. **150**, 125 – 135 (1983).

[5] E. Schrohe, "Spectral invariance, ellipticity, and the Fredholm property for pseudodifferential operators on weighted Sobolev spaces", Annals of Global Analysis and Geometry, **10**, no. 3, 237 – 254 (1992).
[6] L. A. Bagirov, "A priori estimates, existence theorems, and the behavior at infinity of solutions of quasielliptic equations in $\mathbb{R}^n$", Sbornik: Mathematics. **110** (152):4(12), 475 – 492 (1979).
[7] G. A. Karapetyan, A. A. Darbinyan, "Index of semi-elliptic operator in $\mathbb{R}^n$", Proceedings of the NAS Armenia: Mathematics, **42**, no. 5, 33 – 50 (2007).
[8] A. A. Darbinyan, A. G. Tumanyan, "On a priori estimates and the Fredholm property of differential operators in anisotropic spaces", Journal of Contemporary Mathematical Analysis, **53**:2, 61 – 70 (2018).
[9] A. A. Darbinyan, A. G. Tumanyan, "On necessary and sufficient conditions for Noethericity of semielliptic operators with special coefficients" [in Russian], Proceedings of Russian-Armenian (Slavonic) University (Physical, mathematical and natural sciences), no. 2, 5 – 15 (2017).
[10] G. V. Demidenko, "On quasielliptic operators in $\mathbb{R}^n$", Siberian Mathematical Journal, **39**, no. 5, 1028 – 1037 (1998).
[11] G. V. Demidenko, "Quasielliptic operators and Sobolev type equations", Siberian Math. Journal, **49**, no. 5, 842 – 851 (2008).
[12] S. S. Kutateladze, Fundamentals of Functional Analysis, Kluwer Texts in the Mathematical Sciences, **12**, Dordrecht: Kluwer Academic Publishers Group (1996).
[13] A. G. Tumanyan, "On the invariance of index of semielliptical operator on the scale of anisotropic spaces", Journal of Contemporary Mathematical Analysis, **51**, no. 4, 167 – 178 (2016).
[14] S. G. Kreyn, Linear operators in Banach spaces [in Russian], Nauka, Moscow (1971).

Поступила 15 июля 2020

После доработки 25 августа 2020

Принята к публикации 25 октября 2020