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VOLTERRA INTEGRAL OPERATORS FROM CAMPANATO SPACES INTO GENERAL FUNCTION SPACES

R. QIAN AND X. ZHU

Lingnan normal University, Guangdong, P. R. China[1]
University of Electronic Science and Technology of China, Zhongshan, P. R. China
E-mails: *qianruishen@sina.cn; jyuzxl@163.com; xiangling-zhu@163.com*

Abstract. In this paper, the boundedness and compactness of embedding from Campanato spaces $\mathcal{L}_{p,\lambda}$ into tent spaces $\mathcal{T}_{p,s}(\mu)$ are investigated. As an application, we give a characterization for the boundedness of the Volterra integral operator J_g from $\mathcal{L}_{p,\lambda}$ to general function spaces $F(p,p-1-\lambda,s)$. Meanwhile, the operator I_g and the multiplication operator M_g from $\mathcal{L}_{p,\lambda}$ to $F(p,p-1-\lambda,s)$ are studied. Furthermore, the essential norm of J_g and I_g from $\mathcal{L}_{p,\lambda}$ to $F(p,p-1-\lambda,s)$ are also considered.

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1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and $\partial\mathbb{D}$ its boundary. Let $H(\mathbb{D})$ denote the space of all analytic functions in $\mathbb{D}$. For $0 < p < \infty$, the Hardy space H^p is the set of all $f \in H(\mathbb{D})$ satisfying (see [1])

$$\|f\|_{H^p}^p = \sup_{0<r<1} \int_{\partial\mathbb{D}} |f(r\zeta)|^p d\zeta < \infty.$$

For $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space, denoted by A^p_α, consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{A^p_\alpha}^p = (\alpha+1) \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dA(z) < \infty,$$

where dA is the normalized Lebesgue area measure in $\mathbb{D}$ such that $A(\mathbb{D}) = 1$. When $\alpha = 0$, A^p_α is the Bergman space, denoted by A^p. As usual, H^∞ denotes the space of bounded analytic function.

In 1996, Zhao [26] introduced the general family of function spaces $F(p,q,s)$. Namely, for $0 < p < \infty$, $-2 < q < \infty$, $0 \le s < \infty$, the space $F(p,q,s)$ consists of

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functions $f \in H(\mathbb{D})$ satisfying

$$\|f\|^p_{F(p,q,s)} = |f(0)|^p + \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q (1-|\sigma_a(z)|^2)^s dA(z) < \infty,$$

where $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ is a Möbius transformation of $\mathbb{D}$ interchanging a and 0. It is known that, for $p \geq 1$, $F(p,q,s)$ is a Banach space under the above norm. Also, it is known that $F(p,q,s)$ contains only constant functions if $s+q \leq -1$. Thus, it is natural to study $F(p,q,s)$ spaces under the assumption that $s+q > -1$. $F(p,p,0)$ is just the Bergman space. When $p=2$ and $q=0$, it gives the Q_s space (see [22]). Especially, Q_1 is the $BMOA$ space, the space of analytic functions in the Hardy space whose boundary functions have bounded mean oscillation. When $s>1$, Q_s is the Bloch space, denoted by $\mathcal{B}$, which is the space of all $f \in H(\mathbb{D})$ for which

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)| < \infty.$$

The little Bloch space $\mathcal{B}_0$, consists of all $f \in \mathcal{B}$ such that $\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0$. See [13, 26] for more results of $F(p,q,s)$ spaces.

Let I be an arc of $\partial\mathbb{D}$ and $|I|$ be the normalized Lebesgue arc length of I. The Carleson square based on I, denoted by $S(I)$, is defined by

$$S(I) = \left\{ z = re^{i\theta} \in \mathbb{D} : 1-|I| \leq r < 1, e^{i\theta} \in I \right\}.$$

Let $0<p<\infty$, $0<s<\infty$ and μ be a positive Borel measure on $\mathbb{D}$. The tent space $\mathcal{T}_{p,s}(\mu)$ consists of all μ-measurable functions f such that

$$\|f\|^p_{\mathcal{T}_{p,s}(\mu)} = \sup_{I\subseteq\partial\mathbb{D}} \frac{1}{|I|^s} \int_{S(I)} |f(z)|^p d\mu(z) < \infty.$$

Let $p \geq 1$ and $0 \leq \lambda < \infty$. We say that an $f \in H^p$ belongs to the analytic Campanato space $\mathcal{L}_{p,\lambda}$ if (see [25])

$$\|f\|_{\mathcal{L}_{p,\lambda}} = |f(0)| + \left(\sup_{I\subseteq\partial\mathbb{D}} \frac{1}{|I|^\lambda} \int_I |f(\zeta) - f_I|^p \frac{|d\zeta|}{2\pi} \right)^{\frac{1}{p}} < \infty,$$

where

$$f_I = \frac{1}{|I|} \int_I f(\zeta) \frac{|d\zeta|}{2\pi},\ I \subseteq \partial\mathbb{D}.$$

When $p=2$, the space $\mathcal{L}_{2,\lambda}$ is called the Morrey space, which was studied by Wu and Xie in [20]. When $\lambda=0$, $\mathcal{L}_{p,0}$ is just the Hardy space H^p. $\mathcal{L}_{p,1}$ is the $BMOA$ space. Recently, some fundamental function and operator-theoretic properties on $\mathcal{L}_{p,\lambda}$ have been investigated in [5, 10, 14, 18, 19, 20, 21, 24, 25]

Let $f, g \in H(\mathbb{D})$. The Volterra integral operator J_g and the integral operator I_g are defined by

$$J_g f(z) = \int_0^z g'(w) f(w) dw, \quad I_g f(z) = \int_0^z g(w) f'(w) dw, \quad z \in \mathbb{D},$$

respectively. The multiplication operator M_g is defined by $M_g f(z) = g(z)f(z),\ \ f \in H(\mathbb{D}),\ z \in \mathbb{D}$.

The operator J_g was introduced by Pommerenke in [12]. Pommerenke showed that $J_g : H^2 \to H^2$ is bounded if and only if $g \in BMOA$. Furthermore, in [3], Aleman and Siskakis proved that $J_g : H^p \to H^p$ is bounded if and only if $g \in BMOA$. In [4], Aleman and Siskakis showed that $J_g : A^p \to A^p$ is bounded if and only if $g \in \mathcal{B}$. For more information on Volterra integral operators, see [2] - [9], [11, 14, 15, 23] and the references therein.

Recently, Li, Liu and Lou in [5] proved that $J_g : \mathcal{L}_{2,\lambda} \to \mathcal{L}_{2,\lambda}$ is bounded if and only if $g \in BMOA$. In [18], Wang generalized the result in [5] and proved that $J_g : \mathcal{L}_{p,\lambda} \to \mathcal{L}_{2,1-2/p(1-\lambda)}$ is bounded if and only if $g \in BMOA$ under the assumption that $2 \le p < \infty$ and $0 \le \lambda < 1$. An interesting and nature question is to find an analytic function space X for which

$$J_g : \mathcal{L}_{p,\lambda} \to X \ \text{ is bounded if and only if } \ g \in \mathcal{B}.$$

In this paper, we prove that $J_g : \mathcal{L}_{p,\lambda}$ *to* $F(p, p-1-\lambda, s)$ is bounded if and only if $g \in \mathcal{B}$. Moreover, we show that the identity operator $i : \mathcal{L}_{p,\lambda} \to \mathcal{T}_{p,s}(\mu)$ is bounded (resp.compact) if and only if μ is a $s - \lambda + 1$-Carleson measure(resp. a vanishing $s - \lambda + 1$-Carleson measure) under the assumption that $2 \le p < \infty$, $0 \le \lambda < 1$ and $\lambda < s < \infty$. The essential norm of the operator J_g is also investigated. Furthermore, we study the boundedness and compactness of the operators I_g and M_g from $\mathcal{L}_{p,\lambda}$ to $F(p, p-1-\lambda, s)$.

Throughout this paper, we say that $A \lesssim B$, if there exists a constant C such that $A \le CB$. The symbol $A \asymp B$ means that $A \lesssim B \lesssim A$.

2. Embedding from $\mathcal{L}_{p,\lambda}$ to tent spaces

An important tool to study function spaces is Carleson type measure. For $s > 0$, a positive Borel measure μ on $\mathbb{D}$ is said to be an s-Carleson measure if $\sup_{I\subset\partial\mathbb{D}} \frac{\mu(S(I))}{|I|^s} < \infty$. For $s = 1$, we get the classical Carleson measures (see [1]). If μ is an s-Carleson measure, then we set

$$\|\mu\|_s = \sup_{I\subset\partial\mathbb{D}} \frac{\mu(S(I))}{|I|^s}.$$

If $\lim_{|I|\to 0} \frac{\mu(S(I))}{|I|^s} = 0$, then μ is called a vanishing s-Carleson measure. It is well known (see [25]) that μ is an s-Carleson measure if and only if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^s}{|1-\bar{a}z|^{2s}} d\mu(z) < \infty.$$

Moreover,

$$\sup_{I\subset\partial\mathbb{D}}\frac{\mu(S(I))}{|I|^s}\asymp\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{(1-|a|^2)^s}{|1-\bar{a}z|^{2s}}d\mu(z). \tag{2.1}$$

Now we are in a position to state and prove the main results in this section.

Theorem 2.1. *Let $2\le p<\infty$, $0\le\lambda<1$, $\lambda<s<\infty$ and μ be a positive Borel measure on $\mathbb{D}$. Then the identity operator $i:\mathcal{L}_{p,\lambda}\to\mathcal{T}_{p,s}(\mu)$ is bounded if and only if μ is a $(s+1-\lambda)$-Carleson measure.*

Proof. Assume that μ is a $(s+1-\lambda)$-Carleson measure. Let I be any arc on $\partial\mathbb{D}$ and $a=(1-|I|)e^{i\theta}$, where $e^{i\theta}$ is the midpoint of I. Let $f\in\mathcal{L}_{p,\lambda}$. From [18, Lemma 2.5], we get

$$|f(a)|\lesssim\frac{\|f\|_{\mathcal{L}_{p,\lambda}}}{(1-|a|)^{\frac{1-\lambda}{p}}}=\frac{\|f\|_{\mathcal{L}_{p,\lambda}}}{|I|^{\frac{1-\lambda}{p}}}.$$

Then

$$\begin{aligned}\frac{1}{|I|^s}\int_{S(I)}|f(z)|^pd\mu(z)&\lesssim\frac{1}{|I|^s}\int_{S(I)}|f(a)|^pd\mu(z)+\frac{1}{|I|^s}\int_{S(I)}|f(z)-f(a)|^pd\mu(z)\\&=M+N.\end{aligned}$$

It is obvious that

$$M\lesssim\frac{\mu(S(I))}{|I|^{s-\lambda+1}}\|f\|^p_{\mathcal{L}_{p,\lambda}}\lesssim\|f\|^p_{\mathcal{L}_{p,\lambda}}.$$

Now we turn to estimate N. The estimate will be divided into two cases.

Case 1: $s-\lambda\ge1$.

By the assumed condition and Theorem 7.4 in [27], we know that the identity operator $i:A^p_{s-\lambda-1}\to L^p(d\mu)$ is bounded. Then

$$\begin{aligned}N&\asymp\int_{S(I)}\frac{|f(z)-f(a)|^p}{|1-\bar{a}z|^s}d\mu(z)\\&\asymp(1-|a|^2)^{1-\lambda}\int_{S(I)}\frac{|f(z)-f(a)|^p(1-|a|^2)^2}{|1-\bar{a}z|^{3-\lambda+s}}d\mu(z)\\&\lesssim(1-|a|^2)^{1-\lambda}\int_{\mathbb{D}}\frac{|f(z)-f(a)|^p(1-|a|^2)^2}{\left|(1-\bar{a}z)^{\frac{3-\lambda+s}{p}}\right|^p}d\mu(z)\\&\lesssim(1-|a|^2)^{1-\lambda}\int_{\mathbb{D}}\frac{|f(z)-f(a)|^p(1-|a|^2)^2}{|1-\bar{a}z|^{3-\lambda+s}}(1-|z|^2)^{s-\lambda-1}dA(z)\\&\lesssim(1-|a|^2)^{1-\lambda}\int_{\mathbb{D}}\frac{|f(z)-f(a)|^p(1-|a|^2)^2}{|1-\bar{a}z|^4}dA(z)\\&=(1-|a|^2)^{1-\lambda}\int_{\mathbb{D}}|f\circ\sigma_a(w)-f(a)|^pdA(w)\\&\lesssim(1-|a|^2)^{1-\lambda}\int_{\partial\mathbb{D}}|f\circ\sigma_a(\zeta)-f(a)|^pd\zeta\le\|f\|^p_{\mathcal{L}_{p,\lambda}}<\infty.\end{aligned}$$

The last second inequality is come from [25, Theorem 1].

Case 2: $0 < s - \lambda < 1$.

Since $H^p \subseteq A^p_{s-\lambda-1}$, we have

$$\begin{aligned}
N &\asymp (1-|a|^2)^{-s} \int_{S(I)} |f(z)-f(a)|^p d\mu(z) \\
&\asymp (1-|a|^2)^{2-s} \int_{S(I)} \frac{|f(z)-f(a)|^p (1-|a|^2)^2}{|1-\bar{a}z|^4} d\mu(z) \\
&\lesssim (1-|a|^2)^{2-s} \int_{\mathbb{D}} \frac{|f(z)-f(a)|^p (1-|a|^2)^2}{|1-\bar{a}z|^4} d\mu(z) \\
&\lesssim (1-|a|^2)^{2-s} \int_{\mathbb{D}} \frac{|f(z)-f(a)|^p (1-|a|^2)^2}{|1-\bar{a}z|^4} (1-|z|^2)^{s-\lambda-1} dA(z) \\
&= (1-|a|^2)^{2-s} \int_{\mathbb{D}} |f\circ\sigma_a(w)-f(a)|^p (1-|\sigma_a(w)|^2)^{s-\lambda-1} dA(w) \\
&\lesssim (1-|a|^2)^{1-\lambda} \int_{\mathbb{D}} |f\circ\sigma_a(w)-f(a)|^p (1-|w|^2)^{s-\lambda-1} dA(w) \\
&\lesssim (1-|a|^2)^{1-\lambda} \int_{\partial\mathbb{D}} |f\circ\sigma_a(\zeta)-f(a)|^p d\zeta \lesssim \|f\|^p_{\mathcal{L}_{p,\lambda}} < \infty.
\end{aligned}$$

Combining the estimates M and N, we conclude that the identity operator $i : \mathcal{L}_{p,\lambda} \to \mathcal{T}_{p,s}(\mu)$ is bounded.

Conversely, suppose that the identity operator $i : \mathcal{L}_{p,\lambda} \to \mathcal{T}_{p,s}(\mu)$ is bounded. For $a \in \mathbb{D}$, let

$$f_a(z) = \frac{(1-|a|^2)^{1+\frac{\lambda-1}{p}}}{(1-\bar{a}z)}, \quad z \in \mathbb{D}. \tag{2.2}$$

By [18, Lemma 2.3], we have that $f_a \in \mathcal{L}_{p,\lambda}$ with $\sup_{a\in\mathbb{D}} \|f_a\|_{\mathcal{L}_{p,\lambda}} \lesssim 1$. Fixed an arc $I \subseteq \partial\mathbb{D}$. Let $e^{i\theta}$ be the center of I and $a = (1-|I|)e^{i\theta}$. Then

$$|1-\bar{a}z| \asymp 1-|a| = |I|, \quad |f_a(z)|^p \asymp |I|^{\lambda-1},$$

whenever $z \in S(I)$. So

$$\frac{\mu(S(I))}{|I|^{s+1-\lambda}} \asymp \frac{1}{|I|^s} \int_{S(I)} |f_a(z)|^p d\mu(z) \le \|f_a\|^p_{\mathcal{T}_{p,s}(\mu)} < \infty.$$

Consequently, μ is a $(s+1-\lambda)$-Carleson measure. □

Theorem 2.2. *Let $2 \le p < \infty$, $0 \le \lambda < 1$, $\lambda < s < \infty$ and μ be a positive Borel measure on $\mathbb{D}$ such that point evaluation is a bounded functional on $\mathcal{T}_{p,s}(\mu)$. Then the identity operator $i : \mathcal{L}_{p,\lambda} \to \mathcal{T}_{p,s}(\mu)$ is compact if and only if μ is a vanishing $(s-\lambda+1)$-Carleson measure.*

Proof. Assume that μ is a vanishing $(s-\lambda+1)$-Carleson measure. It is clear that μ is a $(s-\lambda+1)$-Carleson measure. Hence $i : \mathcal{L}_{p,\lambda} \to \mathcal{T}_{p,s}(\mu)$ is bounded. For $0 < r < 1$, let $\chi_{\{z:|z|<r\}}$ be the characteristic function of the set $\{z : |z| < r\}$. From

[6] we see that $\lim_{r\to 1}\|\mu-\mu_r\|_{s-\lambda+1}=0$, where $d\mu_r=\chi_{\{z:|z|<r\}}d\mu$. Let $\{f_k\}$ be a bounded sequence in $\mathcal{L}_{p,\lambda}$ such that $\{f_k\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$. We have

$$\begin{aligned}\frac{1}{|I|^s}\int_{S(I)}|f_k(z)|^p d\mu(z) &\lesssim \frac{1}{|I|^s}\int_{S(I)}|f_k(z)|^p d\mu_r(z)+\frac{1}{|I|^s}\int_{S(I)}|f_k(z)|^p d(\mu-\mu_r)(z)\\ &\lesssim \frac{1}{|I|^s}\int_{S(I)}|f_k(z)|^p d\mu_r(z)+\|\mu-\mu_r\|_{s-\lambda+1}\|f_k\|^p_{\mathcal{L}_{p,\lambda}}\\ &\lesssim \frac{1}{|I|^s}\int_{S(I)}|f_k(z)|^p d\mu_r(z)+\|\mu-\mu_r\|_{s-\lambda+1}\to 0,\end{aligned}$$

as $r\to 1$ and $k\to\infty$. Therefore, $\lim_{k\to\infty}\|f_k\|_{\mathcal{T}_{p,s}(\mu)}=0$, which means that the identity operator $i:\mathcal{L}_{p,\lambda}\to\mathcal{T}_{p,s}(\mu)$ is compact.

Conversely, suppose that the identity operator $i:\mathcal{L}_{p,\lambda}\to\mathcal{T}_{p,s}(\mu)$ is compact. Let $\{I_k\}$ be a sequence arcs with $\lim_{k\to\infty}|I_k|=0$. We denote the center of I_k by $e^{i\theta_k}$. Set $a_k=(1-|I_k|)e^{i\theta_k}$ and

$$f_k(z)=\frac{(1-|a_k|^2)^{1+\frac{\lambda-1}{p}}}{(1-\bar{a_k}z)},\quad z\in\mathbb{D}. \tag{2.3}$$

It is easy to check that $\{f_k\}$ is bounded in $\mathcal{L}_{p,\lambda}$ and $\{f_k\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$. Then $\lim_{k\to\infty}\|f_k\|_{\mathcal{T}_{p,s}(\mu)}=0$ by the assumption. Since

$$|f_k(z)|\asymp(1-|a_k|)^{\frac{\lambda-1}{p}}=|I_k|^{\frac{\lambda-1}{p}},$$

when $z\in S(I_k)$, we obtain

$$\frac{\mu(S(I_k))}{|I_k|^{s-\lambda+1}}\asymp\frac{1}{|I_k|^s}\int_{S(I_k)}|f_k(z)|^p d\mu(z)\le\|f_k\|^p_{\mathcal{T}_{p,s}(\mu)}\to 0,\quad k\to\infty,$$

which implies that μ is a vanishing $(s-\lambda+1)$-Carleson measure. □

3. Boundedness of J_g, I_g and M_g

In this section, via the embedding theorem (Theorem 2.1), we provide a characterization for the boundedness of Volterra integral operator J_g from $\mathcal{L}_{p,\lambda}$ to $F(p,p-1-\lambda,s)$. We also study the boundedness of the operators I_g and M_g.

Theorem 3.1. *Let $2\le p<\infty$, $0\le\lambda<1$ and $\lambda<s<\infty$. If $g\in H(\mathbb{D})$, then $J_g:\mathcal{L}_{p,\lambda}\to F(p,p-1-\lambda,s)$ is bounded if and only if $g\in\mathcal{B}$. Moreover, $\|J_g\|\asymp\|g\|_{\mathcal{B}}$.*

Proof. Let $g\in\mathcal{B}$. Using the equivalent norm of Bloch function (see [26]), we obtain

$$\begin{aligned}\|g\|^p_{\mathcal{B}} &\asymp \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|g'(z)|^p(1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^{s-\lambda+1}dA(z)\\ &=\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|g'(z)|^p(1-|z|^2)^{p-1+s-\lambda}\left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^{s-\lambda+1}dA(z)\\ &\asymp\sup_{I\subseteq\partial\mathbb{D}}\frac{1}{|I|^{s-\lambda+1}}\int_{S(I)}|g'(z)|^p(1-|z|^2)^{p-1+s-\lambda}dA(z)\asymp\sup_{I\subseteq\partial\mathbb{D}}\frac{\mu_g(S(I))}{|I|^{s-\lambda+1}},\end{aligned}$$

which implies that $d\mu_g(z) = |g'(z)|^p(1-|z|^2)^{p-1+s-\lambda}dA(z)$ is a $(s-\lambda+1)$-Carleson measure. By Theorem 2.1, the identity operator $i : \mathcal{L}_{p,\lambda} \to \mathcal{T}_{p,s}(\mu_g)$ is bounded. Let $f \in \mathcal{L}_{p,\lambda}$. We deduce that

$$\begin{aligned}
\|J_g f\|^p_{F(p,p-1-\lambda,s)} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^p|g'(z)|^p(1-|z|^2)^{p-1-\lambda}(1-|\sigma_a(z)|^2)^s dA(z) \\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^p|g'(z)|^p(1-|z|^2)^{p-1-\lambda+s}\left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^s dA(z) \\
&\asymp \sup_{I\subseteq\partial\mathbb{D}} \frac{1}{|I|^s}\int_{S(I)} |f(z)|^p d\mu_g(z) \\
&= \|f\|^p_{\mathcal{T}_{p,s}(\mu_g)} \lesssim \|\mu_g\|_{s-\lambda+1}\|f\|^p_{\mathcal{L}_{p,\lambda}} \asymp \|g\|^p_{\mathcal{B}}\|f\|^p_{\mathcal{L}_{p,\lambda}} < \infty.
\end{aligned}$$

That is, $J_g : \mathcal{L}_{p,\lambda} \to F(p,p-1-\lambda,s)$ is bounded and $\|J_g\| \lesssim \|g\|_{\mathcal{B}}$.

Conversely, suppose that $J_g : \mathcal{L}_{p,\lambda} \to F(p,p-1-\lambda,s)$ is bounded. For any $a \in \mathbb{D}$, let f_a be defined as in (2.2). Then $f_a \in \mathcal{L}_{p,\lambda}$ and $\|f_a\|_{\mathcal{L}_{p,\lambda}} \lesssim 1$. Thus,

$$\|J_g f_a\|_{F(p,p-1-\lambda,s)} \leq \|J_g\|\ \|f_a\|_{\mathcal{L}_{p,\lambda}} \lesssim \|J_g\|.$$

By Lemma 4.12 of [27], we have

$$\begin{aligned}
\|J_g f_a\|^p_{F(p,p-1-\lambda,s)} &\geq \int_{\mathbb{D}} |g'(z)|^p \frac{(1-|a|^2)^{p-1+\lambda}}{|1-\bar{a}z|^p}(1-|z|^2)^{p-1-\lambda}(1-|\sigma_a(z)|^2)^s dA(z) \\
&= \int_{\mathbb{D}} |g'(z)|^p \frac{(1-|a|^2)^{p-1+\lambda+s}(1-|z|^2)^{p-1-\lambda+s}}{|1-\bar{a}z|^{2s+p}} dA(z) \\
&\geq \int_{D(a,r)} |g'(z)|^p \frac{(1-|a|^2)^{p-1+\lambda+s}(1-|z|^2)^{p-1-\lambda+s}}{|1-\bar{a}z|^{2s+p}} dA(z) \\
&\gtrsim |g'(a)|^p(1-|a|^2)^p.
\end{aligned}$$

Hence, for any $a \in \mathbb{D}$,

$$|g'(a)|(1-|a|^2) \lesssim \|J_g f_a\|_{F(p,p-1-\lambda,s)} \lesssim \|J_g\|,$$

which implies that $g \in \mathcal{B}$ and $\|g\|_{\mathcal{B}} \lesssim \|J_g\|$. □

Theorem 3.2. *Suppose that* $2 \leq p < \infty$, $0 \leq \lambda < 1 < s < \infty$ *or* $p = 2$, $0 \leq \lambda < 1$ *and* $\lambda < s < \infty$. *If* $g \in H(\mathbb{D})$, *then* $I_g : \mathcal{L}_{p,\lambda} \to F(p,p-1-\lambda,s)$ *is bounded if and only if* $g \in H^\infty$. *Furthermore,* $\|I_g\| \asymp \|g\|_{H^\infty}$.

Proof. Assume that $I_g : \mathcal{L}_{p,\lambda} \to F(p,p-1-\lambda,s)$ is bounded. For any $a \in \mathbb{D}$, set $h_a = \frac{(1-|a|^2)^{1+\frac{\lambda-1}{p}}}{\bar{a}(1-\bar{a}z)}$. It is easy to see that $h_a \in \mathcal{L}_{p,\lambda}$ and $\sup_{a\in\mathbb{D}} \|h_a\|_{\mathcal{L}_{p,\lambda}} \lesssim 1$. Hence

$$\|I_g h_a\|_{F(p,p-1-\lambda,s)} \leq \|I_g\|\ \|h_a\|_{\mathcal{L}_{p,\lambda}} \lesssim \|I_g\|.$$

Lemma 4.12 of [27] gives

$$\begin{aligned}\|I_g h_a\|^p_{F(p,p-1-\lambda,s)} &\gtrsim \int_{\mathbb{D}} |g(z)|^p \frac{(1-|a|^2)^{p+\lambda-1}}{|1-\bar{a}z|^{2p}}(1-|z|^2)^{p-1-\lambda}(1-|\sigma_a(z)|^2)^s dA(z)\\ &\gtrsim \int_{D(a,r)} |g(z)|^p \frac{(1-|a|^2)^{p+\lambda-1}}{|1-\bar{a}z|^{2p}}(1-|z|^2)^{p-1-\lambda}(1-|\sigma_a(z)|^2)^s dA(z)\\ &\gtrsim |g(a)|^p,\end{aligned}$$

which implies that $g \in H^\infty$ and $\|g\|_{H^\infty} \lesssim \|I_g\|$.

Conversely, suppose that $g \in H^\infty$. First we consider the case $2 \le p < \infty$, $0 \le \lambda < 1 < s < \infty$. Let $f \in \mathcal{L}_{p,\lambda}$. Then by [18, Lemma 2.4],

$$|f'(z)|^p \lesssim \frac{\|f\|^p_{\mathcal{L}_{p,\lambda}}}{(1-|z|^2)^{p+1-\lambda}}.$$

Combined with Lemma 3.10 of [27], we have

$$\begin{aligned}\|I_g f\|^p_{F(p,p-1-\lambda,s)} &\le \|g\|^p_{H^\infty} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p-1-\lambda}(1-|\sigma_a(z)|^2)^s dA(z)\\ &\le \|g\|^p_{H^\infty}\|f\|^p_{\mathcal{L}_{p,\lambda}} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (1-|z|^2)^{-2}(1-|\sigma_a(z)|^2)^s dA(z)\\ &\le \|g\|^p_{H^\infty}\|f\|^p_{\mathcal{L}_{p,\lambda}} \sup_{a\in\mathbb{D}}(1-|a|^2)^s \int_{\mathbb{D}} \frac{(1-|z|^2)^{s-2}}{|1-\bar{a}z|^{2s}} dA(z) \quad (s>1)\\ &\le \|g\|^p_{H^\infty}\|f\|^p_{\mathcal{L}_{p,\lambda}}.\end{aligned}$$

Thus, $I_g : \mathcal{L}_{p,\lambda} \to F(p, p-1-\lambda, s)$ is bounded and $\|I_g\| \le \|g\|_{H^\infty}$.

When $p = 2$, $0 \le \lambda < 1$ and $\lambda < s < \infty$. From above, we have

$$\begin{aligned}\|I_g f\|^2_{F(2,1-\lambda,s)} &\le \|g\|^2_{H^\infty} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)^{1-\lambda}(1-|\sigma_a(z)|^2)^s dA(z)\\ &\le \|g\|^2_{H^\infty} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)^{1-\lambda}(1-|\sigma_a(z)|^2)^\lambda dA(z)\\ &\lesssim \|g\|^2_{H^\infty} \sup_{I\subseteq\partial\mathbb{D}} \frac{1}{|I|^\lambda} \int_{\mathbb{D}} |f'(z)|^2(1-|z|^2) dA(z) \le \|g\|^2_{H^\infty}\|f\|^2_{\mathcal{L}_{2,\lambda}}.\end{aligned}$$

The proof is complete. □

Using Theorems 3.1 and 3.2, we get the characterization of the boundedness of the multiplication operator $M_g : \mathcal{L}_{p,\lambda} \to F(p, p-1-\lambda, s)$.

Theorem 3.3. *Suppose that* $2 \le p < \infty$, $0 \le \lambda < 1 < s < \infty$ *or* $p = 2$, $0 \le \lambda < 1$ *and* $\lambda < s < \infty$. *Then* $M_g : \mathcal{L}_{p,\lambda} \to F(p, p-1-\lambda, s)$ *is bounded if and only if* $g \in H^\infty$.

Proof. Assume that $M_g : \mathcal{L}_{p,\lambda} \to F(p, p-1-\lambda, s)$ is bounded. Let $h \in F(p, p-1-\lambda, s)$ and $b \in \mathbb{D}$. We have (see [26])

$$|h'(b)| \lesssim \frac{\|h\|_{F(p,p-1-\lambda,s)}}{(1-|b|^2)^{1+\frac{1-\lambda}{p}}},$$

and hence

$$|h(b)| \lesssim \frac{\|h\|_{F(p,p-1-\lambda,s)}}{(1-|b|^2)^{\frac{1-\lambda}{p}}}.$$

For any $a \in \mathbb{D}$, let f_a be defined as in (2.2). Then $\{f_a\}$ is bounded in $\mathcal{L}_{p,\lambda}$. By the assumption we see that $M_g f_a \in F(p, p-1-\lambda, s)$. Hence

$$|M_g f_a(z)| \lesssim \frac{\|M_g f_a\|_{F(p,p-1-\lambda,s)}}{(1-|z|^2)^{\frac{1-\lambda}{p}}} \lesssim \frac{\|M_g\| \, \|f_a\|_{\mathcal{L}_{p,\lambda}}}{(1-|z|^2)^{\frac{1-\lambda}{p}}} \lesssim \frac{\|M_g\|}{(1-|z|^2)^{\frac{1-\lambda}{p}}},$$

which implies that

$$\left| \frac{1-|a|^2}{(1-\bar{a}z)^{1+\frac{1-\lambda}{p}}} g(z) \right| \lesssim \frac{\|M_g\|}{(1-|z|^2)^{\frac{1-\lambda}{p}}}.$$

By the arbitrariness of $z, a \in \mathbb{D}$, let $a = z$, we obtain that $g \in H^\infty$ and $\|g\|_{H^\infty} \lesssim \|M_g\|$.

Conversely, assume that $g \in H^\infty$. It follows from Theorems 3.1 and 3.2 that

$$J_g : \mathcal{L}_{p,\lambda} \to F(p, p-1-\lambda, s) \ \text{ and } \ I_g : \mathcal{L}_{p,\lambda} \to F(p, p-1-\lambda, s)$$

are bounded. So by the following relation

$$J_g f + I_g f = M_g f - f(0)g(0),$$

we see that $M_g : \mathcal{L}_{p,\lambda} \to F(p, p-1-\lambda, s)$ is bounded. □

4. Essential norm of J_g and I_g

In this section, we give an estimation of the essential norm of J_g and I_g. First, let us recall the definition of the essential norm of a operator. Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. The essential norm of $T : X \to Y$, denoted by $\|T\|_{e,X\to Y}$, is defined by

$$\|T\|_{e,X\to Y} = \inf_S \{\|T - S\|_{X\to Y} : \quad S \text{ is compact from } X \text{ to } Y\}.$$

Lemma 4.1. [17] *If $f \in \mathcal{B}$, then*

$$\limsup_{|z|\to 1} (1-|z|^2)|f'(z)| \asymp \limsup_{r\to 1} \|f - f_r\|_{\mathcal{B}}.$$

Here $f_r(z) = f(rz)$, $0 < r < 1, z \in \mathbb{D}$.

Lemma 4.2. *Let $2 \le p < \infty$, $0 \le \lambda < 1$ and $\lambda < s < \infty$. If $0 < r < 1$ and $g \in \mathcal{B}$, then $J_{g_r} : \mathcal{L}_{p,\lambda} \to F(p, p-1-\lambda, s)$ is compact.*

Proof. Let $\{f_k\}$ be a bounded sequence in $\mathcal{L}_{p,\lambda}$ such that $\{f_k\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ and $\sup_k \|f_k\|_{\mathcal{L}_{p,\lambda}} \le 1$. Then

$$\begin{aligned}\|J_{g_r} f_k\|^p_{F(p,p-1-\lambda,s)} &\le \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_k(z)|^p |g_r'(z)|^p (1-|z|^2)^{p-1-\lambda}(1-|\sigma_a(z)|^2)^s dA(z)\\ &\lesssim \frac{\|g\|^p_{\mathcal{B}}}{(1-r^2)^p} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_k(z)|^p (1-|z|^2)^{p-1-\lambda}(1-|\sigma_a(z)|^2)^s dA(z)\\ &\lesssim \frac{\|g\|^p_{\mathcal{B}}}{(1-r^2)^p} \int_{\mathbb{D}} |f_k(z)|^p (1-|z|^2)^{p-1-\lambda} dA(z)\\ &\lesssim \frac{\|g\|^p_{\mathcal{B}}\|f_k\|^p_{\mathcal{L}_{p,\lambda}}}{(1-r^2)^p} \int_{\mathbb{D}} (1-|z|^2)^{p-2} dA(z).\end{aligned}$$

By the dominated convergence theorem, we get the result. □

Theorem 4.1. *Let $2 \le p < \infty$, $0 \le \lambda < 1$ and $\lambda < s < \infty$. If $g \in H(\mathbb{D})$ such that $J_g : \mathcal{L}_{p,\lambda} \to F(p, p-1-\lambda, s)$ is bounded, then*

$$\|J_g\|_{e,\mathcal{L}_{p,\lambda}\to F(p,p-1-\lambda,s)} \asymp \limsup_{|z|\to 1} (1-|z|^2)|g'(z)|.$$

Proof. By Lemma 4.2, $J_{g_r} : \mathcal{L}_{p,\lambda} \to F(p, p-1-\lambda, s)$ is compact. Hence

$$\begin{aligned}\|J_g\|_{e,\mathcal{L}_{p,\lambda}\to F(p,p-1-\lambda,s)} &\le \|J_g - J_{g_r}\|_{\mathcal{L}_{p,\lambda}\to F(p,p-1-\lambda,s)}\\ &= \|J_{g-g_r}\|_{\mathcal{L}_{p,\lambda}\to F(p,p-1-\lambda,s)} \asymp \|g-g_r\|_{\mathcal{B}}.\end{aligned}$$

Using Lemma 4.1, we have

$$\|J_g\|_{e,\mathcal{L}_{p,\lambda}\to F(p,p-1-\lambda,s)} \lesssim \limsup_{r\to 1} \|g-g_r\|_{\mathcal{B}} \asymp \limsup_{|z|\to 1} (1-|z|^2)|g'(z)|.$$

Next we prove that

$$\|J_g\|_{e,\mathcal{L}_{p,\lambda}\to F(p,p-1-\lambda,s)} \gtrsim \limsup_{|z|\to 1} (1-|z|^2)|g'(z)|.$$

Let $\{a_k\}$ be a sequence in $\mathbb{D}$ such that $\lim_{k\to\infty} |a_k| = 1$ and f_k be defined as in (2.3). Then $\{f_k\}$ is bounded in $\mathcal{L}_{p,\lambda}$ and converges to zero uniformly on each compact subset of $\mathbb{D}$. For any given compact operator $S : \mathcal{L}_{p,\lambda} \to F(p, p-1-\lambda, s)$, by [16, Lemma 2.10] we have $\lim_{k\to\infty} \|Sf_k\|_{F(p,p-1-\lambda,s)} = 0$. Then

$$\begin{aligned}&\|J_g - S\|_{\mathcal{L}_{p,\lambda}\to F(p,p-1-\lambda,s)} \gtrsim \limsup_{k\to\infty} \|(J_g - S)f_k\|_{F(p,p-1-\lambda,s)}\\ &\gtrsim \limsup_{k\to\infty} \Big(\|J_g f_k\|_{F(p,p-1-\lambda,s)} - \|Sf_k\|_{F(p,p-1-\lambda,s)}\Big)\\ &\ge \limsup_{k\to\infty} \left(\int_{\mathbb{D}} |f_k(z)|^p |g'(z)|^p (1-|z|^2)^{p-1-\lambda}(1-|\sigma_{a_k}(z)|^2)^s dA(z)\right)^{\frac{1}{p}}\\ &\gtrsim \limsup_{k\to\infty} (1-|a_k|^2)|g'(a_k)|,\end{aligned}$$

which implies the desired result. □

Using Theorem 4.1 and the well-known result that $T : X \to Y$ is compact if and only if $\|T\|_{e,X\to Y} = 0$, we easily get the following corollary.

Corollary 4.1. *Let $2 \le p < \infty$, $0 \le \lambda < 1$ and $\lambda < s < \infty$. If $g \in H(\mathbb{D})$, then $J_g : \mathcal{L}_{p,\lambda} \to F(p, p-1-\lambda, s)$ is compact if and only if $g \in \mathcal{B}_0$.*

Theorem 4.2. *Suppose that $2 \le p < \infty$, $0 \le \lambda < 1 < s < \infty$ or $p = 2$, $0 < \lambda < 1$ and $\lambda < s < \infty$. If $g \in H(\mathbb{D})$ and $I_g : \mathcal{L}_{p,\lambda} \to F(p, p-1-\lambda, s)$ is bounded, then*

$$\|I_g\|_{e,\mathcal{L}_{p,\lambda}\to F(p,p-1-\lambda,s)} \asymp \|g\|_{H^\infty}.$$

Proof. First, Theorem 3.2 gives

$$\|I_g\|_{e,\mathcal{L}_{p,\lambda}\to F(p,p-1-\lambda,s)} = \inf_S \|I_g - S\|_{\mathcal{L}_{p,\lambda}\to F(p,p-1-\lambda,s)} \le \|I_g\|_{\mathcal{L}_{p,\lambda}\to F(p,p-1-\lambda,s)} \lesssim \|g\|_{H^\infty}.$$

Now we prove that

$$\|I_g\|_{e,\mathcal{L}_{p,\lambda}\to F(p,p-1-\lambda,s)} \gtrsim \|g\|_{H^\infty}.$$

Let $\{a_k\}$, $\{f_k\}$ and S be defined as in the proof of Theorem 4.1. Since $S : \mathcal{L}_{p,\lambda} \to F(p, p-1-\lambda, s)$ is compact, by [16, Lemma 2.10] we get $\lim_{k\to\infty} \|Sf_k\|_{F(p,p-1-\lambda,s)} = 0$. Hence,

$$\begin{aligned}\|I_g - S\|_{\mathcal{L}_{p,\lambda}\to F(p,p-1-\lambda,s)} &\gtrsim \limsup_{k\to\infty} \|(I_g - S)f_k\|_{F(p,p-1-\lambda,s)} \\ &\gtrsim \limsup_{k\to\infty} \Big(\|I_g f_k\|_{F(p,p-1-\lambda,s)} - \|Sf_k\|_{F(p,p-1-\lambda,s)}\Big) \\ &= \limsup_{k\to\infty} \|I_g f_k\|_{F(p,p-1-\lambda,s)}.\end{aligned}$$

Similarly to the proof of Theorem 3.2, we get $\|I_g f_k\|_{F(p,p-1-\lambda,s)} \gtrsim |g(a_k)|$, which implies the desired result. □

Using Theorem 4.2, we easily get the following corollary.

Corollary 4.2. *Suppose that $2 \le p < \infty$, $0 \le \lambda < 1 < s < \infty$ or $p = 2$, $0 \le \lambda < 1$ and $\lambda < s < \infty$. If $g \in H(\mathbb{D})$, then $I_g : \mathcal{L}_{p,\lambda} \to F(p, p-1-\lambda, s)$ is compact if and only if $g = 0$.*

Remark. We conclude the article with a remark. There is a class of Möbius invariant spaces that are closely related to the Bloch space and BMOA, namely, the Q_s space. Let $2 \le p < \infty$, $0 \le \lambda < 1$ and $0 < s < 1$. An interesting and nature question is to find an analytic function space X for which

$$J_g : \mathcal{L}_{p,\lambda} \to X \quad \text{is bounded if and only if} \quad g \in Q_s.$$

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