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A UNIQUENESS THEOREM FOR MULTIPLE ORTHONORMAL SPLINE SERIES

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Abstract. In this paper we obtain recovery formulas for coefficients of multiple Ciesielski series by means of its sum, if the square partial sums of a Ciesielski series converge in measure to a function f and the majorant of partial sums satisfies some necessary condition.

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1. Introduction

The uniqueness problem and reconstruction of coefficients of series by various orthogonal systems has been considered in a number of papers. Uniqueness theorems for almost everywhere convergent or summable trigonometric series were obtained in the papers [1] and [4], under some additional conditions imposed on the series. Results on uniqueness and restoration of coefficients for series by Haar and Franklin systems have been obtained, for instance, in the papers [3], [6], [7] and [11]-[14]. Here we quote a result by G. Gevorkyan [3] on restoration of coefficients of series by Franklin system.

Specifically, in [3] it was proved that if the Franklin series $\sum_{n=0}^{\infty} a_n f_n(x)$ converges a.e. to a function $f(x)$ and

$$\lim_{\lambda\to\infty}\left(\lambda\cdot|\{x\in[0,1]:\sup_{k\in\mathbb{N}}|S_k(x)|>\lambda\}|\right)=0,$$

where $|A|$ denotes the Lebesgue measure of a set A and

$$S_k(x)=\sum_{j=0}^{k} a_j f_j(x),$$

then the coefficients a_n of the Franklin series can be reconstructed by the following formula

$$a_n=\lim_{\lambda\to\infty}\int_0^1 \left[f(x)\right]_\lambda f_n(x)dx,$$

where

$$\left[f(x)\right]_\lambda=\begin{cases} f(x), & \text{if } |f(x)|\le\lambda,\\ 0, & \text{if } |f(x)|>\lambda.\end{cases}$$

Similar result on uniqueness is also obtained for the Haar system (see [5]). Afterwards Gevorkyan's result was extended by V. Kostin [12] to the series by generalized Haar system.

Consider the d-dimensional Franklin series

$$\sum_{\mathbf{n}\in\mathbb{N}_0^d} a_{\mathbf{n}} f_{\mathbf{n}}(\mathbf{x}),$$

where $\mathbf{n} = (n_1, \cdots, n_d) \in \mathbb{N}_0^d$ is a vector with non-negative integer coordinates, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbf{x} = (x_1, \cdots, x_d) \in [0,1]^d$ and

$$f_{\mathbf{n}}(\mathbf{x}) = f_{n_1}(x_1) \cdot \ldots \cdot f_{n_d}(x_d).$$

The following theorem for multiple Franklin series was proved in [7].

Theorem A.([7]) If the partial sums

$$\sigma_{2^k}(\mathbf{x}) = \sum_{\mathbf{n}: n_i \le 2^k, i=1,\cdots,d} a_{\mathbf{n}} f_{\mathbf{n}}(\mathbf{x})$$

converge in measure to a function f and

$$\lim_{m\to\infty}\left(\lambda_m \cdot |\{\mathbf{x} \in [0,1]^d : \sup_k |\sigma_{2^k}(\mathbf{x})| > \lambda_m\}|\right) = 0$$

for some sequence $\lambda_m \to +\infty$, then for any $\mathbf{n} \in \mathbb{N}_0^d$

$$a_{\mathbf{n}} = \lim_{m\to\infty} \int_{[0,1]^d} \left[f(\mathbf{x})\right]_{\lambda_m} f_{\mathbf{n}}(\mathbf{x}) d\mathbf{x}.$$

In this theorem instead of the partial sums $\sigma_{2^k}(\mathbf{x})$ one can take cubic partial sums $\sigma_{q_k}(\mathbf{x})$, where $\{q_k\}$ is any increasing sequence of natural numbers, for which the ratio q_{k+1}/q_k is bounded. The following theorem is proved in [13].

Theorem B.([13]) Let $\{q_k\}$ be an increasing sequence of natural numbers such that the ratio q_{k+1}/q_k is bounded. If the partial sums $\sigma_{q_k}(\mathbf{x})$ converge in measure to a function f and there exists a sequence $\lambda_m \to +\infty$ so that

$$\lim_{m\to\infty}\left(\lambda_m \cdot |\{\mathbf{x} \in [0,1]^d : \sup_k |\sigma_{q_k}(\mathbf{x})| > \lambda_m\}|\right) = 0,$$

then for any $\mathbf{n} \in \mathbb{N}_0^d$

$$a_{\mathbf{n}} = \lim_{m\to\infty} \int_{[0,1]^d} \left[f(\mathbf{x})\right]_{\lambda_m} f_{\mathbf{n}}(\mathbf{x}) d\mathbf{x}.$$

Note that similar questions for series by Franklin system was considered by K. Keryan in [11].

In this paper we generalize the Theorem A for multiple Ciesielski series.

We are concerned with orthonormal spline systems of order k with dyadic partitions. Let $k \ge 2$ be an integer. For n in the range $-k+2 \le n \le 1$, let $\mathcal{S}_n^{(k)}$ be the space of polynomials of order not exceeding $n+k-1$ (or degree not exceeding $n+k-2$) on the interval $[0,1]$ and $\{f_n^{(k)}\}_{n=-k+2}^{1}$ be the collection of orthonormal polynomials in $L^2 \equiv L^2[0,1]$ such that the degree of $f_n^{(k)}$ is $n+k-2$. For $n \geq 2$, let $n = 2^\nu + j$, where $\nu \geq 0$, $1 \leq j \leq 2^\nu$. Denote

$$s_{n,i} = \begin{cases} 0, \quad -k+1 \leq i \leq 0 \\ \frac{i}{2^{\nu+1}}, \quad 1 \leq i \leq 2j \\ \frac{i-j}{2^\nu}, \quad 2j+1 \leq i \leq n-1 \\ 1, \quad n \leq i \leq n+k-1, \end{cases}$$

and let $\mathcal{T}_n$ be the ordered sequence of points $s_{n,i}$. Note that $\mathcal{T}_n$ is obtained from $\mathcal{T}_{n-1}$ by adding the point $s_{n,2j-1}$. In that case, we also define $\mathcal{S}_n^{(k)}$ to be the space of polynomial splines of order k with grid points $\mathcal{T}_n$. For each $n \geq 2$, the space $\mathcal{S}_{n-1}^{(k)}$ has codimension 1 in $\mathcal{S}_n^{(k)}$ and, therefore, there exists a function $f_n^{(k)} \in \mathcal{S}_n^{(k)}$, that is orthogonal to the space $\mathcal{S}_{n-1}^{(k)}$ and $\|f_n^{(k)}\|_2 = 1$. Observe that this function $f_n^{(k)}$ is unique up to the sign.

The system of functions $\{f_n^{(k)}\}_{n=-k+2}^{\infty}$ is called the Ciesielski system of order k.

Let us note that the case $k=2$ corresponds to orthonormal systems of piecewise linear functions, i.e., the Franklin system.

Let d be a natural number. Consider the d-dimensional Ciesielski series

$$\sum_{\mathbf{n} \in \Lambda^d} a_{\mathbf{n}} f_{\mathbf{n}}(\mathbf{x}), \tag{1.1}$$

where $\mathbf{n} = (n_1, \cdots, n_d) \in \Lambda^d$ is a vector with integer coordinates,
$\Lambda := \{n \in \mathbb{Z} \mid n \geq -k+1\}$, $\mathbf{x} = (x_1, \cdots, x_d) \in [0,1]^d$ and

$$f_{\mathbf{n}}(\mathbf{x}) = f_{n_1}(x_1) \cdot \ldots \cdot f_{n_d}(x_d).$$

Denote by $\sigma_{2^\mu}(\mathbf{x})$ the cubic partial sums of the series (1.1) with indices 2^μ, that is

$$\sigma_{2^\mu}(\mathbf{x}) = \sum_{\mathbf{n}: n_i \leq 2^\mu, i=1,\cdots,d} a_{\mathbf{n}} f_{\mathbf{n}}(\mathbf{x}). \tag{1.2}$$

The main result of this paper is the following theorem:

Theorem 1.1. *If the partial sums $\sigma_{2^\mu}(\boldsymbol{x})$ converge in measure to a function f and*

$$\lim_{q\to\infty} \left(\lambda_q \cdot |\{\boldsymbol{x} \in [0,1]^d : \sup_\mu |\sigma_{2^\mu}(\boldsymbol{x})| > \lambda_q\}| \right) = 0 \tag{1.3}$$

for some sequence $\lambda_q \to +\infty$, then for any $\boldsymbol{n} \in \Lambda^d$

$$a_{\boldsymbol{n}} = \lim_{q\to\infty} \int_{[0,1]^d} \left[f(\boldsymbol{x})\right]_{\lambda_q} f_{\boldsymbol{n}}(\boldsymbol{x}) d\boldsymbol{x}. \tag{1.4}$$

2. Properties of B-spline functions and auxiliary lemmas

We define the functions $(N_{n,i})_{i=-k+1}^{n-1}$ to be the collection of L^∞-normalized B-spline functions of order k corresponding to the partition $\mathcal{T}_n$. The functions $(N_{n,i})_{i=-k+1}^{n-1}$ form a basis for $\mathcal{S}_n^{(k)}$. Those functions are non-negative and are normalized in such a way that they form a partition of unity, i.e.,

$$N_{n,i}(x) \geq 0 \ \text{ and } \sum_{i=-k+1}^{n-1} N_{n,i}(x) = 1 \ \text{ for all } \ x \in [0,1].$$

Moreover

$$\delta_{n,i} := \operatorname{supp} N_{n,i} = [s_{n,i}, s_{n,i+k}] \ \text{ and } \int_0^1 N_{n,i}(x)dx = \frac{|\delta_{n,i}|}{k}.$$

The L^1-normalized B-spline functions $M_{n,i}$ in $\mathcal{S}_n^{(k)}$ are given by the formula

$$M_{n,i}(x) = \frac{k}{|\delta_{n,i}|} N_{n,i}(x),$$

and satisfy the inequalities

$$0 \leq M_{n,i}(x) \leq \frac{k}{|\delta_{n,i}|}, \ \ x \in [0,1].$$

Let $n = 2^\mu + j$, with $\mu \geq 0$, $1 \leq j \leq 2^\mu$. Clearly we have that $N_{n-1,i}(x) = N_{n,i}(x)$, if $-k+1 \leq i \leq 2j-k-2$ and $N_{n-1,i}(x) = N_{n,i+1}(x)$, if $2j-1 \leq i \leq n-2$. Böhm formula (see [15]) gives us the following relationship between the B-splines $N_{n,i}$ and $N_{n-1,i}$, if $2j-k-1 \leq i \leq 2j-2$

$$N_{n-1,i}(x) = a_{n,i}N_{n,i}(x) + (1-a_{n,i+1})N_{n,i+1}(x). \tag{2.1}$$

Later we shall mostly deal with the $n = 2^\mu$, so let us introduce the following notation

$$N_i^{(\mu)}(x) := N_{2^\mu,i}(x), \ \ M_i^{(\mu)}(x) := M_{2^\mu,i}(x), \ \ \delta_i^{(\mu)} := \delta_{2^\mu,i}.$$

For any natural μ we set

$$\Lambda_\mu := \{-k+1, \cdots, 2^\mu\}.$$

It is clear that

$$\sigma_{2^\mu}(\mathbf{x}) = \sum_{\mathbf{n}\in\Lambda_\mu^d} a_{\mathbf{n}} f_{\mathbf{n}}(\mathbf{x}).$$

For any vector $\mathbf{i} = (i_1, \cdots, i_d) \in \Lambda_\mu^d$ denote

$$\Delta_{\mathbf{i}}^{(\mu)} := \delta_{i_1}^{(\mu)} \times \cdots \times \delta_{i_d}^{(\mu)},$$

$$N_{\mathbf{i}}^{(\mu)}(\mathbf{x}) = N_{\mathbf{i}}^{(\mu)}(x_1, \cdots, x_d) = N_{i_1}^{(\mu)}(x_1) \cdot \ldots \cdot N_{i_d}^{(\mu)}(x_d).$$

Obviously

$$\operatorname{supp}(N_{\mathbf{i}}^{(\mu)}) = \Delta_{\mathbf{i}}^{(\mu)}.$$

Let us notice that

$$\int_{[0,1]^d} N_{\mathbf{i}}^{(\mu)}(\mathbf{x})d\mathbf{x} = \int_{\Delta_{\mathbf{i}}^{(\mu)}} N_{\mathbf{i}}^{(\mu)}(\mathbf{x})d\mathbf{x} = \prod_{j=1}^{d} \int_{\delta_{i_j}^{(\mu)}} N_{i_j}^{(\mu)}(x_j)dx_j = \prod_{j=1}^{d} \frac{|\delta_{i_j}^{(\mu)}|}{k} = \frac{|\Delta_{\mathbf{i}}^{(\mu)}|}{k^d}.$$

Hence for $M_{\mathbf{i}}^{(\mu)}(\mathbf{x})$ we have

$$0 \le M_{\mathbf{i}}^{(\mu)}(\mathbf{x}) \le \frac{k^d}{|\Delta_{\mathbf{i}}^{(\mu)}|}, \quad \mathbf{x} \in [0,1]^d.$$

To prove Theorem 1 we will need the following two lemmas.

Lemma 2.1. *Let $P_k(\mathbf{x})$ be a polynomial of degree k defined on $\Delta := [a_1, b_1] \times \cdots \times [a_d, b_d]$, $d \in \mathbb{N}$, then*

$$\left|\left\{\mathbf{x} \in \Delta : \ \ |P_k(\mathbf{x})| \ge \frac{\max_{\mathbf{x}\in\Delta} |P_k(\mathbf{x})|}{2^d}\right\}\right| \ge \frac{|\Delta|}{(4k^2)^d}.$$

This lemma is a generalization of Corollary 3.1 from [9] and the proof of one dimensional case can be found in [8].

Proof. The proof will be carried out by induction. The case $d = 1$ coincides with Corollary 3.1 ([9]). Suppose that lemma is valid for dimension d, and let us prove it for dimension $d + 1$.

Let the function $P_k(\mathbf{x})$ be defined on $\Delta := [a_1, b_1] \times \cdots \times [a_d, b_d]$ and let $|P_k(\mathbf{x})|$ attains its greatest value at the point $(\alpha_1, \cdots, \alpha_{d+1})$. The function $P_k(\alpha_1, \cdots, \alpha_d, x)$, $x \in [a_{d+1}, b_{d+1}]$, satisfies the assumptions of Corollary 3.1 from [9]. Therefore

$$\left|\left\{x \in [a_{d+1}, b_{d+1}] : |P_k(\alpha_1, \cdots, \alpha_d, x)| \ge \frac{1}{2} \cdot \max_{\mathbf{x}\in\Delta} |P_k(\mathbf{x})|\right\}\right| \ge \frac{b_{d+1} - a_{d+1}}{4k^2}. \tag{2.2}$$

For a fixed $x \in [a_{d+1}, b_{d+1}]$, the function

$$P_k(x_1, \cdots, x_d, x), \ \ (x_1, \cdots, x_d) \in [a_1, b_1] \times \cdots \times [a_d, b_d]$$

satisfies the induction assumption. Therefore

$$\left|\left\{(x_1, \cdots, x_d) : x_i \in [a_i, b_i], |P_k(x_1, \cdots, x_d, x)| \ge \frac{|P_k(\alpha_1, \cdots, \alpha_d, x)|}{2^d}\right\}\right| \ge \frac{(b_1 - a_1) \cdots (b_d - a_d)}{(4k^2)^d}. \tag{2.3}$$

It follows from relations (2.2) and (2.3) that

$$\left|\left\{(x_1, \cdots, x_d, x_{d+1}) : x_i \in [a_i, b_i], |P_k(x_1, \cdots, x_d, x_{d+1})| \ge \frac{\max_{\mathbf{x}\in\Delta} |P_k(\mathbf{x})|}{2^{d+1}}\right\}\right| \ge \frac{(b_1 - a_1) \cdots (b_d - a_d)(b_{d+1} - a_{d+1})}{(4k^2)^{d+1}}.$$

The proof is complete. $\square$

Repeatedly using Böhm formula (2.1) one can proof the following lemma (see [9]), which is the generalization of Lemma 2 from [8].

Lemma 2.2. ([9]) *There exist* $\alpha_{ij}^{(\mu)} \geq 0$ *so that*

$$M_i^{(\mu)}(x) = \sum_{j=-k+1}^{2^{\mu+1}-1} \alpha_{ij} M_j^{(\mu+1)}(x), \quad \textit{with} \quad \alpha_{ij} > 0 \quad \textit{iff} \quad \delta_j^{(\mu+1)} \subset \delta_i^{(\mu)}.$$

Lemma 2.3. *There exist* $\alpha_{\mathbf{j}} \geq 0$ *so that*

$$M_{\mathbf{i}}^{(\mu)}(\mathbf{x}) = \sum_{\mathbf{j} \in \Lambda_\mu^d} \alpha_{\mathbf{j}} M_{\mathbf{j}}^{(\mu+1)}(\mathbf{x}), \quad \textit{with} \quad \alpha_{\mathbf{j}} > 0 \quad \textit{iff} \quad \Delta_{\mathbf{j}}^{(\mu+1)} \subset \Delta_{\mathbf{i}}^{(\mu)}.$$

This lemma is the generalization of the previous lemma (for d-dimensional case).

3. The proof of the main theorem

Let the partial sums (1.2) converge in measure to a function f and the series (1.1) satisfy the condition (1.3). First let's prove that for an arbitrary μ_0 and $\mathbf{i}_0 \in \Lambda_{\mu_0}^d$, the following statement is true:

$$\int_{[0,1]^d} \sigma_{2^{\mu_0}}(\mathbf{x}) M_{\mathbf{i}_0}^{(\mu_0)}(\mathbf{x}) d\mathbf{x} = \lim_{q \to \infty} \int_{[0,1]^d} \left[f(\mathbf{x})\right]_{\lambda_q} M_{\mathbf{i}_0}^{(\mu_0)}(\mathbf{x}) d\mathbf{x}. \tag{3.1}$$

Denote

$$E_q := \{\mathbf{x} \in \operatorname{supp}(M_{\mathbf{i}_0}^{(\mu_0)}) = \Delta_{\mathbf{i}_0}^{(\mu_0)} : \sup_\mu |\sigma_{2^\mu}(\mathbf{x})| > \lambda_q\}.$$

Let ε be an arbitrary positive number. Under the conditions of the theorem, one can take the natural number q_0 such that the following inequalities hold:

$$2^{5d} \cdot 2^{\mu_0 d} \cdot k^{2d} \cdot \lambda_q \cdot |E_q| < \varepsilon, \quad \text{when} \quad q \geq q_0, \tag{3.2}$$

and

$$|E_q| < \frac{1}{2^{3d} \cdot k^{3d}} \cdot |\Delta_{\mathbf{i}_0}^{(\mu_0)}|, \quad \text{when} \quad q \geq q_0. \tag{3.3}$$

Suppose $\mu \geq \mu_0$. We set

$$\Omega_\mu := \left\{ A : A = \left[\frac{i_1}{2^\mu}, \frac{i_1+1}{2^\mu}\right] \times \cdots \times \left[\frac{i_d}{2^\mu}, \frac{i_d+1}{2^\mu}\right], \quad A \subset \operatorname{supp}(M_{\mathbf{i}_0}^{(\mu_0)}) \right\}.$$

Notice, that if for some $A \in \Omega_\mu$, $\mu \geq \mu_0$, the inequality

$$|E_q \cap A| < \frac{1}{2 \cdot 4^d \cdot k^{2d}} \cdot |A| \tag{3.4}$$

holds, then

$$|\sigma_{2^\mu}(\mathbf{x})| \leq 2^d \lambda_q, \quad \text{for} \quad \mathbf{x} \in A. \tag{3.5}$$

Let suppose that $A \in \Omega_\mu$ and for some point $\mathbf{x}' \in A$ the inequality (3.5) does not hold, i.e.

$$|\sigma_{2^\mu}(\mathbf{x}')| > 2^d \lambda_q.$$

According to the Lemma (2.1), we obtain that

$$\left|\left\{\mathbf{x} \in A : |\sigma_{2^\mu}(\mathbf{x})| > \lambda_q\right\}\right| \geq \frac{|A|}{4^d \cdot k^{2d}},$$

which contradicts (3.4). From (3.3) we have

$$|E_q \cap A| < \frac{1}{2^{3d} \cdot k^{3d}} \cdot |\Delta_{\mathbf{i}_0}^{(\mu_0)}| = \frac{1}{2^{3d} \cdot k^{2d}} \cdot |A|, \quad \text{when} \quad A \in \Omega_{\mu_0}. \tag{3.6}$$

Now let's define by induction the families Ω_μ^1 and Ω_μ^2, $\mu \geq \mu_0$. If $\mu = \mu_0$, then we set

$$\Omega_{\mu_0}^1 := \left\{A \in \Omega_{\mu_0} : |A \cap E_q| > \frac{1}{2^{3d} \cdot k^{2d}} \cdot |A|\right\}, \quad Q_{\mu_0} := \bigcup_{A \in \Omega_{\mu_0}^1} A,$$

and

$$\Omega_{\mu_0}^2 := \{A \in \Omega_{\mu_0} : A \not\subset Q_{\mu_0}\}, \quad P_{\mu_0} := \bigcup_{A \in \Omega_{\mu_0}^2} A.$$

From (3.6) we have, that $Q_{\mu_0} = \emptyset$ and the closure of P_{μ_0} is the $\operatorname{supp}(M_{\mathbf{i}_0}^{(\mu_0)})$. Now suppose we have defined the sets $\Omega_{\mu'}^1$, $\Omega_{\mu'}^2$, $Q_{\mu'}$ and $P_{\mu'}$ for all $\mu' < \mu$.
Let's denote

$$\Omega_\mu^1 := \left\{A \in \Omega_\mu : |A \cap E_q| > \frac{1}{2^{3d} \cdot k^{2d}} \cdot |A| \ \text{ and } \ A \not\subset \bigcup_{\mu' < \mu} Q_{\mu'}\right\}, \tag{3.7}$$

$$Q_\mu := \bigcup_{A \in \Omega_\mu^1} A, \quad \Omega_\mu^2 := \left\{A \in \Omega_\mu : A \not\subset \bigcup_{\mu' \leq \mu} Q_{\mu'}\right\}, \quad P_\mu := \bigcup_{A \in \Omega_\mu^2} A.$$

Thus we have defined the families $\Omega_\mu^1, \Omega_\mu^2$ and the sets Q_μ, P_μ, satisfying to the following conditions

$$\Omega_\mu^1 \subset \Omega_\mu, \quad \Omega_\mu^2 \subset \Omega_\mu,$$

$$\operatorname{supp}(M_{\mathbf{i}_0}^{(\mu_0)}) = P_\mu \cup \left(\bigcup_{\mu' \leq \mu} Q_{\mu'}\right), \quad P_\mu \cap \left(\bigcup_{\mu' \leq \mu} Q_{\mu'}\right) = \emptyset, \tag{3.8}$$

$$Q_{\mu'} \cap Q_{\mu''} = \emptyset, \quad \text{if} \quad \mu' \neq \mu''. \tag{3.9}$$

Next, it follows from (3.7) and (3.9) that

$$\left|\bigcup_{\mu' \leq \mu} Q_{\mu'}\right| < 2^{3d} \cdot k^{2d} \cdot |E_q|, \quad \text{for any} \quad \mu \geq \mu_0. \tag{3.10}$$

For any $\mu > \mu_0$ denote

$$I_\mu = \{\mathbf{i} \in \Lambda_\mu^d : \ \Delta_{\mathbf{i}}^{(\mu)} \cap Q_\mu \neq \emptyset \ \text{ and } \ \Delta_{\mathbf{i}}^{(\mu)} \subset P_{\mu-1}\},$$

and observe that if $\mathbf{i} = (i_1, \cdots, i_d) \in I_\mu$ then, for any set B, $B \subset \Delta_{\mathbf{i}}^{(\mu)}$, $B \in \Omega_\mu$ the following inequality holds:

$$|E_q \cap B| < \frac{1}{4^d \cdot k^{2d}} \cdot |B|. \tag{3.11}$$

Indeed, if for some B the inequality (3.11) is not satisfied, then for a cube $D \in \Omega_{\mu-1}$, with $B \subset D$, we would have

$$|E_q \cap D| \geq \frac{1}{2^d \cdot 4^d \cdot k^{2d}} \cdot |D|, \tag{3.12}$$

because

$$|D| = 2^d \cdot |B|.$$

Then, it follows from (3.12) that

$$D \subset \bigcup_{\mu'<\mu} Q_{\mu'}, \quad \text{therefore} \quad B \subset \bigcup_{\mu'<\mu} Q_{\mu'} \quad \text{and} \quad \Delta_{\mathbf{i}}^{(\mu)} \cap \left(\bigcup_{\mu'<\mu} Q_{\mu'} \right) \neq \emptyset,$$

which contradicts the condition $\Delta_{\mathbf{i}}^{(\mu)} \subset P_{\mu-1}$ and the relation (3.12).
Therefore

$$|\sigma_{2^\mu}(\mathbf{x})| \leq 2^d \cdot \lambda_q, \quad \text{if} \quad \mathbf{x} \in \Delta_{\mathbf{i}}^{(\mu)}, \mathbf{i} \in I_\mu. \tag{3.13}$$

Similarly, we can obtain that if $\Delta_{\mathbf{i}}^{(\mu)} \subset P_\mu$, then

$$\left| E_q \cap \Delta_{\mathbf{i}}^{(\mu)} \right| \leq \frac{1}{2^{3d} \cdot k^{2d}} \cdot |\Delta_{\mathbf{i}}^{(\mu)}|, \tag{3.14}$$

therefore

$$|\sigma_{2^\mu}(\mathbf{x})| \leq 2^d \cdot \lambda_q, \quad \text{if} \quad \mathbf{x} \in \Delta_{\mathbf{i}}^{(\mu)} \subset P_\mu. \tag{3.15}$$

Now by induction we define expansions ψ_μ for $M_{\mathbf{i}_0}^{(\mu_0)}$, satisfying the conditions:

$$M_{\mathbf{i}_0}^{(\mu_0)} = \psi_\mu = \sum_{n=\mu_0}^{\mu} \sum_{\mathbf{i} \in I_\mu} \alpha_{\mathbf{i}}^{(n)} M_{\mathbf{i}}^{(n)} + \sum_{\mathbf{i}:\Delta_{\mathbf{i}}^{(\mu)} \subset P_\mu} \beta_{\mathbf{i}}^{(\mu)} M_{\mathbf{i}}^{(\mu)}, \tag{3.16}$$

where

$$\sum_{n=\mu_0}^{\mu} \sum_{\mathbf{i} \in I_\mu} \alpha_{\mathbf{i}}^{(n)} + \sum_{\mathbf{i}:\Delta_{\mathbf{i}}^{(\mu)} \subset P_\mu} \beta_{\mathbf{i}}^{(\mu)} = 1, \quad \alpha_{\mathbf{i}}^{(n)} \geq 0, \ \beta_{\mathbf{i}}^{(\mu)} \geq 0. \tag{3.17}$$

Since $P_{\mu_0} = \operatorname{supp}(M_{\mathbf{i}_0}^{(\mu_0)})$, then $\psi_{\mu_0} = M_{\mathbf{i}_0}^{(\mu_0)}$. Assuming that ψ_μ, satisfying the conditions (3.16), (3.17), is already defined, we define $\psi_{\mu+1}$. By Lemma (2.3), we have

$$M_{\mathbf{i}}^{(\mu)} = \sum_{\mathbf{i}:\Delta_{\mathbf{i}}^{(\mu+1)} \subset \operatorname{supp} M_{\mathbf{i}}^{(\mu)}} \gamma_{\mathbf{i}}^{(\mu+1)} M_{\mathbf{i}}^{(\mu+1)}, \quad \gamma_{\mathbf{i}}^{(\mu+1)} \geq 0. \tag{3.18}$$

Inserting the expressions (3.18) in (3.16) and grouping similar terms, we obtain

$$M_{\mathbf{i}_0}^{(\mu_0)} = \psi_{\mu+1} = \sum_{n=\mu_0}^{\mu+1} \sum_{\mathbf{i} \in I_\mu} \alpha_{\mathbf{i}}^{(n)} M_{\mathbf{i}}^{(n)} + \sum_{\mathbf{i}:\Delta_{\mathbf{i}}^{(\mu+1)} \subset P_{\mu+1}} \alpha_{\mathbf{i}}^{(\mu+1)} M_{\mathbf{i}}^{(\mu+1)}. \tag{3.19}$$

Since the integrals of all functions $M_{\mathbf{i}}^{(\mu)}$ are equal to one, from (3.19) we obtain

$$\sum_{n=\mu_0}^{\mu+1}\sum_{\mathbf{i}\in I_\mu}\alpha_{\mathbf{i}}^{(n)}+\sum_{\mathbf{i}:\Delta_{\mathbf{i}}^{(\mu+1)}\subset P_{\mu+1}}\alpha_{\mathbf{i}}^{(\mu+1)}=1,\quad \alpha_{\mathbf{i}}^{(n)}\geq 0,\alpha_{\mathbf{i}}^{(\mu+1)}\geq 0.$$

Thus, the possibility of representation (3.16) with coefficients satisfying (3.17), is proved.

Suppose we are given a number $\mu\geq\mu_0$ and $\mathbf{p}=(p_1,\cdots,p_d)$ such that $\max_i\{p_i\}>2^\mu$. Then, according to the definition of functions $f_{\mathbf{p}}$ and $M_{\mathbf{i}}^{\mu}$, we get

$$(f_{\mathbf{p}},M_{\mathbf{i}}^{(\mu)})=\int_{[0,1]^d}f_{\mathbf{p}}(\mathbf{x})M_{\mathbf{i}}^{(\mu)}(\mathbf{x})d\mathbf{x}=0,\quad \text{for any }\ \mathbf{i}\in\Lambda_\mu^d.$$

Therefore, for any $n\geq\mu$ and for all $\mathbf{i}\in\Lambda_\mu^d$ one can write

$$(\sigma_{2^n},M_{\mathbf{i}}^{(\mu)})=\sum_{\mathbf{p}\in\Lambda_n^d}a_{\mathbf{p}}(f_{\mathbf{p}},M_{\mathbf{i}}^{(\mu)})=\sum_{\mathbf{p}\in\Lambda_\mu^d}a_{\mathbf{p}}(f_{\mathbf{p}},M_{\mathbf{i}}^{(\mu)})=(\sigma_{2^\mu},M_{\mathbf{i}}^{(\mu)}).$$

Taking into account (3.16), for $n>\mu_0$ we can write

$$(\sigma_{2^{\mu_0}},M_{\mathbf{i}_0}^{(\mu_0)})=\int_{[0,1]^d}\sigma_{2^{\mu_0}}(\mathbf{x})M_{\mathbf{i}_0}^{(\mu_0)}(\mathbf{x})d\mathbf{x}=\int_{\Delta_{\mathbf{i}_0}^{(\mu_0)}}\sigma_{2^\mu}(\mathbf{x})M_{\mathbf{i}_0}^{(\mu_0)}(\mathbf{x})d\mathbf{x}$$

$$=\sum_{n=\mu_0}^{\mu}\sum_{\mathbf{i}\in I_\mu}\alpha_{\mathbf{i}}^{(n)}(\sigma_{2^n},M_{\mathbf{i}}^{(n)})+\sum_{\mathbf{i}:\Delta_{\mathbf{i}}^{(\mu)}\subset P_\mu}\beta_{\mathbf{i}}^{(\mu)}(\sigma_{2^\mu},M_{\mathbf{i}}^{(\mu)})=:I_{\mu,1}+I_{\mu,2}.$$

For $I_{\mu,1}$ we will have the inequality

$$|I_{\mu,1}|\leq\sum_{n=\mu_0}^{\mu}\sum_{\mathbf{i}\in I_\mu}\alpha_{\mathbf{i}}^{(n)}|(\sigma_{2^n},M_{\mathbf{i}}^{(n)})|\leq 2^d\lambda_q\sum_{n=\mu_0}^{\mu}\sum_{\mathbf{i}\in I_\mu}\alpha_{\mathbf{i}}^{(n)}\int_{\Delta_{\mathbf{i}}^{(n)}}M_{\mathbf{i}}^{(n)}(\mathbf{x})d\mathbf{x}.$$

Denote

$$D_\mu=\bigcup_{n=\mu_0}^{\mu}\bigcup_{\mathbf{i}\in I_\mu}\Delta_{\mathbf{i}}^{(n)}. \tag{3.20}$$

From the definition of the set I_μ, it follows that

$$|\Delta_{\mathbf{i}}^{(n)}\cap Q_\mu|\geq k^{-d}\Delta_{\mathbf{i}}^{(n)}.$$

The last relation and (3.9), (3.10) imply

$$|D_\mu|\leq k^d\cdot 2^{3d}\cdot k^{2d}\cdot|E_q|. \tag{3.21}$$

We obtain

$$\sum_{n=\mu_0}^{\mu}\sum_{\mathbf{i}\in I_\mu}\alpha_{\mathbf{i}}^{(n)}\int_{\Delta_{\mathbf{i}}^{(n)}}M_{\mathbf{i}}^{(n)}(\mathbf{x})d\mathbf{x}\leq\int_{D_\mu}M_{\mathbf{i}_0}^{(\mu_0)}(\mathbf{x})d\mathbf{x}\leq||M_{\mathbf{i}_0}^{(\mu_0)}||_\infty|D_\mu|. \tag{3.22}$$

It is clear that

$$|I_{\mu,1}|\leq 2^d\cdot\lambda_q\cdot||M_{\mathbf{i}_0}^{(\mu_0)}||_\infty|D_\mu|. \tag{3.23}$$

Hence, from (3.23), (3.21) and (3.2), we obtain

$$|I_{\mu,1}| \le 2^d \cdot \lambda_q \cdot \frac{k^d}{|\Delta_{\mathbf{i}_0}^{(\mu_0)}|} \cdot k^d \cdot 2^{3d} \cdot k^{2d} \cdot |E_q| \le \varepsilon \cdot \left(\frac{k}{2}\right)^d. \tag{3.24}$$

For $I_{\mu,2}$ we will have the representation

$$I_{\mu,2} = (\sigma_{2^\mu}, \sum_{\mathbf{i}:\Delta_{\mathbf{i}}^{(\mu)} \subset P_\mu} \beta_{\mathbf{i}}^{(\mu)} M_{\mathbf{i}}^{(\mu)}) = (\sigma_{2^\mu} - \big[f\big]_{\lambda_q}, \sum_{\mathbf{i}:\Delta_{\mathbf{i}}^{(\mu)} \subset P_\mu} \beta_{\mathbf{i}}^{(\mu)} M_{\mathbf{i}}^{(\mu)})$$
$$+(\big[f\big]_{\lambda_q}, \sum_{\mathbf{i}:\Delta_{\mathbf{i}}^{(\mu)} \subset P_\mu} \beta_{\mathbf{i}}^{(\mu)} M_{\mathbf{i}}^{(\mu)}) =: I_{\mu,3} + I_{\mu,4}.$$

Denote

$$H_\mu = \bigcup_{\mathbf{i}:\Delta_{\mathbf{i}}^{(\mu)} \subset P_\mu} \Delta_{\mathbf{i}}^{(\mu)} \quad \text{and} \quad T_q = \{\mathbf{x} \in \Delta_{\mathbf{i}_0}^{(\mu_0)} : \quad |f(\mathbf{x})| > \lambda_q\}.$$

It is clear that $T_q \subset E_q$, therefore $|T_q| < |E_q|$. From (3.2) we get

$$|T_q| < \frac{\varepsilon}{2^{5d} \cdot 2^{\mu_0 d} \cdot k^{2d} \cdot \lambda_q}. \tag{3.25}$$

Next from (3.15) we have $|\sigma_{2^\mu}(\mathbf{x})| \le 2^d \cdot \lambda_q$, for $\mathbf{x} \in H_\mu$, and hence

$$|\sigma_{2^\mu}(\mathbf{x}) - \big[f(\mathbf{x})\big]_{\lambda_q}| \le (2^d+1) \cdot \lambda_q, \quad \text{for} \quad \mathbf{x} \in H_\mu. \tag{3.26}$$

It is clear that

$$|I_{\mu,3}| \le (|\sigma_{2^\mu}(\mathbf{x}) - \big[f(\mathbf{x})\big]_{\lambda_q}|, \sum_{\mathbf{i}:\Delta_{\mathbf{i}}^{(\mu)} \subset P_\mu} \beta_{\mathbf{i}}^{(\mu)} M_{\mathbf{i}}^{(\mu)}) \tag{3.27}$$
$$\le \int\limits_{H_\mu} |\sigma_{2^\mu}(\mathbf{x}) - \big[f(\mathbf{x})\big]_{\lambda_q}| M_{\mathbf{i}_0}^{(\mu_0)}(\mathbf{x}) d\mathbf{x} \le 2^{\mu_0 d} \int\limits_{H_\mu} |\sigma_{2^\mu}(\mathbf{x}) - \big[f(\mathbf{x})\big]_{\lambda_q}| d\mathbf{x}$$
$$= 2^{\mu_0 d} \left(\int\limits_{H_\mu \setminus T_q} |\sigma_{2^\mu}(\mathbf{x}) - \big[f(\mathbf{x})\big]_{\lambda_q}| d\mathbf{x} + \int\limits_{H_\mu \cap T_q} |\sigma_{2^\mu}(\mathbf{x}) - \big[f(\mathbf{x})\big]_{\lambda_q}| d\mathbf{x} \right).$$

From (3.25) and (3.26) for the second integral on the right-hand side of (3.27), we have

$$2^{\mu_0 d} \int\limits_{H_\mu \cap T_q} |\sigma_{2^\mu}(\mathbf{x}) - \big[f(\mathbf{x})\big]_{\lambda_q}| d\mathbf{x} \le \frac{\varepsilon}{2^{5d}}.$$

From (3.26) we have that the $|\sigma_{2^\mu}(\mathbf{x}) - \big[f(\mathbf{x})\big]_{\lambda_q}|$ is bounded on H_μ and it tends to zero in measure outside the set T_q, since

$$|\sigma_{2^\mu}(\mathbf{x}) - \big[f(\mathbf{x})\big]_{\lambda_q}| = |\sigma_{2^\mu}(\mathbf{x}) - f(\mathbf{x})| \ \text{ on } \ T_q^c.$$

Hence

$$\int\limits_{H_\mu \setminus T_q} |\sigma_{2^\mu}(\mathbf{x}) - \big[f(\mathbf{x})\big]_{\lambda_q}| d\mathbf{x} \xrightarrow[\mu \to \infty]{} 0.$$

Therefore, for sufficiently large μ we have

$$|I_{\mu,3}| < \frac{\varepsilon}{2}. \tag{3.28}$$

For $I_{\mu,4}$, from (3.16) we have

$$I_{\mu,4} = (\left[f\right]_{\lambda_q}, M_{\mathbf{i}_0}^{(\mu_0)}) - (\left[f\right]_{\lambda_q}, \sum_{n=\mu_0}^{\mu} \sum_{\mathbf{i}\in I_\mu} \alpha_{\mathbf{i}}^{(n)} M_{\mathbf{i}}^{(n)}) \tag{3.29}$$

$$= (\left[f\right]_{\lambda_q}, M_{\mathbf{i}_0}^{(\mu_0)}) + I_{\mu,5}.$$

The relations (3.2),(3.21),(3.22) imply that

$$|I_{\mu,5}| \leq \varepsilon \cdot \left(\frac{k}{2}\right)^d. \tag{3.30}$$

Therefore by (3.24), (3.28), (3.29), (3.30) we get

$$\left|\int_{[0,1]^d} \sigma_{2^{\mu_0}}(\mathbf{x}) M_{\mathbf{i}_0}^{(\mu_0)}(\mathbf{x}) d\mathbf{x} - \int_{[0,1]^d} \left[f(\mathbf{x})\right]_{\lambda_q} M_{\mathbf{i}_0}^{(\mu_0)}(\mathbf{x}) d\mathbf{x}\right| < C_{k,d} \cdot \varepsilon \ \text{ for } \ q \geq q_0.$$

Now let's prove that for any $\mathbf{n} \in \Lambda^d$ the coefficient $a_{\mathbf{n}}$ can be reconstructed by (1.4). First let's fix a number μ satisfying $\max_{1\leq i\leq d} n_i \leq 2^\mu$. Since $f_{\mathbf{n}} \in \mathcal{S}_{2^\mu}$ and the system of functions $\{M_{\mathbf{i}}^{(\mu)}\}_{\mathbf{i}\in\Lambda_\mu^d}$ is a basis in the space $\mathcal{S}_{2^\mu}$, then one can find numbers $\beta_{\mathbf{i}}$, $\mathbf{i} \in \Lambda_\mu^d$, such that

$$f_{\mathbf{n}}(\mathbf{x}) = \sum_{\mathbf{i}\in\Lambda_\mu^d} \beta_{\mathbf{i}} M_{\mathbf{i}}^{(\mu)}(\mathbf{x}).$$

Therefore

$$a_{\mathbf{n}} = (\sigma_{2^\mu}, f_{\mathbf{n}}) = \sum_{\mathbf{i}\in\Lambda_\mu^d} \beta_{\mathbf{i}} (\sigma_{2^\mu}, M_{\mathbf{i}}^{(\mu)}) = \sum_{\mathbf{i}\in\Lambda_\mu^d} \beta_{\mathbf{i}} \lim_{q\to\infty} \int_{[0,1]^d} \left[f(\mathbf{x})\right]_{\lambda_q} M_{\mathbf{i}}^{(\mu)}(\mathbf{x}) d\mathbf{x}$$

$$= \lim_{q\to\infty} \int_{[0,1]^d} \left[f(\mathbf{x})\right]_{\lambda_q} f_{\mathbf{n}}(\mathbf{x}) d\mathbf{x},$$

which finishes the proof of Theorem 1.1.

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