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On the Minimal Fragment of S5 Modal Logic

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1. Introduction. In the theory of automated theorem proving systems of constructive logic are of a special interest due to an ability to extract rigorous information from the constructed proof. Minimal logic being part of intuitionistic logic attracts a special interest of a research community. In this paper we construct new propositional systems for minimal fragment of modal logics by introducing modality rules. Intuitionistic modal logics originate from different sources and have different areas of application. They include philosophy (see, e.g., [1]), foundation of mathematics [2], and computer science [3]. One of the considered systems is based on the sequential system of minimal logic introduced earlier in [4]. Two kinds of logical symbols are added to GM : $\Box$ (necessary) and $\Diamond$(possible). Using them the notion of formula is extended as follows: if α is a formula, then $\Box\alpha$ and $\Diamond\alpha$ are also formulas.

New modality rules are:

$$\frac{\Gamma \to \alpha}{\Gamma \to \Diamond\alpha}(\to \Diamond) \qquad \frac{\alpha, \Gamma \to \perp}{\Diamond\alpha, \Gamma \to \perp}(\Diamond \to)$$

$$\frac{\alpha, \Gamma \to \perp}{\Box\alpha, \Gamma \to \perp}(\Box \to) \qquad \frac{\Gamma \to \alpha}{\Gamma \to \Box\alpha}(\to \Box)$$

In these rules Γ is a set of formulae as in GM [7]. $\Box\Gamma$ ($\Diamond\Gamma$) means the series of formulae, which is formed by prefixing $\Box$ ($\Diamond$) in front of each formulae of Γ. As a result, a sequential calculus based on GM is constructed, which we call $S5^*_{Min}$.

It is obvious that $\Diamond$ and $\Box$ rules are symmetric in this system. As $\Diamond\alpha$ is $\neg\Box\neg\alpha$ it can be easily verified that the rules of $\Box$ can be derived from the corresponding rules of $\Diamond$ and vice versa.

2. Minimal fragment of S5 modal logic. In this section using notions and concepts from [5-8] the definition of the minimal fragment of the S5 modal logic (further in the text it is denoted as $S5_{Min}$) will be provided.

1. Signs:
   1.1. The constants $\neg$, & ,$\vee$, $\supset$ and $\leftrightarrow$ of propositional logic.
   1.2. The constants $\Box$ and $\Diamond$ of modal logic.
   1.3. Sentence-variables , $b, c, \dots$.
2. Rules of Formation:
   2.1. A sentence-variable is a formula.
   2.2. A formula preceded by $\neg$, by $\Box$ or by $\Diamond$ is a formula.
   2.3. Two formulae joined by & ,$\vee$, $\supset$ or $\leftrightarrow$ constitute a formula.
3. Axioms:
   3.1. A set of axioms for minimal fragment of propositional logic.
   3.2. $\alpha \supset \Diamond\alpha$ (The axiom of Possibility).
   3.3. $\Diamond(\alpha \vee \beta) \leftrightarrow \Diamond\alpha \vee \Diamond\beta$ (The axiom of Distribution).
   3.4. $\Diamond\neg\Diamond\alpha \supset \neg\Diamond\alpha$ (The axiom of Reduction).
4. Definition of "necessary":
   The constant $\Box$ we introduce by the definition $\Box\alpha = \neg\Diamond\neg\alpha$.
5. Rules of Transformation:
   5.1. The rules of transformation of minimal fragment of propositional logic.
   5.2. If $f_1 \leftrightarrow f_2$ is provable, then $\Diamond f_1 \leftrightarrow \Diamond f_2$ is also provable. (The rule of Extentionality)
   5.3. If f is provable, then $\Box f$ is also provable. (The rule of Tautology)

Equivalence of the modal systems. In this section we prove the equivalence of the above formulated systems. We follow the notion of systems equivalence as in [7]. The proof of equivalence between $S5_{Min}$ and $S5^*_{Min}$ will be divided into two parts. The first part will show that the axioms and the rules of $S5_{Min}$ system can be derived directly from the axioms and rules of $S5^*_{Min}$ system, while the second one will show that the axioms and rules of $S5^*_{Min}$ system can be derived from axioms and rules of $S5_{Min}$.

Theorem 3.1. *If a sequence $\rightarrow \alpha$ is deducible in $S5^*_{Min}$ then α is deducible in $S5_{Min}$.*

Proof. The theorem may be proved by showing that all the rules described in definition of section 2 are satisfied in $S5^*_{Min}$ system. 1, 2 and 4 parts of the definition are the same for both systems, so one only needs to prove points 3 and 5 of the definition to be satisfied in $S5^*_{Min}$ system.

Axioms of group 3.1 are the same as they are in propositional fragment of minimal logic described in [9].

It is obvious that the axiom 3.2 is provable in $S5^*_{Min}$ system as:

$$\frac{\alpha \to \alpha}{\alpha \to \Diamond\alpha}\,(\to \Diamond)$$

Now it is necessary to show that Axiom 3.3 can be proved in $S5^{*}_{Min}$. It goes in the following way:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \to \alpha}{\Box\neg\alpha \& \Box\neg\beta,\ \alpha \vee \beta,\ \alpha \to \alpha}}{\Box\neg\alpha \& \Box\neg\beta,\ \neg\alpha,\ \alpha \vee \beta,\ \alpha \to \bot}}{\Box\neg\alpha \& \Box\neg\beta,\ \Box\neg\alpha,\ \alpha \vee \beta,\ \alpha \to \bot}}{\Box\neg\alpha \& \Box\neg\beta,\ \alpha \vee \beta,\ \alpha \to \bot} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\beta \to \beta}{\Box\neg\alpha \& \Box\neg\beta,\ \alpha \vee \beta,\ \beta \to \beta}}{\Box\neg\alpha \& \Box\neg\beta,\ \neg\beta,\ \alpha \vee \beta,\ \beta \to \bot}}{\Box\neg\alpha \& \Box\neg\beta,\ \Box\neg\beta,\ \alpha \vee \beta,\ \beta \to \bot}}{\Box\neg\alpha \& \Box\neg\beta,\ \alpha \vee \beta,\ \beta \to \bot}}{\Box\neg\alpha \& \Box\neg\beta,\ \alpha \vee \beta \to \bot}}{\Box\neg\alpha \& \Box\neg\beta,\ \Diamond(\alpha \vee \beta) \to \bot}}{\Diamond(\alpha \vee \beta) \to \neg(\Box\neg\alpha \& \Box\neg\beta)}}{\Diamond(\alpha \vee \beta) \to \neg\Box\neg\alpha \vee \neg\Box\neg\beta}}{\Diamond(\alpha \vee \beta) \to \Diamond\alpha \vee \Diamond\beta}$$

Now the left part of the equivalence needs to be proved.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \to \alpha}{\alpha \to \alpha \vee \beta}}{\alpha, \neg(\alpha \vee \beta) \to \bot}}{\alpha, \Box\neg(\alpha \vee \beta) \to \bot}}{\Diamond\alpha, \Box\neg(\alpha \vee \beta) \to \bot}}{\Diamond\alpha \to \neg\Box\neg(\alpha \vee \beta)}}{\Diamond\alpha \to \Diamond(\alpha \vee \beta)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\beta \to \beta}{\beta \to \alpha \vee \beta}}{\beta, \neg(\alpha \vee \beta) \to \bot}}{\beta, \Box\neg(\alpha \vee \beta) \to \bot}}{\Diamond\beta, \Box\neg(\alpha \vee \beta) \to \bot}}{\Diamond\beta \to \neg\Box\neg(\alpha \vee \beta)}}{\Diamond\beta \to \neg\Box\neg(\alpha \vee \beta)}}{\Diamond\alpha \vee \Diamond\beta \to \Diamond(\alpha \vee \beta)}$$

So, the axiom of distribution is proved.

Axiom 3.4:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \to \alpha}{\alpha \to \Diamond\alpha}\,(\to \Diamond)}{\alpha,\ \neg\Diamond\alpha \to \bot}\,(\neg\to)}{\Diamond\alpha,\ \neg\Diamond\alpha \to \bot}\,(\Diamond \to)}{\Diamond\alpha,\ \Diamond\neg\Diamond\alpha \to \bot}\,(\Diamond \to)}{\Diamond\neg\Diamond\alpha \to \neg\Diamond\alpha}\,(\to \neg)$$

The axiom of reduction is proved.

Axioms of group 5.1 are the same as they are in propositional fragment of minimal logic described in [4].

Axiom 5.2:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\alpha \to \beta}{\alpha, \neg\beta \to \bot}\,(\neg\to)}{\alpha, \Box\neg\beta \to \bot}\,(\Box \to)}{\Diamond\alpha, \Box\neg\beta \to \bot}\,(\Diamond \to)}{\Diamond\alpha \to \neg\Box\neg\beta}\,(\to \neg)}{\Diamond\alpha \to \Diamond\beta}$$

So, it becomes obvious, that formula $\Diamond f_1 \leftrightarrow \Diamond f_2$ is provable in system $S5^*_{Min}$, only if $f_1 \leftrightarrow f_2$ is also provable in that system. The rule of extentionality is proved.

Axiom 5.3:

$$\cfrac{\cfrac{\cfrac{\cfrac{\to \alpha}{\neg\alpha \to \bot}\,(\neg\to)}{\Diamond\neg\alpha \to \bot}\,(\Diamond \to)}{\to \neg\Diamond\neg\alpha}\,(\to \neg)}{\to \Box\alpha}$$

Formula $\Box f$ is provable only if f is provable. The rule of tautology is proved.

Theorem 3.2. *If a formula α is deducible in the $S5_{Min}$ system then a sequence $\to \alpha$ is deducible in the $S5^*_{Min}$ system.*

Proof. The theorem will be proved by showing that the rules $(\to \Diamond)$, $(\Diamond \to)$, $(\to \Box)$, and $(\Box \to)$ can be derived from rules of the system $S5_{Min}$.

Firstly, let's prove that:

$$\frac{\Gamma \to \alpha}{\Gamma \to \Diamond\alpha}\,(\to \Diamond)$$

That is:

$$\frac{\alpha,\ \beta \to \gamma}{\alpha,\ \beta \to \Diamond\gamma}\,(\to \Diamond)$$

In other words, we only need to prove that if $(\alpha \vee \beta) \to \gamma$ then $(\alpha \vee \beta) \to \Diamond\gamma$ in $S5_{Min}$. Suppose that $(\alpha \vee \beta) \to \gamma$. Using the axiom of Possibility $(\gamma \to \Diamond\gamma)$ and applying the modus ponens rule we get $(\alpha \vee \beta) \to \Diamond\gamma$.

Next the rule $(\to \Box)$ needs to be proved:

$$\frac{\Gamma \to \alpha}{\Gamma \to \Box\alpha}\,(\to \Box)$$

So, it needs to be proved that:

$$\frac{\alpha,\ \beta \to \gamma}{\alpha,\ \beta \to \Box\gamma}\,(\to \Box)$$

In other words, we only need to prove that if $(\alpha \vee \beta) \to \gamma$ then $(\alpha \vee \beta) \to \Box\gamma$.

So, $(\alpha \vee \beta) \to \gamma$. On the other hand, using rule 5.3 we get $\gamma \to \Box\gamma$. Using modus ponens rule we can conclude that $(\alpha \vee \beta) \to \Box\gamma$ in $S5_{Min}$.

It can be easily verified that the rules $(\Box \to)$, $(\Diamond \to)$ can be derived from the corresponding rules of $(\to \Diamond)$, $(\to \Box)$ as:

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha, \Gamma \to \perp}{\Gamma \to \neg\alpha}}{\Gamma \to \Diamond\neg\alpha}}{\neg\Diamond\neg\alpha, \Gamma \to \perp}}{\Box\alpha, \Gamma \to \perp} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\alpha, \Gamma \to \perp}{\Gamma \to \neg\alpha}}{\Gamma \to \Box\neg\alpha}}{\neg\Box\neg\alpha, \Gamma \to \perp}}{\Diamond\alpha, \Gamma \to \perp}$$

From theorems 2.1 and 2.2, we can conclude the following one.

Theorem 3.3. *A formula α is deducible in $S5_{Min}$ if and only if $\to \alpha$ is deducible in $S5^*_{Min}$.*

Conclusion. Two systems of minimal modal logic are introduced and their equivalence is proved. Those systems may serve as a basis for automated provers in minimal systems for modal logic.

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On the Minimal Fragment of S5 Modal Logic

Two systems of propositional fragment of modal logic are constructed. One of the systems is Herbrand type while the other one is a sequential system. It is proved that every formula deducible in one of them is also provable in the other system.

Հ. Ռ. Բոլիբեկյան, Ա. Ռ. Բաղդասարյան

S5 մոդալ տրամաբանության նվազագույն ֆրագմենտի վերաբերյալ

Սահմանված են մոդալ տրամաբանության երկու ասույթային համակարգեր: Նրանցից առաջինը հերբրանյան տիպի է, իսկ մյուսը՝ սեկվենսային: Ապացուցված է այդ համակարգերի համարժեքությունը:

О. Р. Болибекян, А. Р. Багдасарян

О минимальном фрагменте S5 модальной логики

Сформулированы две пропозициональные системы модальной логики: эрбрановского типа и секвенциальная. Доказана формульная равнообъёмность рассмотренных систем.

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