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ON VALUE DISTRIBUTION OF A CLASS OF ENTIRE FUNCTIONS

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Abstract. We study uniqueness problems in terms of shared values or shared sets for a large class of entire functions representable as Dirichlet series in some right half-plane. In this article, we obtain a result that extends a recent result due to Oswald and Steuding [Annales Univ. Sci. Budapest., Sect. Comp., 48 (2018), 117-128]. Our result is also a variant of a result of Yuan-Li-Yi [Lithuanian Math. J., 58 (2018), 249-262], and a result of the present authors [Lithuanian Math. J., 60 (2020), 80-91] for the said class of functions.

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1. Introduction, definitions and results

Suppose that f and g are either meromorphic or entire functions in the complex plane $\mathbb{C}$. Let $c \in \mathbb{C} \cup \{\infty\}$. The functions f and g are said to share the value c IM (ignoring multiplicities) if $f - c$ and $g - c$ have the same set of zeros, or equivalently, if $f^{-1}(c) = g^{-1}(c)$, where $f^{-1}(c)$ denotes the set of preimages of c under f, defined as $f^{-1}(c) := \{s \in \mathbb{C} : f(s) - c = 0\}$. Moreover, f and g are said to share the value c CM (counting multiplicities) if f and g have the same set of zeros and the multiplicities of the corresponding zeros are also equal. In connection to the shared values one must recall a much celebrated result due to R. Nevanlinna (known as Nevanlinna's five value theorem or uniqueness theorem) which tells that two nonconstant meromorphic functions are identical whenever they share five distinct values IM; the number "five" is the best possible, as shown by Nevanlinna (see [5, 11, 19]). Besides Nevanlinna's uniqueness theorem Pólya's theorem [13] can be mentioned as another fundamental result and a forerunner of the above theorem. In [13], the author showed that four distinct shared CM values are required for the uniqueness of entire functions of finite order. For any set $S \subset \mathbb{C} \cup \{\infty\}$, we define

$$E_f(S) := \bigcup_{c \in S} \{s \in \mathbb{C} : f(s) - c = 0\},$$

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where each zero of $f-c$ is counted with multiplicities, that is, $E_f(S)$ is a multi-set. Also, by $\overline{E}_f(S)$ we mean the collection of distinct elements in $E_f(S)$. If $E_f(S) = E_g(S)$, we say f and g share the set S CM; if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that they share the set S IM. Clearly, sharing a singleton set and sharing a value have the same meaning by all means. There are meromorphic functions which have importance in number theory, and so their value distribution is also valuable. During the last decade, shared value problems related to these functions, such as zeta functions and more generally the Selberg class L-functions have been studied extensively (see [3, 8, 10, 16, 18]).

In [16], Steuding investigated on the possible number of shared values for the Selberg class functions. A function of the said class generally means a Dirichlet series $\mathcal{L}(s) = \sum_{n\geq 1} \frac{a(n)}{n^s}$ with coefficients $a(n) \ll n^{\epsilon}$ (for each $\epsilon > 0$) which has a meromorphic continuation of finite order to the entire complex plane $\mathbb{C}$ with only possible pole at $s = 1$, satisfies a Riemann type functional equation, and also might have an Euler product over primes (see [15, 16] for precise definition).

In view of Gross's question for two sets (see [4]), Yuan, Li and Yi [20] asked: *What can be said about the relationship between a meromorphic function f and an L-function $\mathcal{L}$ of Selberg class when they share two finite sets?* The authors [20] also resolved this question by proving the following theorem.

Theorem A. *Let f be a meromorphic function having finitely many poles in $\mathbb{C}$, and let $\mathcal{L}$ be a nonconstant L-function of Selberg class. Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$, where α_1, α_2, ..., α_l are all distinct roots of the algebraic equation $\omega^p + a\omega^q + b = 0$. Here l is a positive integer satisfying $1 \leq l \leq p$, p and q are relatively prime positive integers with $p \geq 5$ and $p > q$, and a, b, c are three finite nonzero constants, where $c \neq \alpha_j$ for $1 \leq j \leq l$. If f and $\mathcal{L}$ share S CM and c IM, then $f = \mathcal{L}$.*

Recently, in [14], the present authors proved an IM analogue of Theorem A, as shown in the following result.

Theorem B. *Let f be a meromorphic function having finitely many poles in $\mathbb{C}$, and let $\mathcal{L}$ be a nonconstant L-function of Selberg class. Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$, where $\alpha_1, \alpha_2, \ldots, \alpha_l$ are all distinct roots of the algebraic equation $P(\omega) = \omega^p + a\omega^q + b = 0$. Here l is a positive integer satisfying $1 \leq l \leq p$, p and q are relatively prime positive integers with $p > 4k + 9$ and $k = p - q \geq 1$, and a, b, c are three finite nonzero constants, where $c \neq \alpha_j$ for $1 \leq j \leq l$. If f and $\mathcal{L}$ share S IM and c IM, then $f = \mathcal{L}$.*

In [12], Oswald and Steuding considered a more general class of functions, namely the class of entire functions of the form

$$L(s;f)=\sum_{n\geq 1}\frac{f(n)}{n^s}, \tag{1.1}$$

which are representable as Dirichlet series in some right half-plane. Here the coefficients are given by an arithmetical function $f:\mathbb{N}\to\mathbb{C}$. For such functions the authors [12] proved the following result.

Theorem C. *Let $L(s;f_1)$ and $L(s;f_2)$ be two entire functions of finite order so that each of them has a convergent Dirichlet series representation of the form* (1.1) *in some right half-plane. If $L(s;f_1)$ and $L(s;f_2)$ share two distinct complex values a and b CM, then $L(s;f_1)=L(s;f_2)$.*

As Theorem C deals with only the shared values, it would be desirable to explore the problem on the shared sets for the same pair of functions. Moreover, it becomes interesting to investigate how far the conclusions of Theorem A and Theorem B hold for these functions. We prove the following theorem in this regard.

Theorem 1.1. *Let $L(s;f_1)$ and $L(s;f_2)$ be two nonconstant entire functions having convergent Dirichlet series representations of the form* (1.1) *in certain right half-plane, and one of them is of finite order. Let $S=\{\alpha_1,\alpha_2,\ldots,\alpha_l\}$, where $\alpha_1,\alpha_2,\ldots,\alpha_l$ are all distinct roots of the algebraic equation $P(\omega)=\omega^p+a\omega^q+b=0$. Here l is a positive integer satisfying $1\leq l\leq p$, p and q are relatively prime positive integers with $p>2$ and $p>q$, and a, b are two finite nonzero constants. If $L(s;f_1)$ and $L(s;f_2)$ share S IM and they assume a common complex value $c\ (\neq\alpha_j)\ (1\leq j\leq l)$ for some $s_0\in\mathbb{C}$, then $L(s;f_1)=L(s;f_2)$ in some right half-plane.*

It is assumed that the readers are accustomed with Nevanlinna theory, and so with its standard notations for a meromorphic (entire) function f, such as $T(r,f)$ (the Nevanlinna characteristic function), $m(r,f)$ (the proximity function), $N(r,f)$ (the counting function) and $\overline{N}(r,f)$ (the reduced counting function) (for details, we refer the reader to [5], [7], [19]). The notion $S(r,f)$, often used in this theory, will mean any quantity that equals $O(\log(rT(r,f)))$, $(r\to\infty)$ except possibly a set of r of finite Lebesgue measure. In particular, if $\rho(f)<+\infty$ ($\rho(f)$ denotes the order of f), then $S(r,f)=O(\log r)$, $(r\to\infty)$ holds without any exceptional set.

2. Lemmas

The following results are important for the proof of our main theorem.

Lemma 2.1. [6, Satz 12] *Let $F(s)$ be a function represented by a Dirichlet series* $F(s)=\sum_{n\geq 1}\frac{f(n)}{n^s}$, *convergent and non-vanishing in some right half-plane Re $s>\sigma_0$. Then its reciprocal also obeys a Dirichlet series representation* $\frac{1}{F(s)}=\sum_{n\geq 1}\frac{g(n)}{n^s}$ *in the same half-plane Re $s>\sigma_0$.*

Lemma 2.2. [7, p. 5] *Let $g,h:(0,+\infty)\to\mathbb{R}$ be monotonically increasing real functions such that $g(r)\leq h(r)$ outside an exceptional set M of finite linear measure. Then, for any $\kappa>1$, there exists $r_0>0$ such that $g(r)\leq h(\kappa r)$ for all $r>r_0$.*

Lemma 2.3. [21, Lemma 8] *Let $p(>0)$ and q be two relatively prime integers, and let a be a finite complex number satisfying $a^p=1$. Then the expressions ω^p-1 and ω^q-a have a unique common zero.*

Lemma 2.4. [9, Lemma 2.7] *Let $P(\omega)=\omega^p+a\omega^q+b$, where p and q are positive integers satisfying $p>q$, $a(\neq 0)$ and $b(\neq 0)$ are finite complex numbers. Then the following cases occur:*

(i) The algebraic equation $P(\omega)=0$ has no root of multiplicity ≥ 3.

(ii) If

$$\frac{b^{p-q}}{a^p}\neq\frac{(-1)^p q^q (p-q)^{p-q}}{p^p}, \tag{2.1}$$

then the algebraic equation $P(\omega)=0$ has exactly p distinct roots which are all simple, and no multiple root exists.

(iii) If

$$\frac{b^{p-q}}{a^p}=\frac{(-1)^p q^q (p-q)^{p-q}}{p^p}, \tag{2.2}$$

and p and q are relatively prime, then the algebraic equation $P(\omega)=0$ has exactly $p-1$ distinct roots which include $p-2$ simple roots and only one double root.

Lemma 2.5. *Let $L(s;f_1)$ and $L(s;f_2)$ be two entire functions of finite order so that each of them has a convergent Dirichlet series representation of the form* (1.1) *in some right half-plane. Let $R(\omega)=0$ be an algebraic equation with $l(\geq 1)$ distinct roots, where $R(\omega)$ is a monic polynomial. If $L(s;f_1)$ and $L(s;f_2)$ share $S=\{\omega: R(\omega)=0\}$ IM and they assume a common complex value c for some $s_0\in\mathbb{C}$ such that $R(c)\neq 0$, then $R(L(s;f_1))\equiv R(L(s;f_2))$ for all sufficiently large Re s.*

Proof. Suppose that $F(s;f_1)=R(L(s;f_1))$ and $F(s;f_2)=R(L(s;f_2))$. Since $L(s;f_1)$ and $L(s;f_2)$ share S IM, then $F(s;f_1)$ and $F(s;f_2)$ share 0 IM.

We now need an explicit form of $R(\omega)$ to proceed further. Suppose that $R(\omega)$ has the form: $R(\omega) = (\omega - \gamma_1)^{l_1}(\omega - \gamma_2)^{l_2} \dots (\omega - \gamma_k)^{l_k}$, where $\gamma_j \in \mathbb{C}$ are all distinct, $\sum_{j=1}^{k} l_j = l$, and $l_j \in \mathbb{N}$. Therefore

$$F(s; f_i) = (L(s; f_i) - \gamma_1)^{l_1}(L(s; f_i) - \gamma_2)^{l_2} \dots (L(s; f_i) - \gamma_k)^{l_k}, \quad i = 1, 2.$$

Let us define a function $\varepsilon : \mathbb{N} \to \mathbb{C}$ by

$$\varepsilon(n) = \begin{cases} 0, & \text{if } n > 1 \\ 1, & \text{if } n = 1 \end{cases}$$

so that $\varepsilon = \mu * u$ as a Dirichlet convolution of the Möbius μ-function μ with the function u (see [1, p. 31]). Here u is the arithmetical function defined as $u(n) = 1$ for all $n \geq 1$. Then $L(s; f_i) - \gamma_j = L(s; f_i - \gamma_j \varepsilon)$ for $i = 1, 2;\ j = 1, 2, \dots, k$.

Now from the uniqueness theorem for Dirichlet series (see [1, p. 227], [17, p. 309]), it follows that any convergent Dirichlet series is non-vanishing in another right half-plane, and hence $L(s; f_i) - \gamma_j$ $(i = 1, 2; j = 1, 2, \dots, k)$ is also a zero-free Dirichlet series for all s with sufficiently large Re s. Therefore, for the shared value zero, we see that there exists a suitable right half-plane in which each of $F(s; f_1)$ and $F(s; f_2)$ is zero-free. Moreover, $F(s; f_i)$ is an entire function as $L(s; f_i)$ is so.

Let

$$W(s) = \frac{F(s; f_1)}{F(s; f_2)}.$$

Then for all s with sufficiently large Re s, W is an entire function without any zeros. Note that the orders of both $F(s; f_1)$ and $F(s; f_2)$ are finite. If $\hat{\rho}$ =max $\{\rho(F(s; f_1)), \rho(F(s; f_2))\}$, then by Hadamard Factorization Theorem (see [2, p. 384], [17, p. 250]), $W(s)$ must take the form

$$W(s) = \frac{F(s; f_1)}{F(s; f_2)} = e^{P_1(s)}, \tag{2.3}$$

for some polynomial $P_1(s)$ with $\deg(P_1(s)) \leq \hat{\rho}$.

Since $L(s; f_2) - \gamma_j$ is a zero-free Dirichlet series for all s with sufficiently large real part, using Lemma 2.1, we have for all these s with large Re s, $[L(s; f_2) - \gamma_j]^{-1} = [L(s; f_2 - \gamma_j \varepsilon)]^{-1} = L(s; g)$, where $(f_2 - \gamma_j \varepsilon) * g = \varepsilon$. As the set of Dirichlet series is closed under multiplication, in view of Dirichlet convolution $*$, we obtain

$$\frac{L(s; f_1) - \gamma_j}{L(s; f_2) - \gamma_j} = L(s; f_1 - \gamma_j \varepsilon) L(s; g) = L(s; h_j),$$

where $h_j = (f_1 - \gamma_j \varepsilon) * g$ for $j = 1, 2, \dots, k$. This implies

$$\left(\frac{L(s; f_1) - \gamma_j}{L(s; f_2) - \gamma_j} \right)^{l_j} = [L(s; h_j)]^{l_j} = L(s; \hat{h}_j),$$

where $\hat{h}_j = h_j * h_j * \ldots * h_j$ (l_j times) for $j = 1, 2, \ldots, k$. Therefore, if $x = \hat{h}_1 * \hat{h}_2 * \ldots * \hat{h}_k$, then

$$\frac{F(s; f_1)}{F(s; f_2)} = \prod_{1 \le j \le k} L(s; \hat{h}_j) = L(s; x) = \sum_{n \ge 1} \frac{x(n)}{n^s} = \sum_{n \ge m_1} \frac{x(n)}{n^s},$$

where m_1 is the minimum of all $n \in \mathbb{N}$ such that $x(n) \neq 0$. From (2.3) we have

$$\begin{aligned} P_1(s) &= \log \left[\sum_{n \ge m_1} \frac{x(n)}{n^s} \right] \\ &= \log \frac{x(m_1)}{{m_1}^s} + \log \left[1 + \sum_{n > m_1} \frac{x(n)}{x(m_1)} \left(\frac{m_1}{n} \right)^s \right]. \end{aligned}$$

Clearly, the series on the right-hand side is convergent for all sufficiently large Re s. Since $P_1(s)$ is a polynomial, the series must be identically zero and so

$$P_1(s) = \log \left[\frac{x(m_1)}{{m_1}^s} \right] = -s \log m_1 + \log\{x(m_1)\}, \tag{2.4}$$

which means $P_1(s)$ is a linear polynomial or constant. Now for $s = \sigma + it$, we can write

$$Re\ P_1(\sigma + it) = A(t)\sigma + B(t), \tag{2.5}$$

a polynomial in σ with $A(t), B(t)$ being polynomials in t. We now show that $A(t) \equiv 0$. For this, we first note that $\lim_{\sigma \to +\infty} F(s; f_1) = d_1$ and $\lim_{\sigma \to +\infty} F(s; f_2) = d_2$ for some nonzero constants $d_1, d_2 \in \mathbb{C}$ as $F(s; f_1)$ and $F(s; f_2)$ are non-vanishing and convergent for all sufficiently large Re s. Therefore we get

$$\lim_{\sigma \to +\infty} \frac{F(s; f_1)}{F(s; f_2)} = d_3, \tag{2.6}$$

where $d_3 (\neq 0) \in \mathbb{C}$. Again, from (2.3) and (2.5) we obtain that

$$\left| \frac{F(s; f_1)}{F(s; f_2)} \right| = e^{A(t)\sigma + B(t)}. \tag{2.7}$$

If we assume that $A(t_0) > 0$ for some $t_0 \in \mathbb{C}$, then from (2.6) and (2.7), it follows that for the limit $\sigma \to +\infty$ and $t = t_0$, $|d_3| = \infty$, which is a contradiction. Similarly, if we suppose that $A(t_1) < 0$ for some $t_1 \in \mathbb{C}$, then we get $|d_3| = 0$ as $\sigma \to +\infty$, that is, a contradiction. Therefore $A(t) \equiv 0$ and so from (2.7) we obtain

$$\left| \frac{F(s; f_1)}{F(s; f_2)} \right| = e^{B(t)}. \tag{2.8}$$

Since $e^{B(t)}$ is independent of σ, it has the same value for any arbitrary σ. Taking $\sigma \to +\infty$, we see from (2.6) that the left-hand side of (2.8) is $|d_3|$ for any value of t and hence $e^{B(t)} = |d_3|$. Therefore, we have $|\frac{F(s;f_1)}{F(s;f_2)}| = |d_3|$, which implies that the function $\frac{F(s;f_1)}{F(s;f_2)}$ is a constant. Therefore, from (2.6) we get

$$\frac{F(s;f_1)}{F(s;f_2)} = d_3. \tag{2.9}$$

Since $L(s;f_1)$ and $L(s;f_2)$ assume the common value c at some $s = s_0$ with $R(c) \neq 0$, by (2.9) we deduce that $d_3 = 1$. Therefore $F(s;f_1) = F(s;f_2)$. This completes the proof of the lemma. □

3. Proof of the theorem

Proof of Theorem 1.1. Let $F(s;f_1) = P(L(s;f_1))$ and $F(s;f_2) = P(L(s;f_2))$. Then $F(s;f_1)$ and $F(s;f_2)$ share 0 IM. We first show that $\rho(L(s;f_1)) = \rho(L(s;f_2))$. In view of Lemma 2.4, we know that $P(\omega) = 0$ has at least $p-1$ distinct roots, say $\alpha_1, \alpha_2, \ldots, \alpha_{p-1}$. Since the entire functions $L(s;f_1)$ and $L(s;f_2)$ share S IM and $p > 2$, we get by Nevanlinna's second fundamental theorem that

$$\begin{aligned} (p-2)T(r, L(s;f_1)) &\leq \sum_{j=1}^{p-1} \overline{N}(r, \alpha_j; L(s;f_1)) + O(\log r + \log T(r, L(s;f_1))) \\ &= \sum_{j=1}^{p-1} \overline{N}(r, \alpha_j; L(s;f_2)) + O(\log r + \log T(r, L(s;f_1))), \end{aligned}$$

Therefore

$$T(r, L(s;f_1)) \leq \frac{p-1}{p-2} T(r, L(s;f_2)) + O(\log r + \log T(r, L(s;f_1))) \tag{3.1}$$

as $r \to \infty$ and $r \notin M$, where M is a set of positive real numbers of finite linear measure.

Similarly,

$$T(r, L(s;f_2)) \leq \frac{p-1}{p-2} T(r, L(s;f_1)) + O(\log r + \log T(r, L(s;f_2))) \tag{3.2}$$

as $r \to \infty$ and $r \notin M$.

Using Lemma 2.2, we can remove the exceptional set in (3.1) and (3.2) and thus the inequalities hold for all $r > r_0$ for some $r_0 > 0$. Therefore, we get $\rho(L(s;f_1)) \leq \rho(L(s;f_2))$ and $\rho(L(s;f_2)) \leq \rho(L(s;f_1))$. Consequently, we obtain that both the orders of $L(s;f_1)$ and $L(s;f_2)$ are equal and finite as well. Therefore, by Lemma 2.5 it follows that

$$L^p(s;f_1) - L^p(s;f_2) = -a(L^q(s;f_1) - L^q(s;f_2)), \tag{3.3}$$

and so

$$L^{p-q}(s;f_2) = -a\frac{G^q - 1}{G^p - 1}, \tag{3.4}$$

for all s having sufficiently large real part, where $G = \frac{L(s;f_1)}{L(s;f_2)}$ is a non-vanishing entire function for such s. Now in the common right half-plane of $F(s; f_1)$ and $F(s; f_2)$, we consider the following two cases:

Case 1: Suppose that $G^p = 1$. Then $L^p(s; f_2) = L^p(s; f_1)$. Substituting this in (3.3), we obtain $L^q(s; f_2) = L^q(s; f_1)$. Applying Lemma 2.3, we have $L(s; f_1) = L(s; f_2)$.

Case 2: Suppose that $G^p \neq 1$. Since p and q are relatively prime positive integers, we get by Lemma 2.3 that the numerator and the denominator of right-hand side of (3.4) has exactly one common zero. Therefore, the zeros of the denominator (if exist) produces $p - 1$ distinct poles of $L^{p-q}(s; f_2)$ on the left hand-side of (3.4). Since $L^{p-q}(s; f_2)$ has no pole, $p > 2$, and that a nonconstant entire function can possess at most one Picard exceptional value, then it follows that G should have $p-1$ Picard exceptional values. Thus G is a constant and so from (3.4) we get that $L(s; f_2)$ is constant, which is clearly a contradiction.

This completes the proof of Theorem 1.1. □

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