АСТРОФИЗИКА

ТОМ 63 ФЕВРАЛЬ, 2020 ВЫПУСК 1

TWO EFFICIENT COMPUTATIONAL ALGORITHMS TO SOLVE THE NON-LINEAR SINGULAR LANE-EMDEN EQUATIONS

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Received 25 March 2019
Accepted 18 December 2019

In this paper, two efficient computational algorithms based on Rational and Exponential Bessel (RB and EB) functions are compared to solve several well-known class of non-linear Lane-Emden type models. The problems, which define in some models of non-Newtonian fluid mechanics and mathematical physics, are nonlinear ordinary differential equations of second-order over the semi-infinite interval and have a singularity at $x = 0$. The non-linear Lane-Emden equations are converted to a sequence of linear differential equations by utilizing the quasilinearization method (QLM) and then, these linear equations are solved by RB and EB collocation methods. Afterward, the obtained results are compared with the solution of other methods for demonstrating the efficiency and applicability of the proposed methods.

Keywords: *rational Bessel functions: exponential Bessel functions: Lane-Emden type equations: nonlinear ODE: quasilinearization method: collocation method*

1. *Introduction*. The investigation of singular initial/boundary value problem for non-linear second order differential equations has been attracted by some astrophysicist, mathematicians, and physicists. Lane-Emden type equations describe the temperature variation of a spherical gas cloud under the mutual attraction of its molecules and subject to the laws of classical thermodynamics. Let $P(r)$ denote the total pressure at a distance r from the center of spherical gas cloud. The total pressure is due to the usual gas pressure and a contribution from radiation:

$$P = \frac{1}{3}\xi T^4 + \frac{RT}{v},$$

where ξ, T, R, and v are the radiation constant, the absolute temperature, the gas constant, and the specific volume, respectively [1]. Let $M(r)$ be the mass within a sphere of radius r and G the constant of gravitation, the equilibrium equation for the configuration are

$$\frac{dP}{dr} = -\rho \frac{GM(r)}{r^2}, \quad \frac{dM(r)}{dr} = 4\pi\rho r^2, \tag{1}$$

where ρ is the density at a distance r from the center of a spherical star. To eliminate M, the previous equations should be written in a dimensional form as follows [1,2]:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right)=-4\pi G\rho\,,$$

We already know that in the case of a degenerate electron gas, the pressure and density are $\rho = P^{3/5}$, assuming that such a relation exists in other states of the star, we are led to consider a relation of the form $P = K\rho^{1+1/m}$, where K and m are constants.

We can insert this relation into Eq. (1) for the hydrostatic equilibrium condition and, from this, we can rewrite the equation as follows:

$$\left[\frac{K(m+1)}{4\pi G}\lambda^{1/m-1}\right]\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dy}{dr}\right)=-y^m\,,$$

where λ represents the central density of the star and y denotes the dimensionless quantity, which are both related to ρ through the following relation [1,2]:

$$\rho=\lambda\, y^m(x),$$

and let

$$r = ax\,,\quad a=\left[\frac{K(m+1)}{4\pi G}\lambda^{1/m-1}\right]^{1/2}.$$

Inserting these relations into our previous relation, we obtain the Lane-Emden equation [1,2]:

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right)=-y^m\,,$$

or

$$y''(x)+\frac{2}{x}y'(x)+y^m(x)=0\,,\quad x>0\,, \tag{2}$$

where the initial conditions are as follows:

$$y(0)=1\,,\quad y'(0)=0\,. \tag{3}$$

Eq. (2) with the initial conditions (3) is known as the standard Lane-Emden equation.

The values of m, which are physically interesting, lie in the interval [0, 5]. The main difficulty in analyzing this type of equation is the singularity behaviour occurring at $x = 0$.

The solutions of the Lane-Emden equation could be exact only for $m = 0$, 1 and 5. For the other values of m, the Lane-Emden equation is to be integrated numerically [2]. Thus, we decided to present a new and efficient technique to solve it numerically for $m = 1.5,\ 2,\ 2.5,\ 3,$ and 4.

1.1. *Previous works*. Recently, some analytical, semi-analytical, and numerical techniques have been applied to solve Lane-Emden equations. The main difficulty arises in the singularity of the equations at $x=0$. We have introduced several techniques as follow:

Bender et al. [3] proposed a new perturbation technique based on an artificial parameter δ, the method is often called δ-method. Wazwaz [4] employed the Adomian decomposition technique with an alternate framework designed, He [5] employed Ritz's method to obtain an analytical solution, Parand et al. [6,7] applied Spectral methods based on the fractional order of rational Bernoulli functions and the fractional order of Chebyshev functions, Ramos [8,9] presented linearzation methods to utilize an analytical solutions and globally smooth solutions, and the obtained series solutions of the Lane-Emden type equation, Yousefi [10] applied Legendre Wavelet approximations and used integral operator and converted Lane-Emden equations to integral equations, Chowdhury and Hashim [11] used analytical solutions of the generalized Emden-Fowler type equations by Homotopy perturbation method (HPM), Aslanov [12] introduced a further development in the Adomian decomposition technique, Dehghan and Shakeri [13] investigated Lane-Emden equations by applying the variational iteration method (VIM), Marzban et al. [14] used a method based upon hybrid of block-pulse functions and Lagrange interpolating polynomials together with the operational integration matrix to approximate solution of the problem, Adibi and Rismani [15] proposed the approximate solutions of singular the Lane-Emden via modified Legendre-spectral method, Vanani and Aminataei [16] provided a numerical method which produces an approximate polynomial solution, they used an integral operator and convert Lane-Emden equations into integral equations and then convert the acquired integral equations into a power series and finally, transforming the power series into padé series form, Kaur et al. [17] obtained the Haar wavelet approximate solution.

Furthermore, other researchers trying to solve the Lane-Emden type equations with several methods, For example, Yildirm and Öziş [18] by using HPM method, Iqbal and Javad [19] by using Optimal HAM, Boubaker and Van Gorder [20] by using boubaker polynomials expansion scheme, Daşcoğlu and Yaslan [21] by using Chebyshev collocation method, Yüzbaş [22] by using Bessel matrix method, Boyd [23] by using Chebyshev spectral method, Bharwy and Alofi [24] by using Jacobi-Gauss collocation method, Pandey et al. [25] by using Legendre operation matrix, Rismani and Monfared [26] by using Modified Legendre spectral method, Delkhosh et al. [27] by using the fractional order of rational Euler collocation methods, Nazari-Golshan et al. [28] by using Homotopy perturbation with Fourier transform, Doha et al. [29] by using second kind Chebyshev operation matrix algorithm, Mall and Chakaraverty [30] by using Chebyshev Neural Network based model, Gürbüz and Sezer [31] by using Laguerre polynomial and Kazemi-Nasab et al. [32] by using Chebyshev wavelet finite difference method. In this paper, we attempt to introduce two efficient computational algorithms based on Rational and Exponential Bessel (RB and EB) functions for solving non-linear singular Lane-Emden equations.

The rest of this paper is arranged as follows: Section 2 introduces new rational and exponential Bessel (RB and EB) functions and their properties. Section 3 describes a brief formulation of quasilinearization method (QLM) [38]. In section 4 at first, by utilizing the QLM over the Lane-Emden equation a sequence of linear differential equations is obtained, and then at each iteration, the linear differential equation is solved by RB and EB collocation methods that we name RB-QLM and EB-QLM methods. Comparison between these two methods with some well-known results in section 5, show that using rational functions is highly accurate, and we also describe our results via tables and figures. Finally, we give a brief conclusion in section 6.

2. *Properties of rational and exponential Bessel functions*. The Bessel functions arise in many problems in physics possessing cylindrical symmetry, such as the vibrations of circular drumheads and the radial modes in optical fibers. Bessel functions are usually defined as a particular solution of a linear differential equation of the second order which known as Bessel's equation [33]. Bessel functions first defined by Daniel Bernoulli on heavy chains (1738) and then generalized by Friedrich Bessel. More general Bessel functions were studied by Leonhard Euler in 1781 and in his study of the vibrating membrane in 1764.

Bessel differential equation of order $n \in \mathbb{R}$ is:

$$x^2 \frac{d^2 y(x)}{dx^2} + x\frac{dy(x)}{dx} + \left(x^2 - n^2\right)y(x) = 0, \quad x \in (-\infty, \infty). \tag{4}$$

One of the solutions of equation (4) by applying the method of Frobenius as follows [34]:

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(n+r)!}\left(\frac{x}{2}\right)^{2r+n}, \tag{5}$$

where series (5) is convergent for all $x \in (-\infty, \infty)$.

Bessel polynomials have been introduced as follows [35]:

$$B_n(x) = \sum_{r=0}^{[N-n]/2} \frac{(-1)^r}{r!(n+r)!}\left(\frac{x}{2}\right)^{2r+n}, \quad x \in [0,1], \tag{6}$$

where $n \in \mathbb{N}$, [.] denotes the floor of a number, and N is the number of basis of Bessel polynomials.

2.1. *Rational Bessel functions*. The new basis functions, "Rational Bessel

(RB) functions" denote by $RB_n(x, L)$ which are generated from well known Bessel polynomials by using the algebraic mapping of $\phi(x) = x/(x+L)$ as follow:

$$RB_n(x, L) = B_n(\phi(x)), \quad n = 0, 1, ..., N,$$

or

$$RB_n(x, L) = \sum_{r=0}^{[N-n]/2} \frac{(-1)^r}{r!(n+r)!}\left(\frac{x}{2(x+L)}\right)^{2r+n}, \quad n = 0, 1, ..., N, \tag{7}$$

where $x \in [0, \infty)$, $B_n(x)$ is Bessel polynomials of order n, and the constant parameter $L > 0$ is a scaling/stretching factor which can be used to fine tune the spacing of collocation points. For a problem whose solution decays at infinity, there is an effective interval outside of which the solution is negligible, and collocation points which fall outside of this interval are essentially wasted. On the other hand, if the solution is still far from negligible at the collocation points with largest magnitude, one cannot expect a very good approximation. Hence, the performance of spectral methods in unbounded domains can be significantly enhanced by choosing a proper scaling parameter such that the extreme collocation points are at or close to the endpoints of the effective interval [36]. Boyd [37] offered guidelines for optimizing the map parameter L for rational Chebyshev functions which is also useful for the RB functions.

Let us define $\Gamma = \{x \mid 0 \le x < \infty\}$ and
$L^2_{w_r}(\Gamma) = \{v : \Gamma \to \mathbb{R} \mid v$ is measurable and $\|v\|_{w_r} < \infty\}$, where

$$\|v\|_{w_r} = \left(\int_0^\infty |v(x)|^2 w_r(x, L)dx\right)^{1/2},$$

with $w_r(x, L) = L/(x+L)^2$, is the norm induced by inner product of the space $L^2_{w_r}(\Gamma)$ as follows:

$$\langle v, u\rangle_{w_r} = \int_0^\infty v(x)u(x)w_r(x, L)dx.$$

Now, suppose that

$$\mathfrak{S} = \text{span}\{RB_0(x), RB_1(x), ..., RB_N(x)\},$$

where $\mathfrak{S}$ is a finite-dimensional subspace of $L^2_w(\Gamma)$, $\dim(\mathfrak{S}) = N+1$, so $\mathfrak{S}$ is a closed subspace of $L^2(\Gamma)$. Therefore, $\mathfrak{S}$ is a complete subspace of $L^2(\Gamma)$. Assume that $f(x)$ be an arbitrary element in $L^2(\Gamma)$. Thus $f(x)$ has a unique best approximation in $\mathfrak{S}$ subspace, say $\hat{b}(x) \in \mathfrak{S}$, that is

$$\exists \hat{b}(x) \in \mathfrak{S}, \quad \forall b(x) \in \mathfrak{S}(x), \quad \|f(x) - \hat{b}(x)\|_{w_r} \le \|f(x) - b(x)\|_{w_r}.$$

Notice that we can write $b(x)$ vector as a combination of the basis vectors of $\mathfrak{S}$ subspace.

We know function of $f(x)$ can be expanded by $N+1$ terms of RB functions as:

$$f(x)=f_N(x)+R(x),$$

that is

$$f_N(x)=\sum_{n=0}^{N} a_n RB_n(x)=A^T RB(x), \tag{8}$$

where $RB(x)$ is vector $[RB_0(x), RB_1(x), ..., RB_N(x)]^T$ and $R \in \mathfrak{S}^\perp$ that $\mathfrak{S}^\perp$ is the orthogonal complement. So $f(x)-f_N(x) \in \mathfrak{S}^\perp$ and $b(x) \in \mathfrak{S}$ are orthogonal which we denote it by:

$$f(x)-f_N(x) \perp b,$$

thus $f(x)-f_N(x)$ vector is orthogonal over all of basis vectors of $\mathfrak{S}$ subspace as:

$$\langle f(x)-f_N(x), RB_i(x)\rangle_{w_r}=\langle f(x)-A^T RB(x), RB_i(x)\rangle_{w_r}=0, \quad i=0,1,...,N,$$

hence

$$\langle f(x)-A^T RB(x), RB^T(x)\rangle_{w_r}=0,$$

and A can be obtained by

$$\langle f(x), RB^T(x)\rangle_{w_r}=\langle A^T RB(x), RB^T(x)\rangle_{w_r},$$

$$A^T=\langle f(x), RB^T(x)\rangle_{w_r}\langle RB(x), RB^T(x)\rangle_{w_r}^{-1}.$$

2.2. *Exponentioal Bessel functions*. Exclusive of rational functions, we can use exponential transformation to have new functions which are also defined on the semi-infinite interval. The exponential Bessel (EB) functions can be defined by

$$EB_n(x)=B_n\left(1-e^{-x/L}\right), \quad n=0,1,...,N.$$

or

$$EB_n(x,L)=\sum_{r=0}^{[N-n]/2}\frac{(-1)^r}{r!(n+r)!}\left(1-e^{-x/L}\right)^{2r+n}, \quad n=0,1,...,N \tag{9}$$

where parameter L is a constant parameter and, like rational functions, it sets the length scale of the mapping. All of the above relations can also be used to EB functions with respect to the weight function $w_e(x,L)=e^{-x/L}/L$ in the interval $[0,\infty)$.

3. *The quasilinearization method (QLM)*. The QLM is a generalization of the Newton-Raphson method [38] to solve the nonlinear differential equation as a limit of approximating the nonlinear terms by an iterative sequence of linear expressions. The QLM techniques are based on the linearization of the higher order ordinary/partial differential equation and require the solution of a linear ordinary differential equation at each iteration. Mandelzweig and Tabakin [39] have determined general conditions for the quadratic, monotonic, and uniform convergence

of the QLM to solve both initial and boundary value problems in nonlinear ordinary n-th order differential equations in N-dimensional space. And also, Canuto et al. [40] have proved the stability and convergence analysis of spectral methods, and, we will show that our numerical results are convergent.

Let us assume that a second-order nonlinear ordinary differential equation in one variable on the interval $[0,\infty)$ as follows:

$$\frac{d^2u}{dx^2} = F(u'(x), u(x), x). \tag{10}$$

with the initial conditions: $u(0) = A$, $u'(0) = B$, where A and B are real constants and F is a nonlinear function.

By utilizing the QLM to solve Eq. (10) determines the $(I+1)$-th iterative approximation $u_{I+1}(t)$ as a solution of the linear differential equation:

$$\frac{d^2u_{I+1}}{dx^2} = F(u'_I, u_I, x) + (u_{I+1} - u_I)F_u(u'_I, u_I, x) + (u'_{I+1} - u'_I)F_{u'}(u'_I, u_I, x), \tag{11}$$

with the initial conditions:

$$u_{I+1}(0) = A, \quad u'_{I+1}(0) = B, \tag{12}$$

where $I = 0, 1, 2, \ldots$ and the functions $F_u = \partial F/\partial u$ and $F_{u'} = \partial F/\partial u'$ are functional derivatives of the functional of $F(u'_I, u_I, x)$.

4. *Application of methods*. In this paper, two methods based on RB collocation method and EB collocation method for solving Eq. (2) with initial conditions of Eq. (3) have been considered.

First, by utilizing the QLM technique on Eq. (2), we have

$$xy''_{I+1}(x) + 2y'_{I+1}(x) - (m-1)xy_I^m(x) + mxy_{I+1}(x)y_I^{m-1}(x) = 0 \tag{13}$$

with the initial conditions:

$$y_{I+1}(0) = 1, \quad y'_{I+1}(0) = 0, \tag{14}$$

where $I = 0, 1, 2, \ldots$

For rapid convergence is actually enough that the initial guess is sufficiently good to ensure the smallness of just one of the quantity $q_r = k\|y_{I+1} - y_I\|$, where k is a constant independent of I. Usually, it is advantageous that $y_0(t)$ would satisfy at least one of the initial conditions Eq. (3) [39], thus set $y_0(x) = 1$ for the initial guess of the Lane-Emden equation.

Then, we can approximate $y_{I+1}(x)$ by $N+1$ basis of RB and EB functions as follows:

1. approximating $y_{I+1}(x)$ by $N+1$ basis of RB functions:

$$y_{I+1}(x) \approx u_{N,I+1}(x) = 1 + x^2 \sum_{n=0}^{N} \hat{b}_n RB_n(x, L). \tag{15}$$

where $r = 0, 1, 2, \ldots$ and two terms 1 and x^2 are to satisfy initial conditions Eq. (14).

To apply the collocation method, we have constructed the residual function for $(I+1)$-th iteration in QLM by substituting $y_{I+1}(x)$ by $u_{N,I+1}(x)$ into Eq. (13) as follows:

$$RESr_{I+1}(x) = xu''_{N+1,I+1}(x) + 2u'_{N+1,I+1}(x) - \\ -(m-1)xu^{m}_{N+1,I}(x) + mxu_{N+1,I+1}(x)u^{m-1}_{N+1,I}(x) = 0. \quad (16)$$

2. approximating $y_{I+1}(x)$ by $N+1$ basis of EB functions:

$$y_{I+1}(x) \approx w_{N,I+1}(x) = \frac{1}{x^2+1} + \frac{x^2}{x+1}\sum_{n=0}^{N} \hat{c}_n EB_n(x, L) \quad (17)$$

where $r = 0, 1, 2, \ldots$.

Two terms of $1/(x^2+1)$ and $x^2/(x+1)$ are considered to satisfy initial conditions Eq. (14). Also, like above, to apply the collocation method, we have constructed the residual function for $(I+1)$-th iteration in QLM by substituting $y_{I+1}(x)$ by $w_{N,I+1}(x)$ into Eq. (13) as follows:

$$RESe_{I+1}(x) = xw''_{N+1,I+1}(x) + 2w'_{N+1,I+1}(x) - \\ -(m-1)xw^{m}_{N+1,I}(x) + mxw_{N+1,I+1}(x)w^{m-1}_{N+1,I}(x) = 0. \quad (18)$$

In all of the spectral methods, the purpose is to find the $\hat{b}_n$ and $\hat{c}_n$ unknown coefficients.

A method for forcing the residual functions Eq. (16) and Eq. (18) to zero can be defined as collocation algorithm. There is no limitation to choose points in the collocation method. The $N+1$ collocation points have been substituted in the equations of $RESr_{I+1}(x)$ and $RESe_{I+1}(x)$, therefore:

$$RESr_{I+1}(x_i) = 0, \quad i = 0, 1, \ldots, N, \quad (19)$$

$$RESe_{I+1}(x_i) = 0, \quad i = 0, 1, \ldots, N, \quad (20)$$

Table 1

COMPARISON OF THE FIRST ZEROS OF STANDARD LANE-EMDEN EQUATIONS, WITH VALUES GIVEN BY HOREDT [2] AND THE PRESENT METHODS WITH N = 75 AND ITERATION 15

m	RB	EB	Horedt [2]
1.5	3.65375373622763424836747856706295570	3.653753736227530116708951	3.65375374
2.0	4.35287459594612467697357006152614262	4.352874595946124676973570	4.35287460
2.5	5.35527545901076012377857991160851840	5.355275459010769844745925	5.35527546
3.0	6.89684861937696037545452818712314053	6.896848619376960375436984	6.89684862
4.0	14.9715463488380950976509645543077611	14.97154634883796085494984	14.9715463

which x_i are roots of the shifted Chebyshev functions on the finite interval [7]. Finally, a linear system of equations has been obtained, all of these equations can be solved by a suitable method such as the Newton method for calculating the unknown coefficients.

5. *Results and discussion*. The Lane-Emden type equations describe the variation of density as a function of the radial distance for a poly-trope. They

Table 2

NUMERICAL RESULTS OF FIRST ZEROS BY BASIS OF RB WITH VARIOUS VALUES OF *m*, *N* AND ITERATIONS, ACCURATE DIGITS ARE BOLD

m	*N*	Iteration	RB
1.5	50	05	**3.6537537362**5072342590
		10,15,20	**3.6537537362**5071853754
	75	05	**3.65375373622**763914172
		10,15,20	**3.65375373622**763424836
	100	05	**3.65375373622**225950682
		10,15,20	**3.65375373622**225461061
2	50	05	**4.35**41023191782544510394699271974639349588062470049419121696397470
		10,15,20	**4.3528745959461246769735700615261433**9487342457587311708331752
	75	05	**4.35287459**7893199784546816142774753394907169932534281348066892095
		10,15,20	**4.352874595946124676973570061526142628112365363213**147181521
	100	05	**4.35287459**7893199784546816142774753394907169932542963806389373524
		10,15,20	**4.352874595946124676973570061526142628112365363213**008835302
2.5	50	05	**5.35529645**45076443677
		10,15,20	**5.3552754590107**44925
	75	05	**5.3552**9645450764436772
		10,15,20	**5.3552754590107**601237
	100	05	**5.3552**9645450764436772
		10,15,20	**5.3552754590107**873176
3	50	05	7.1216938046517305045330727094680858444666907392
		10,15,20	**6.896848619376960375454**2796110144170369244612
	75	05	7.1216938046404145204995503800811081360235860196
		10,15,20	**6.8968486193769603754545281871231**4053555203
	100	05	7.1216938046404152911963760032858519494248670403
		10,15,20	**6.8968486193769603754545281871231**2127697218
4	50	05	**1**6.711045707072842315340457798698905988740559701
		10	**14.97154**867059731700938111496437106672775015032
		15,20	**14.9715463488380950976**50964554307761107155441
	75	05	16.402670239960775259418702056564527058250944781
		10	**14.97154**289318059650158197244640609252173187180
		15,20	**14.9715463488380950976**50964554307761107155441
	100	05	16.172787459355139190211994543646969560813181439
		10	**14.97154**439717955256111887952830248179390503419
		15,20	**14.9715463488380950976**11066133148254587457821

were first studied by the astrophysicists Jonathan Homer Lane and Robert Emden, which considered the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics [1]. In the Lane-Emden type equations, the first zero of $y(x)$ is an important point of the function, so we have computed $y(x)$ to calculate this zero. In this paper, the equation is solved for $m = 1.5$, 2, 2.5, 3 and 4, which

Table 3

NUMERICAL RESULTS OF FIRST ZEROS BY BASIS OF EB WITH VARIOUS VALUES OF *m*, *N* AND ITERATIONS, ACCURATE DIGITS ARE BOLD

m	*N*	Iteration	EB
1.5	50	05	**3.6537537362**5083916424
		10,15,20	**3.6537537362**5083427589
	75	05	**3.65375373622**753501010
		10,15,20	**3.65375373622**753011670
	100	05	**3.65375373622**227432714
		10,15,20	**3.65375373622**226943093
2	50	05	**4.352874597**89319978533890331059465 2764
		10,15,20	**4.35287459594612467**77656573583430 9221
	75	05	**4.352874595946124676 97**472244822039342
		10,15,20	**4.3528745959461246769 73570**1033024306
	100	05	**4.352874597**893199784546816142774 7526020
		10,15,20	**4.35287459594612467697 3570**0615261418
2.5	50	05	**5.35529645**450764438595
		10,15,20	**5.3552754590107**203902
	75	05	**5.35529645**450764436772
		10,15,20	**5.3552754590107**698447
	100	05	**5.35529645**450764436772
		10,15,20	**5.3552754590107**770840
3	50	05	7.1216937188899311101353842743 7823
		10,15,20	**6.8968486193769 6**9505160794512467
	75	05	7.121693804669123395399030474 82119
		10,15,20	**6.896848619376960 3754**3698467213
	100	05	**6.896848619376960 37**791871227973
		10,15,20	**6.896848619376960 3754**5452817312
4	50	05	**1**6.2649173119023736994 3385
		10	**14.97154**73275763026931076
		15,20	**14.9715463**522353010587855
	75	05	**1**6.0521001192445702611 5446
		10	**14.97154**72743172097800824
		15,20	**14.97154634883**7960854949
	100	05	**1**6.0321860978545652701 0395
		10	**14.97154**72744622920651685
		15,20	**14.97154634883**8095104708

does not have exact solutions.

The comparison of the initial slope $y'(0)$ calculated by RB-QLM ($N=75$ and iteration 15) with values obtained by Horedt [2] is given in Table 1.

Table 4

OBTAINED VALUES OF $y(x)$ AND $y'(x)$ OF STANDARD LANE-EMDEN EQUATIONS FOR $m=1.5$ BY BASIS OF RB WITH $N=75$ AND ITERATIONS 15

x	$y(x)$	$y'(x)$
0.1	0.998334582651024	-0.033283374960220
0.2	0.993353288961344	-0.066267995319313
0.3	0.985100745872271	-0.098660068556290
0.4	0.973650509840501	-0.130175582648867
0.5	0.959103856956817	-0.160544891813613
0.6	0.941588132070691	-0.189516931926819
0.7	0.921254699087677	-0.216862968455471
0.8	0.898276543103152	-0.242379797978458
0.9	0.872845582616537	-0.265892334576062
1.0	0.845169755493675	-0.287255540026184
2.0	0.495936764048973	-0.372832141746160
3.0	0.158857608676200	-0.284252727750886
3.6	0.011090994555729	-0.209392664698195

Table 5

OBTAINED VALUES OF $y(x)$ AND $y'(x)$ OF STANDARD LANE-EMDEN EQUATIONS FOR $m=2.5$ BY BASIS OF RB WITH $N=75$ AND ITERATIONS 15

x	$y(x)$	$y'(x)$
0.1	0.998335414189491	-0.033250148555062
0.2	0.993366508668235	-0.066004732702853
0.3	0.985166960607077	-0.097785664864449
0.4	0.973856692696194	-0.128148702313160
0.5	0.959597754464204	-0.156697706048055
0.6	0.942588917282480	-0.183095996800778
0.7	0.923059301998553	-0.207074283925069
0.8	0.901261395554722	-0.228434944738734
0.9	0.877463820286722	-0.247052726803513
1.0	0.851944199128236	-0.262872200779799
2.0	0.558372334987405	-0.290313683599236
3.0	0.306675101717593	-0.208571050779423
4.0	0.137680733022609	-0.134053438395795
5.0	0.029019186649369	-0.087473533084964
5.3	0.004259543533703	-0.077863974396729

Table 6

OBTAINED VALUES OF $y(x)$ AND $y'(x)$ OF STANDARD LANE-EMDEN EQUATIONS FOR $m=4$ BY BASIS OF RB WITH $N=75$ AND ITERATIONS 15

x	$y(x)$	$y'(x)$
0.1	0.99833665953957353917	-0.03320042731101602052
0.2	0.99338621353236887458	-0.06561355430127865539
0.3	0.98526489445824457228	-0.09650144694916813609
0.4	0.97415840895070184085	-0.12521904232653407185
0.5	0.96031090234222125391	-0.15124704523040264218
0.6	0.94401129085560210481	-0.17421139290379387733
0.7	0.92557835269653368985	-0.19388869549916036586
0.8	0.90534592383779093911	-0.21019908106443456806
0.9	0.88364932397603694257	-0.22318930318706216396
1.0	0.86081381220831175185	-0.23300964460615518736
2.0	0.62294077167068319754	-0.21815323531073192916
3.0	0.44005069158766127850	-0.14895436785082222650
4.0	0.31804242903566436744	-0.09886802020831413214
5.0	0.23592273104248679739	-0.06788810347440624083
6.0	0.17838426534298279218	-0.04865643577466167176
7.0	0.13635230535983164961	-0.03626805424834208635
8.0	0.10450408207160914867	-0.02795075318477840998
9.0	0.07961946745395432400	-0.02214833117831084820
10	0.05967274158948932881	-0.01796142023434323612
11	0.04334009538193507922	-0.01485063006054293705
12	0.02972593235798682964	-0.01248033393137584648
13	0.01820540390617142867	-0.01063445527740952134
14	0.00833052669542489543	-0.00916953946501606750
14.9	0.00057641886621354664	-0.00809526559361695336

Tables 2 and 3 present some numerical examples to illustrate the accuracy and convergence of our suggested methods by increasing the number of points and iterations.

Tables 4-6 show the obtained values of $y(x)$ and $y'(x)$ by the approach which based on RB collocation method, for $m=1.5$, 2.5, and 4 with the values of $N=75$ and iteration 15.

The resulting graphs of the standard Lane-Emden equation obtained by the present methods for $m=1.5$, 2, 2.5, 3, and 4 are shown in Fig.1.

Finally, Fig.2-6 show the residual errors for approximation solutions by basis of the rational and exponential functions with $N=50$, 75, and 100. Note that the residual error decreases with the increase of the collocation points.

6. *Conclusion*. The fundamental goal of this paper was to introduce novel

hybrid basis of Rational Bessel and Exponential (RB and EB) functions with the quasilinearization method (QLM) to construct an approximation for solving nonlinear Lane-Emden type equations. These problems describe a variety of phenomena in theoretical physics and astrophysics, including aspects of stellar structure, the thermal history of a spherical cloud of gas, isothermal gas spheres, and thermionic currents [1]. To achieve this goal at first, a sequence of linear differential equations is obtained by utilizing the QLM over Lane-Emden equation. Second, at each iteration of QLM, the linear differential equation is solved by new RB and EB collocation methods. This paper has been shown that the present works have provided two acceptable approaches for solving Lane-Emden type equations caused by the following reasons:

1. Cause of simplicity to solve problems and convergence of approximation

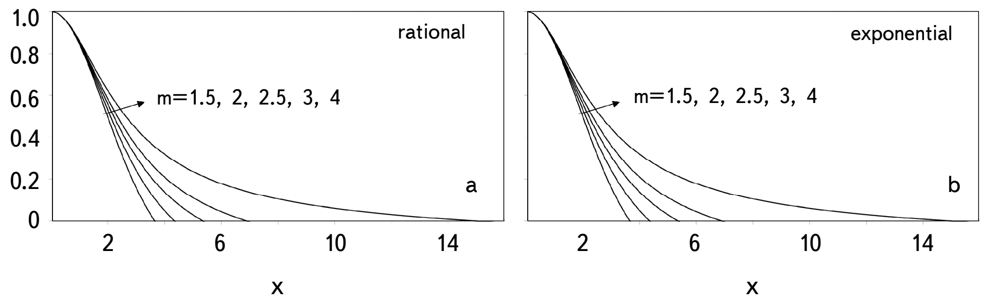

Fig.1. The obtained graphs of solutions of Lane-Emden standard equations by basis of RB and EB with $m = 1.5,\ 2,\ 2.5,\ 3,\ 4$.

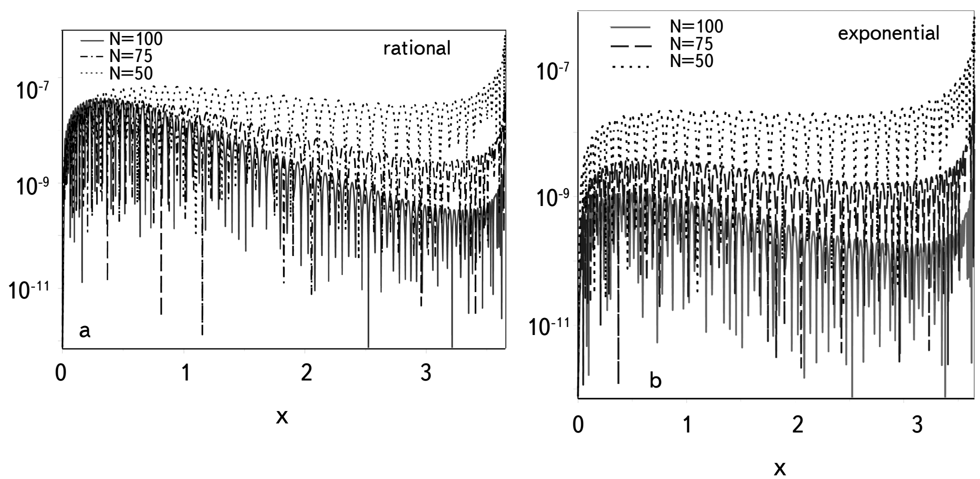

Fig.2. Logarithmic graph of residual error by present works with $N = 50,\ 75,\ 100$ and iteration 15 when $m = 1.5$.

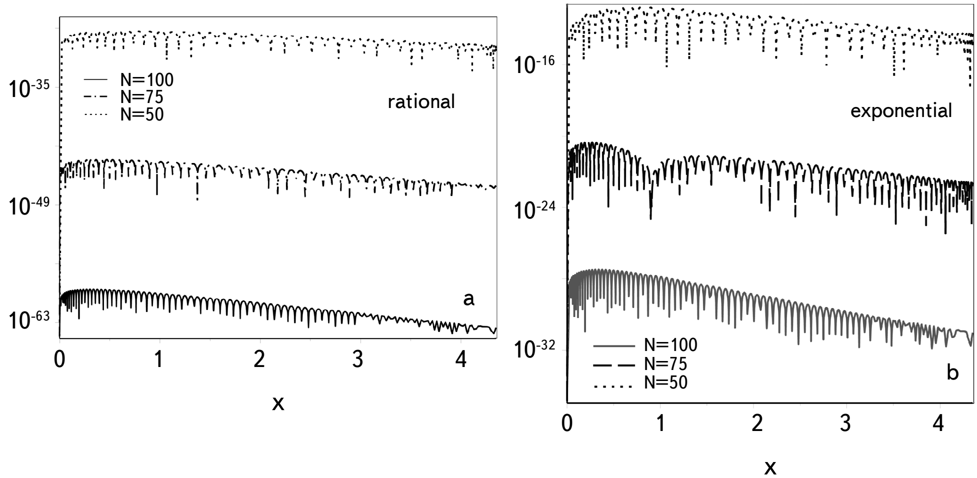

Fig.3. Logarithmic graph of residual error by present works with $N = 50, 75, 100$ and iteration 15 when $m = 2$.

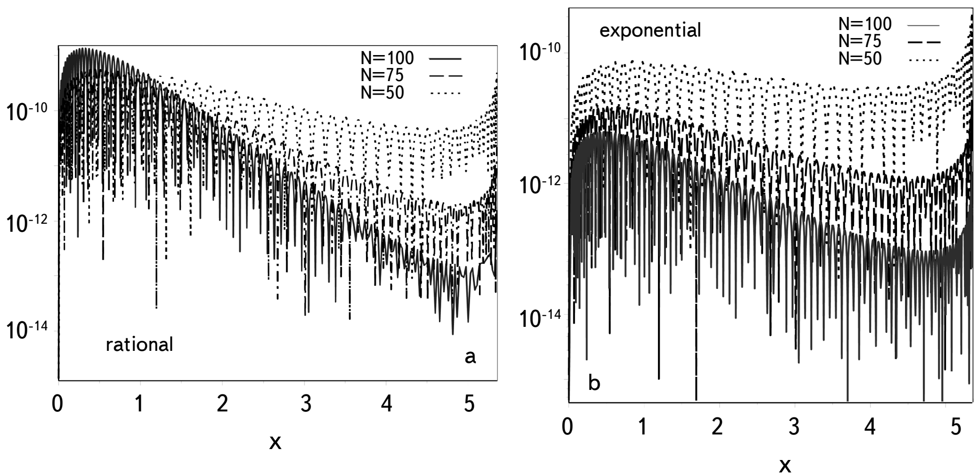

Fig.4. Logarithmic graph of residual error by present works with $N = 50, 75, 100$ and iteration 15 when $m = 2.5$.

functions, we convert the nonlinear problems to a sequence linear equations using the QLM.

2. Numerical results indicate effectiveness, applicability, and accuracy of the present approaches.

3. Present paper describes shortly bibliography of different methods utilized in previous works for solving Lane-Emden-type equations.

4. The approaches applied to solve the problems without reformulating the equation to bounded domains.

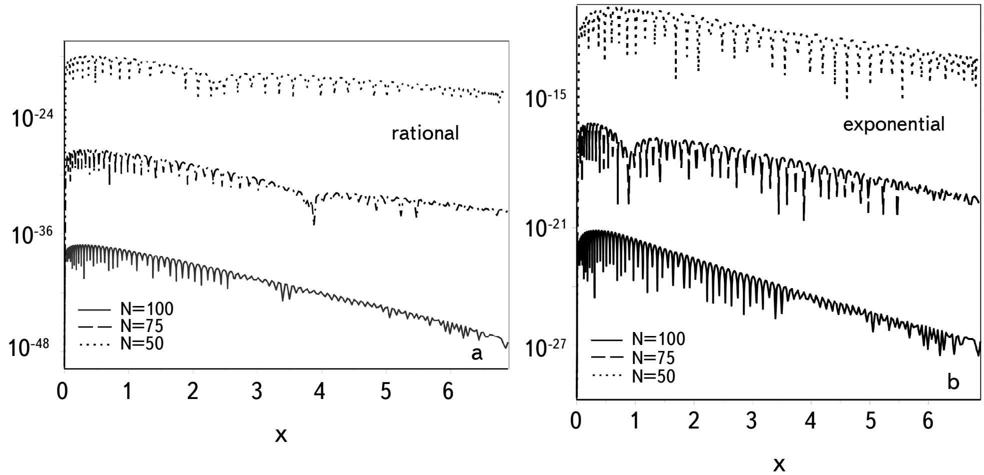

Fig.5. Logarithmic graph of residual error by present works with $N=50$, 75, 100 and iteration 15 when $m=3$.

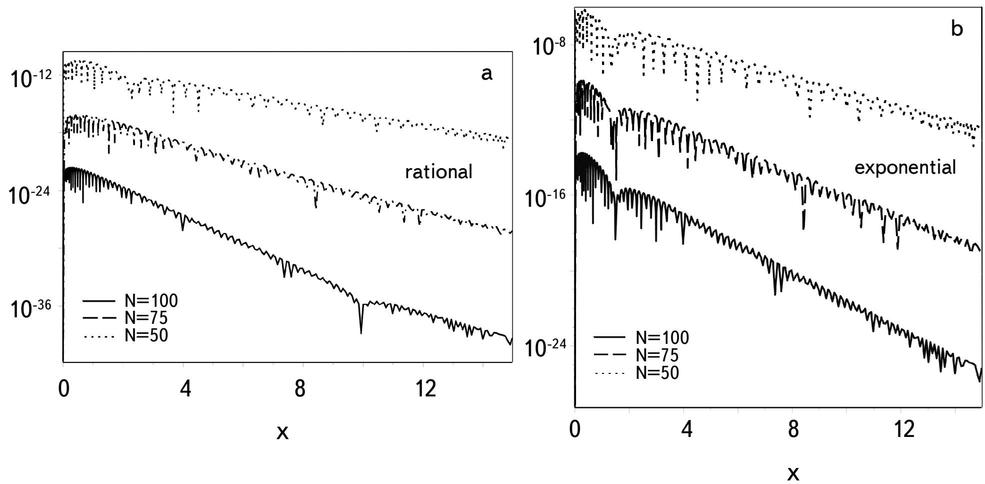

Fig.6. Logarithmic graph of residual error by present works with $N=50$, 75, 100 and iteration 15 when $m=4$.

5. The approaches have been displayed converges when increasing the number of collocation points by tabular reports.

6. At the first time, Rational and Exponential Bessel functions have been to obtain numerical outcomes of the nonlinear exponent m of the standard Lane-Emden equations.

7. Moreover, a very good approximation solution of $y(x)$ for Lane-Emden type equations with the various values of parameter m after only fifteen iterations are

obtained. So, these methods are a good experience and method for the other sciences.

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ДВА ЭФФЕКТИВНЫХ ВЫЧИСЛИТЕЛЬНЫХ АЛГОРИТМА ДЛЯ РЕШЕНИЯ НЕЛИНЕЙНЫХ СИНГУЛЯРНЫХ УРАВНЕНИЙ ЛЕЙНА-ЭМДЕНА

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В статье сравниваются два эффективных вычислительных алгоритма, основанные на рациональных и экспоненциальных функциях Бесселя (RB и EB), для решения некоторых хорошо известных классов нелинейных моделей типа Лейна-Эмдена. Задачи, которые встречаются в ряде моделей не-ньютоновской механики жидкости и математической физики, являются нелинейными обыкновенными дифференциальными уравнениями второго порядка на полубесконечном интервале и имеют особенность при $x=0$. Нелинейные уравнения Лейна-Эмдена преобразуются в последовательность линейных дифференциальных уравнений с использованием метода квазилинеаризации (QLM), а затем эти линейные уравнения решаются методами коллокации RB и EB. После этого полученные результаты сравниваются с решением других методов для демонстрации эффективности и применимости предложенных методов.

Ключевые слова: *рациональные функции Бесселя: экспоненциальные функции Бесселя: уравнения типа Лейна-Эмдена: нелинейное ОДУ: метод квазилинеаризации: метод коллокации*

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