АСТРОФИЗИКА

ТОМ 64 ФЕВРАЛЬ, 2021 ВЫПУСК 1

NON-SINGULAR COORDINATES OF SOME BLACK HOLE IN f(R) GRAVITY

S.M.J.RIAZ, R.HUSSAIN

Received 2 December 2019
Accepted 23 December 2020

Non-singular Kruskal-like coordinates of some black holes space-times in $f(R)$ gravity are presented in this research paper, and are also removed by establishing Kruskal-Szekeres coordinates for Non-extremal case. Carter-like coordinates can also be built for its extreme case.

Keywords: *Kruskal coordinates: black hole in $f(R)$ gravity: Carter coordinates*

1. *Introduction.* In practical sciences, it is well-thought-out that physics is the science of dimension. Thus, the position possesses the fundamental standing in the centre of all the quantities. For this purpose, to calculate precisely the exact place of some particles in space-time, a coordinate system (CS) is chosen. After selecting a CS, a tag is given to all points in space-time geometry. As physics' laws are invariant, selection of any CS will not create any issue, thus, to compare quantities amongst the two different CS's, we required a collection of equations related to the two different points assigned to a like point physically. These types of connections are so-called coordinate transformations which use in two coordinate systems. In physics, a very famous, simple and the popular CS is Cartesian CS, with three axes which are 90° to each other, familiarized as x-axis, y-axis and z-axis. Though, it is not our best choice every time. Generally, it is the most appropriate to select a CS that has alike symmetries with the model under discussion. Hence, for the spherically symmetric models of physics, spherical coordinate-system will be preferred selection. In that condition, again a point is categorized by three tags (quantities), one shows the locus of a point from the centre (distance) and the rest 2 are angular coordinates. Again the set of conversions accorded and narrate as the indicators of a mark of one CS to the indicators of the same mark in another CS.

General relativity (GR), cosmology and astrophysics envisage amazing phenomena's like neutron stars, black holes and gravitational waves. Vast CS's are implemented in the black hole physics for the purpose of removing physical singularity. This is for the purpose to find various geometrical shapes of black holes, see [1-3]. Not theoretically, the number of CS's is also used during practical field of solar-terrestrial relationship to point out the trajectory of different satellites, locations of boundaries, vector field parameter etc. The requirement of more coordinate systems is the arisen question here. Possible answers are, rarely different physical techniques are easy to understand, experimental informatio's are properly ordered, and also easy to manipulate in one coordinate system related to the other CS's. Transforming from one CS to others is necessary for these situations.

By the invention of cosmic acceleration, the phenomenon of a black hole (BH) has become the most interesting illustration with useful physical aspects. Two important existences of vacuum BH solutions under the field general relativity (GR), like uncharged Schwarzschild and charged Reissner-Nordstrom black hole. Schwarzschild's space-time temporary singularity in (t, r) coordinates, when $r=2m$ by Kruskal (or Kruskal-Szekeres-like) coordinates are defined to this space-time by direct substitution of $r=2m$ in metric's coefficients. After substitution we get a finite number ($\pm 16m^2/e$) henceforth, at horizon, the coordinates become regular [4]. The Reissner-Nordstrom BH when $Q<m$ is singular at (r_-, r_+) in the coordinates (t, r). An appropriate redefine an analogue to Kruskal-like coordinates helped to avoid these singularities [5], but unable to vanish both simultaneously. As far as to use that analogue, two independent coordinate patches are required. Although, the number of black holes (BHs) is present in Einstein's general relativity, where the presented case is for non-vacuum [6-13] and needed further illustration.

The presentation order of the research article is as follows. Section 2, consists brief introduction of important cases of some BH in $f(R)$ gravity space time. For a given space time, we also have explained physical as well as curvature singularities. The proceeding subsections contain non-singular coordinates of the non-extremal space-time cases. Also possibility of obtaining non-singular Kruskal-like coordinates for extremal case is explained. Furthermore, we built Carter-like coordinates for the extremal case of the BH in $f(R)$ geometry. The last section is reserved for the final summary and conclusion.

2. *Some BH in $f(R)$ gravity*. The action of GR has been reformed to enlighten the accelerative expansions of space and currently $f(R)$ gravity stands among a good plan reform of GR. It is because of the fact that $f(R)$ gravitational theory is capable of generating an accelerative expansion in a universe [14]. Besides this, if a cosmological constant occurs, it does not possess measurable effects of the most astrophysical phenomena's [15]. Nevertheless, the $f(R)$ gravity can have astrophysical significance. Actually, astrophysical significance are used to govern different types of $f(R)$ gravitational models. Thus, it became more exciting and essential to learn astrophysical phenomena by means of $f(R)$ gravity. Some space-times in $f(R)$ gravity models are studied in [15] debated the constancy of $f(R)$ BH. Additionally, there are numerous claims in $f(R)$ space-time theory, i.e. active

equation approach, gravitational waves, LHC trial, brans models etc.

In the last decade, thought-provoking models of $f(R)$ gravitation are under study and performances of particles bounded by a BH have been invented. The Lagrangian essential for such a $f(R)$ gravity model is,

$$f(R)=R+\Lambda+\frac{R+\Lambda}{d^2\left(6\alpha^2\right)^{-1}R+2\alpha^{-1}}\ln\frac{R+\Lambda}{R_c}, \tag{1}$$

where Λ be a cosmological constant, R_c is integration constant and α and d are unrestricted constraints for the model under consideration. The condition required at the astronomical scale agrees to $R >> \Lambda$ and $d^2\left(6\alpha^2\right)^{-1} R >> 2\alpha$. Using these limits, we have

$$f(R)=R+\Lambda+d^2\left(6\alpha^2\right)^{-1}R\ln\frac{R}{R_c}. \tag{2}$$

The limit which is pertinent at the cosmological scale is $R \sim d^2\left(6\alpha^2\right)^{-1}R \sim \Lambda$, resulted $f(R)=R+\Lambda$. Using that limit, usually constraints the accelerating developments.

It is therefore stimulating to present a parameter $\beta=\alpha/d$ in terms of which both restrictions of the gravity have been discussed. Under that gravity, BH's metric, having a mass *M*, is given by

$$ds^2=-f(r)dt^2+\frac{1}{f(r)}dr^2+r^2d\Omega^2, \tag{3}$$

where $f(r)$ is

$$f(r)=1-\frac{2M}{r}+\beta r-\frac{\Lambda r^2}{3}.$$

Now, take $G=c=1$ and focused on a unique case $\Lambda \neq 0$, describes some BH in $f(R)$ gravity. When we put $f(r)=0$, we get three horizons of the gravity and are denoted by r_a, r_b and r_c. By observing nature of roots, two different cases will generate and are:

1) Non-extremal BH Space-time (NEBHST) which have 3 distinct, but real roots i.e. r_a, r_b and r_c

2) Extremal Space-time like (EBHST) having real and repeating roots, $r_a=r_b=r_c$.

At first, taking non-extremal BH Space-time. The absolute value of 3 roots of $f(R)$ are:

$$r_a=\sqrt{\frac{g_2}{3}}\cos\frac{\pi-\eta}{3}, \tag{4}$$

$$r_b=\sqrt{\frac{g_2}{3}}\cos\frac{\pi+\eta}{3}, \tag{5}$$

$$r_c = \sqrt{\frac{g_2}{3}}\cos\frac{\pi}{3}, \tag{6}$$

where, η is the angle with $0 \le \eta \le \pi$ and r_a, r_b and r_c represents the horizons in $f(R)$ gravity. Given Ricci scalar "R" of the gravity under consideration is

$$R = \frac{6\sigma}{r} - 4\Lambda .$$

Clearly it shows that r_a, r_b and r_c can be physical singularities. Now constructing the 2nd curvature invariants

$$I_1 = R^{\mu}_{\nu} = \frac{2\left(5\beta^2 - 6r\beta\Lambda + 2r^2\Lambda^2\right)}{r^2} \tag{7}$$

and

$$I_2 = R^{\mu}_{\nu}\frac{48M^2}{r^6} + \frac{8\beta^2}{r^2} - \frac{8\beta\Lambda}{r} + \frac{8\Lambda^2}{8}, \tag{8}$$

confirm that they are physical singularities. Hence, we can try to build non-singular coordinates of the space-time. By the roots, r_a, r_b and r_c, we express $f(r)$ as

$$f(r) = \frac{(r-r_a)(r-r_b)(r-r_c)}{3r}. \tag{9}$$

2.1. *Non-singular Kruskal-like coordinates for the non-extremal BH space-time in $f(R)$ gravity*. For elimination of physical singularities, we defining r^* by

$$r^* = \int \frac{1}{f(r)}dr = \tilde{\delta}\left[\ln\left|\frac{r}{r_a}-1\right|^{r_a(r_c-r_b)} + \ln\left|\frac{r}{r_b}-1\right|^{-r_b(r_c-r_a)} + \ln\left|\frac{r}{r_c}-1\right|^{r_c(r_b-r_a)}\right], \tag{10}$$

where

$$\tilde{\delta} = \frac{3}{(r_b-r_a)(r_c-r_a)(r_c-r_b)}.$$

Then Eddington coordinates are

$$v = t + \tilde{\delta}\left[\ln\left|\frac{r}{r_a}-1\right|^{r_a(r_c-r_b)} + \ln\left|\frac{r}{r_b}-1\right|^{-r_b(r_c-r_a)} + \ln\left|\frac{r}{r_c}-1\right|^{r_c(r_b-r_a)}\right], \tag{11}$$

$$u = t - \tilde{\delta}\left[\ln\left|\frac{r}{r_a}-1\right|^{r_a(r_c-r_b)} + \ln\left|\frac{r}{r_b}-1\right|^{-r_b(r_c-r_a)} + \ln\left|\frac{r}{r_c}-1\right|^{r_c(r_b-r_a)}\right]. \tag{12}$$

Using Eddington coordinates, the usual BH metric takes the form

$$ds^2 = -\frac{f(r)}{4}\left[\left(1-\frac{9r^2}{\Delta^2\tilde{\delta}}\right)\left(dv^2+du^2\right) + 2\left(1+\frac{9r^2}{\Delta^2\tilde{\delta}}\right)dvdu\right] + r^2 d\Omega^2 . \tag{13}$$

Using Kruskal-like coordinates, the usual BH metric will be as

$$ds^2 = -\frac{f(r)}{4}\frac{\beta^2}{\Delta^2\tilde{\delta}^2}\left(\Delta^2\tilde{\delta}^2 - 9r^2\right)\left(\frac{1}{V^2}dV^2 + \frac{1}{U^2}dU^2\right) +$$
$$+\frac{f(r)}{4}\frac{\beta^2}{\Delta^2\tilde{\delta}^2}\frac{2}{\alpha^2}\left|\frac{r}{r_a}-1\right|^{2r_a\tilde{\delta}(r_c-r_b)/\beta}\left|\frac{r}{r_b}-1\right|^{-2r_b\tilde{\delta}(r_c-r_a)/\beta}\left|\frac{r}{r_c}-1\right|^{2r_c\tilde{\delta}(r_b-r_a)/\beta} dVdU + r^2 d\Omega^2 , \quad (14)$$

where

$$d\Omega^2 = d\theta^2 + \sin\theta^2 \, d\varphi^2 .$$

It is known that Kruskal-like coordinates [3] can't eliminate all the three singularities simultaneously. To this gravitation geometry, we need two different coordinate patches. First for regions $0<r<r_b$ and $r_b<r<\infty$, another for the regions $r_a<r<r_c$. For $0<r<r_c$ region, we define non-singular coordinates similar to Kruskal's-like coordinates as $V_1 = -\alpha\exp^{-v/\beta}$ and $U_1 = \alpha\exp^{u/\beta}$, where the retarded coordinates [16] are $v = t + r^*$ and $u = t - r^*$, choose $\alpha = r_a r_b r_c$ and $\beta = 2r_a r_b r_c\tilde{\delta}$. The space-time metric (14), according to these coordinates will be as

$$ds^2 = -\frac{f(r)}{\Delta^2}\left[\left(\Delta^2\tilde{\delta} - 9r^2\right)\left(dV_1^2 + dU_1^2\right) - 2\,dV_1 dU_1\right]\times$$
$$\times\left|\frac{r}{r_a}-1\right|^{(r_c-r_b)/r_b r_c}\left|\frac{r}{r_b}-1\right|^{-(r_c-r_a)/r_a r_c}\left|\frac{r}{r_c}-1\right|^{(r_b-r_a)/r_a r_b} + d\Omega^2 . \quad (15)$$

Clearly, singularity exists at $r = r_b$, where

$$V_1 = -r_a r_b r_c \exp^{-t/2r_a r_b r_c\tilde{\delta}}\left|\frac{r}{r_a}-1\right|^{(r_c-r_b)/r_b r_c}\left|\frac{r}{r_b}-1\right|^{-(r_c-r_a)/r_a r_c}\left|\frac{r}{r_c}-1\right|^{(r_b-r_a)/r_a r_b} , \quad (16)$$

$$U_1 = r_a r_b r_c \exp^{t/2r_a r_b r_c\tilde{\delta}}\left|\frac{r}{r_a}-1\right|^{(r_c-r_b)/r_b r_c}\left|\frac{r}{r_b}-1\right|^{-(r_c-r_a)/r_a r_c}\left|\frac{r}{r_c}-1\right|^{(r_b-r_a)/r_a r_b} , \quad (17)$$

$$V_1 U_1 = -\left(r_a r_b r_c\right)^2\left(\frac{r}{r_a}-1\right)^{2(r_c-r_b)/r_b r_c}\left(\frac{r}{r_b}-1\right)^{-2(r_c-r_a)/r_a r_c}\left(\frac{r}{r_c}-1\right)^{2(r_b-r_a)/r_a r_b} \quad (18)$$

and

$$t = 2r_a r_b r_c\tilde{\delta}\tanh^{-1}\frac{V_1+U_1}{V_1-U_1} .$$

As $r \to r_a$ or r_c, $(V_1, U_1) \to 0$, although in metric (15) singularity still exists at $r = r_b$. This coordinate system cover regions $0<r<r_b$ and $r_b<\infty$ of the whole manifold. A CS, analogous to Kruskal-like coordinates for $r_a<r<r_c$ regions are $V_2 = \alpha\exp^{v/\beta}$ and $U_2 = -\alpha\exp^{-u/\beta}$, where the recent and retarded coordinates are $v = t + r^*$ and $u = t - r^*$, choosing $\alpha = r_a r_b r_c$ and $\beta = 2r_a r_b r_c\tilde{\delta}$. The space-time

metric (14) then takes the form as

$$ds^2 = \frac{f(r)}{\Delta^2}\left[\left(\Delta^2\tilde{\delta}^2 + 9r^2\right)\left(dV_2^2 + dU_2^2\right) - 2\,dV_2 dU_2\right]\times$$
$$\times\left|\frac{r}{r_a}-1\right|^{-(r_c-r_b)/r_b r_c}\left|\frac{r}{r_b}-1\right|^{(r_c-r_a)/r_a r_c}\left|\frac{r}{r_c}-1\right|^{(r_b-r_a)/r_a r_b} + r^2 d\Omega^2 . \tag{19}$$

This is non-singular at $r = r_a$ and $r = r_c$, where

$$V_2 = r_a r_b r_c \exp^{t/2 r_a r_b r_c \tilde{\delta}}\left|\frac{r}{r_a}-1\right|^{-(r_c-r_b)/r_b r_c}\left|\frac{r}{r_b}-1\right|^{(r_c-r_a)/r_a r_c}\left|\frac{r}{r_c}-1\right|^{-(r_b-r_a)/r_a r_b} , \tag{20}$$

$$U_2 = -r_a r_b r_c \exp^{-t/2 r_a r_b r_c \tilde{\delta}}\left|\frac{r}{r_a}-1\right|^{-(r_c-r_b)/r_b r_c}\left|\frac{r}{r_b}-1\right|^{(r_c-r_a)/r_a r_c}\left|\frac{r}{r_c}-1\right|^{-(r_b-r_a)/r_a r_b} , \tag{21}$$

$$V_2 U_2 = -(r_a r_b r_c)^2\left(\frac{r}{r_a}-1\right)^{-2(r_c-r_b)/r_b r_c}\left(\frac{r}{r_b}-1\right)^{2(r_c-r_a)/r_a r_c}\left(\frac{r}{r_c}-1\right)^{-2(r_b-r_a)/r_a r_b} \tag{22}$$

and

$$t = 2 r_a r_b r_c\,\tilde{\delta}\tanh^{-1}\frac{V_2+U_2}{V_2-U_2} .$$

As $r \to r_a$ and $r \to r_c$, $(V_2, U_2) \to 0$ but singularity still exists at the horizon $r \to r_b$ in the metric (19). This coordinate system V_2, U_2 only covers the region $r_a < r < r_c$ of the whole manifold.

2.2. *Kruskal-Szekeres like coordinates for the non-extremal BH space-time in $f(R)$ gravity*. Now introducing space-time coordinates for $0 < r < r_b$ and $r_b < r < \infty$ regions as in [17]

$$\xi_1 = V_1 + U_1 , \quad \eta_1 = V_1 - U_1 .$$

These coordinates transforming the metric (15) to

$$ds^2 = -\frac{2 f(r)}{r_a r_b r_c \Delta^2}\left[\left(\Delta^2\tilde{\delta} - 9r^2 - 1\right)d\xi_1^2 + \left(\Delta^2\tilde{\delta} - 9r^2 - 1\right)d\eta_1^2\right]\times$$
$$\times\left|\frac{r}{r_a}-1\right|^{(r_c-r_b)/r_b r_c}\left|\frac{r}{r_b}-1\right|^{-(r_c-r_a)/r_a r_c}\left|\frac{r}{r_c}-1\right|^{(r_b-r_a)/r_a r_b} + d\Omega^2 , \tag{23}$$

where

$$\xi_1 = r_a r_b r_c \sinh\left(\frac{t}{2 r_a r_b r_c}\tilde{\delta}\right)\left|\frac{r}{r_a}-1\right|^{(r_c-r_b)/2 r_b r_c}\left|\frac{r}{r_b}-1\right|^{-(r_c-r_a)/2 r_a r_c}\left|\frac{r}{r_c}-1\right|^{(r_b-r_a)/2 r_a r_b} ,$$

$$\eta_1 = r_a r_b r_c \cosh\left(\frac{t}{2 r_a r_b r_c}\tilde{\delta}\right)\left|\frac{r}{r_a}-1\right|^{(r_c-r_b)/2 r_b r_c}\left|\frac{r}{r_b}-1\right|^{-(r_c-r_a)/2 r_a r_c}\left|\frac{r}{r_c}-1\right|^{(r_b-r_a)/2 r_a r_b} .$$

Their inverse transformation is

$$t = 2r_a r_b r_c \tilde{\delta} \tanh^{-1} \frac{\xi_1}{\eta_1}.$$

ξ_1 and η_1 are relating to r by

$$\xi_1^2 - \eta_1^2 = (r_a r_b r_c)^2 \left|\frac{r}{r_a} - 1\right|^{(r_c - r_b)/r_b r_c} \left|\frac{r}{r_b} - 1\right|^{-(r_c - r_a)/r_a r_c} \left|\frac{r}{r_c} - 1\right|^{(r_b - r_a)/r_a r_b}.$$

Again, introducing space-time coordinates for $r_a < r < r_c$ regions [17].

$$\xi_2 = V_2 + U_2, \quad \eta_2 = V_2 - U_2.$$

Transforming the metric (19) by using above coordinates as:

$$ds^2 = -\frac{2f(r)}{r_a r_b r_c \Delta^2}\left[\left(\Delta^2\tilde{\delta} - 9r^2 - 1\right)d\xi_2^2 + \left(\Delta^2\tilde{\delta} - 9r^2 - 1\right)d\eta_2^2\right]\times$$
$$\times \left|\frac{r}{r_a} - 1\right|^{-(r_c - r_b)/r_b r_c} \left|\frac{r}{r_b} - 1\right|^{(r_c - r_a)/r_a r_c} \left|\frac{r}{r_c} - 1\right|^{-(r_b - r_a)/r_a r_b} + d\Omega^2, \tag{24}$$

where

$$\xi_2 = r_a r_b r_c \sinh\left(\frac{t}{2r_a r_b r_c}\tilde{\delta}\right)\left|\frac{r}{r_a} - 1\right|^{-(r_c - r_b)/2r_b r_c} \left|\frac{r}{r_b} - 1\right|^{(r_c - r_a)/2r_a r_c} \left|\frac{r}{r_c} - 1\right|^{-(r_b - r_a)/2r_a r_b},$$

$$\eta_2 = r_a r_b r_c \cosh\left(\frac{t}{2r_a r_b r_c}\tilde{\delta}\right)\left|\frac{r}{r_a} - 1\right|^{-(r_c - r_b)/2r_b r_c} \left|\frac{r}{r_b} - 1\right|^{(r_c - r_a)/2r_a r_c} \left|\frac{r}{r_c} - 1\right|^{-(r_b - r_a)/2r_a r_b}.$$

Their inverse transformation is

$$t = 2r_a r_b r_c \tilde{\delta} \tanh^{-1} \frac{\xi_2}{\eta_2}.$$

ξ_2 and η_2 are relating to r by

$$\xi_2^2 - \eta_2^2 = (r_a r_b r_c)^2 \left|\frac{r}{r_a} - 1\right|^{-(r_c - r_b)/r_b r_c} \left|\frac{r}{r_b} - 1\right|^{(r_c - r_a)/r_a r_c} \left|\frac{r}{r_c} - 1\right|^{-(r_b - r_a)/r_a r_b}.$$

2.3. *Compactification of Kruskal-Szekeres like coordinates for the non-extreme BH space-time in $f(R)$ gravity*. Now defining the compactified coordinates for $0 < r < r_b$ and $r_b < r < \infty$ regions [17].

$$\zeta_1 = \tan^{-1}(\xi_1 + \eta_1) + \tan^{-1}(\xi_1 - \eta_1),$$

$$\chi_1 = \tan^{-1}(\xi_1 + \eta_1) - \tan^{-1}(\xi_1 - \eta_1)$$

and for the region $r_a < r < r_c$ are defined as

$$\zeta_2 = \tan^{-1}(\xi_2 + \eta_2) + \tan^{-1}(\xi_2 - \eta_2),$$

$$\chi_2 = \tan^{-1}(\xi_2 + \eta_2) - \tan^{-1}(\xi_2 - \eta_2).$$

They implicitly relate with these coordinates and r as

$$\xi_1^2 - \eta_1^2 = (r_a r_b r_c)^2 \left|\frac{r}{r_a} - 1\right|^{(r_c - r_b)/r_b r_c} \left|\frac{r}{r_b} - 1\right|^{-(r_c - r_a)/r_a r_c} \left|\frac{r}{r_c} - 1\right|^{(r_b - r_a)/r_a r_b} =$$

$$= -\tan\frac{\zeta_1 + \chi_1}{2}\tan\frac{\zeta_1 - \chi_1}{2}$$

and

$$\xi_2^2 - \eta_2^2 = (r_a r_b r_c)^2 \left|\frac{r}{r_a} - 1\right|^{-(r_c - r_b)/r_b r_c} \left|\frac{r}{r_b} - 1\right|^{(r_c - r_a)/r_a r_c} \left|\frac{r}{r_c} - 1\right|^{-(r_b - r_a)/r_a r_b} =$$

$$= -\tan\frac{\zeta_2 + \chi_2}{2}\tan\frac{\zeta_2 - \chi_2}{2}.$$

Compactified Kruskal-Szekeres like coordinates shows the geometry of NEBH.

2.4. *Non-existence of Kruskal like coordinates for the extreme BH space-time in $f(R)$ gravity*. For EBHST case, transforming metric (3) to

$$ds^2 = -h(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2\left(d\theta^2 + \sin\theta^2 \, d\varphi^2\right), \tag{25}$$

where h(r) is

$$h(r) = \frac{(r - r_e)^3}{3r}.$$

In metric (25), singularity exists at $r=0$ and $r=r_e$ the former being an essential and the latter a coordinate singularity. To avoid the coordinate singularity for an extremal space-time, one defines r^* as

$$r^* = \int \frac{1}{h(r)} dr. \tag{26}$$

The recent and retarded coordinates [16] are (v, u) as $v = t + r^*$ and $u = t - r^*$. Then, the Kruskal-like coordinates (V, U) are $V = \alpha\exp^{\beta v}$ and $U = -\alpha\exp^{-\beta u}$ having α, β be constants. r^* for EBHST's geometry will be

$$r^* = \frac{-3(2r - r_e)}{(r - r_e)^2}. \tag{27}$$

Here, a type of singularity, named as pole divergence, unlikely non-extreme geometry, and all four are logarithmic divergence there. One more key difference, namely that for non-extreme geometry. The singularity is at 2 various values of r, but for extreme case, singularity exists at $r=r_e$. For non-extreme space-time the singular point can be vanish by setting appropriate values of β. But in extreme space-time, the singular point can't be vanished through ordinary procedures. It is because the extreme metric for Kruskal like coordinates (V, U) will be

$$ds^2 = -\frac{\beta^2 (r-r_e)^3}{3\alpha^2 r} \exp^{3(2r+r_e)/\beta(r-r_e)^2} dVdU + r^2\left(d\theta^2 + \sin\theta^2 d\varphi^2\right), \tag{28}$$

where singularity exists at $r = r_e$. Hence, Kruskal like coordinates have no use in extremal case.

2.5. *Non-singular Carter-like coordinates for the extreme BH space-time in $f(R)$ gravity*. Now introducing Carter-like coordinates for getting rid of coordinate singularities [18].

$$\psi = \tan^{-1}\frac{v}{t} + \cot^{-1}\frac{w}{t}, \quad \xi = \tan^{-1}\frac{v}{t} - \cot^{-1}\frac{w}{t},$$

where $v = t + r^*$, similarly $w = -t + r^*$ having r^* as

$$r^* = \int \frac{(r-r_e)^3}{3r} dr = \frac{-3(2r-r_e)}{2(r-r_e)^2}.$$

The following coordinates (ψ, ξ, θ, ϕ) transformed the line element (25) to

$$ds^2 = \frac{-l^2 (r-r_e)^3}{12r} \sec^2\frac{\psi+\xi}{2} \csc^2\frac{\psi-\xi}{2}\left[d\psi^2 - d\xi^2\right] + r^2\left(d\theta^2 + \sin\theta^2 d\varphi^2\right). \tag{29}$$

Here ψ and ξ can be relating with radial parameter as

$$\tan\frac{\psi+\xi}{2} + \cot\frac{\psi-\xi}{2} + \frac{3(2r-r_e)}{l(r-r_e)^3} = 0.$$

We can find the determinant of (23) as:

$$\left|g_{xy}\right| = -\frac{l^4 r^2 (r-r_e)^6}{144} \sec^4\frac{\psi+\xi}{2} \csc^4\frac{\psi-\xi}{2} \sin^2\theta, \quad (x, y = 0, 1, 2, 3), \tag{30}$$

where

$$\sec^4\frac{\psi+\xi}{2} = \left(1 + \frac{v^2}{l^2}\right)^2 = \left(1 + \frac{1}{l^2}(t + r^*)^2\right)^2, \tag{31}$$

$$\csc^4\frac{\psi-\xi}{2} = \left(1 + \frac{w^2}{l^2}\right)^2 = \left(1 + \frac{1}{l^2}(-t + r^*)^2\right)^2. \tag{32}$$

We get a finite determinant if $r \to r_e$ and $\psi \to \xi$, like the extreme case of BTZ BH. Also the metric (29) becomes non-singular at $r = r_e$. Hence Carter-like coordinates vanishes the coordinate singularity of the EBHST.

3. *Summary and conclusions*. In our observation, all the three singularities can't be removed at a time by single coordinate patch through Kruskal-like coordinates. For gravitation geometry, we need 2 distinct coordinate patches. First is for the regions $0 < r < r_b$ and $r_b < r < \infty$. Similarly for the next patch, the region

is $r_a < r < r_c$. By compactified Kruskal-Szekeres like coordinates, not only we can remove coordinate singularities but in future be able for retrieving of space-time diagram of the whole NEBHST. It also revealed that Kruskal-like coordinates are unable for removing singularity at $r = r_e$ of EBHST's geometry. We try to resolve Einstein's equations related to this case, but couldn't successful till now. Our efforts show that EBHST metric can't be transform to a desired form such as in rotating BTZ BH case [17]. Anyways, that can't be shown formally, hence the results couldn't be relevant. Although to keep the metric away from physical singularity of EBHST, we are successful to build non-singular Carter-like coordinates, regular for extreme case only.

Department of Mathematics, COMSATS University Islamabad, Wah Campus, G.T. Road, Wah Cantt. 47040, Pakistan, e-mail: jawwadriaz@yahoo.com

НЕСИНГУЛЯРНЫЕ КООРДИНАТЫ НЕКОТОРЫХ ЧЕРНЫХ ДЫР В f(R) ГРАВИТАЦИИ

С.М.Дж.РИАЗ, Р.ХУССЕЙН

В статье представлены несингулярные координаты типа Крускала пространства-времени некоторых черных дыр в гравитации $f(R)$. Сингулярности координат могут быть удалены путем установления координат Крускала-Секереша для неэкстремального случая. Для экстремального случая могут быть построены координаты типа Картера.

Ключевые слова: *Координаты Крускала: черная дыра в гравитации* $f(R)$: *координаты Картера*

REFERENCES

1. *A.S.Eddington*, Nature, **113**, 192, 1924.
2. *D.Finkelstein*, Phys. Rev., **1**, 965, 1958.
3. *M.D.Kruskal*, Phys. Rev., **119**, 1743, 1960.
4. *C.W.Misner*, *K.S.Thorne*, *J.A.Wheeler*, Gravitation, W.H.Freeman and Sons, 1973.
5. *A.Qadir, A.A.Siddiqui,* J. Math. Phys., **40**, 4598, 1999.

6. *J.D.Bekenstein*, Ann. Phys., **91**, 75, 1975.
7. *E.Ayon-Beato*, *A.Garcia*, Gen. Relat. Grav., **31**, 629, 1999.
8. *D.Garfinkle*, *G.T.Horowitz*, *A.Strominger*, Phys. Rev., **D43**, 3140, 1991.
9. *D.Garfinkle*, *G.T.Horowitz*, *A.Strominger*, Phys. Rev., **D45**, 3888(E), 1992.
10. *A.Jawad*, *M.U.Shahzad*, Eur. Phys. J., **C76**, 123, 2016.
11. *R.J.Yang*, arXiv: 1605, 02320.
12. *A.Younas et al.*, Phys. Rev., **D92**, 084042, 2015.
13. *B.Majeed*, *M.Jamil*, *P.Pradhan*, Advances in High Energy Physics, 124910, 2015.
14. *V.V.Kiselev*, Class Quantum Grav., **20**, 1187, 2003.
15. *A.Younas*, *S.Hussain*, *M.Jamil et al.*, Phys. Rev., **D92**, 084042, 2015.
16. *A.Qadir*, Einsteins general theory of relativity, (under preparation).
17. *S.M.Jawwad*, *A.A.Siddiqui*, Gen. Relativ. Gravit., **43**, 1167, 2010.
18. *B.Carter*, Phys. Lett., **21**, 423, 1966.