International Journal of Mathematics and
Computer Science, **13**(2018), no. 1, 59–71

On the Generation of Pythagorean Triples and Representation of Integers as a Difference of Two Squares

Emil Asmaryan

Institute of Radiophysics and Electronics of Armenian Nat. Ac. Sci.
Alikhanian Bros. str., 1, 0203, Ashtarak, Armenia

email: emilas@irphe.am

(Received November 12, 2017, Accepted November 27, 2017)

Abstract

We propose general formulas for finding the quantity of all primitive and non-primitive triples generated by a given number x. In addition, we obtain formulas for finding the complete quantity of the representations of the integers as a difference of two squares.

1 Introduction

We consider the equation

$$x^2 + y^2 = z^2. \tag{1.1}$$

The solutions of (1.1), when x,y,z (x,$y > 2$) are positive integers, called **Pythagorean triples**, and (1.1) itself is among the Diophantine equation. If such a triple (x,y,z) is known, then one can obtain the infinite quantity of solutions of (1.1) by multiplying of x,y,z by any nonzero integer. The triples with coprime x,y,z are called **primitive**.

Since the Euclidean era there was a well-known method, which allows to calculate the Pythagorean triples by two integers k, l, representing x, y, and z as

$$x = 2kl, \qquad y = k^2 - l^2, \qquad z = k^2 + l^2; \qquad k > 1$$

Key words and phrases: Pythagorean triple, primitive, integer, square, representation.
AMS (MOS) Subject Classifications: 11D09, 11A67.
ISSN 1814-0432, 2018, http://ijmcs.future-in-tech.net

$$(2kl)^2 + (k^2 - l^2)^2 = (k^2 + l^2)^2 \tag{1.2}$$

If k and l are coprime and of opposite parity, then (1.2) gives the primitive triples.

There are other ways of obtaining Pythagorean triples as special combinations of certain numbers.

In this work, however, the problem of finding the Pythagorean triples is stated as follows: for any integer $x > 2$ we find the total quantity of the primitive and non-primitive triples, in which x is one of the elements on the left side of (1.1) (i.e., generated by x), and also we propose a convenient way for their calculation.

The methods mentioned above are not effective to solve this kind of problem. So, the method (1.2) gives only as many triples as the given x can be represented as $2kl$ (if x is even), or as $k^2 - l^2$ (if x is odd).

The Euclidean method gives the right quantity of primitive triples and also gives some quantity of non-primitive (if the integer numbers k and l are not coprime, or they are both odd). However, as it is shown below, the total quantity of non-primitive triples, generated by x, in reality may be significantly more than the quantity of the triples obtained from (1.2) with the integers k and l.

In the present work the general formulas for finding the quantity of all triples generated by the given number x have been proposed. Also the formulas for finding the complete quantity of the representations of the integers as a difference of two squares have been obtained. Some of these results had been represented in [1].

2 Calculation of triples

Let us turn to the calculation of Pythagorean triples and their quantity. Taking into account, that the numbers x and y can both be even or of different parity, but cannot both be odd, we use the following standard procedure:

(a) Let the odd number $x > 1$ be given. Then y is even, and z is odd: i.e., $z = y + (2m + 1)$, $m \geq 0$.

From (1.1) we have:

$$x^2 + [z - (2m+1)]^2 = z^2,$$

$$z = \frac{1}{2}\left[\frac{x^2}{2m+1} + 2m + 1\right], \qquad y = \frac{1}{2}\left[\frac{x^2}{2m+1} - (2m+1)\right] \tag{2.3}$$

Since y and z are integers, $d := 2m+1$ must be divisor of x^2:

$$y_i = \frac{1}{2}\left(\frac{x^2}{d_i} - d_i\right), \qquad z_i = \frac{1}{2}\left(\frac{x^2}{d_i} + d_i\right) \tag{2.4}$$

Substituting the values of the divisors d_i into (2.4) we obtain corresponding triples $(x,\, y_i,\, z_i)$.

From (2.4), it is easy to show that z will be odd and y will be even.

In order to find the quantity of all different triples with positive elements, only divisors $d_i < x$ must be used in (2.4). Let N_d be the number of all divisors of x^2. It is obvious that the number of divisors $d_i < x$ and $d_i > x$ are the same. Therefore the quantity of $d_i < x$ and so the quantity of Pythagorean triples is:

$$N_{tr} = \frac{N_d - 1}{2} \tag{2.5}$$

Note, that the squares of integers always have the odd quantity of divisors.

(b) Let the even number $x > 2$ be given. In this case y and z are the numbers of the same parity; i.e., $z = y + 2m$, $m > 0$.

Then, from (1.1) we obtain:

$$x^2 + (z - 2m)^2 = z^2,$$
$$x^2 - 4mz + 4m^2 = 0$$

$$y = \frac{(\frac{x}{2})^2}{m} - m, \qquad z = \frac{(\frac{x}{2})^2}{m} + m \tag{2.6}$$

Since y and z must be integers, $m \equiv d$ must be divisor of $(\frac{x}{2})^2$,i.e.

$$y_i = \frac{(\frac{x}{2})^2}{d_i} - d_i, \qquad z_i = \frac{(\frac{x}{2})^2}{d_i} + d_i. \tag{2.7}$$

Substituting the values of d_i into (2.7), we obtain the triples $(x,\, y_i,\, z_i)$. In order to obtain the triples with positive elements only divisors $d_i < \frac{x}{2}$ must be used in (2.7). If N_d is the quantity of all divisors of $(\frac{x}{2})^2$, then the number of divisors $d_i < \frac{x}{2}$ is equal to $\frac{N_d-1}{2}$. Therefore, the quantity of all Pythagorean triples, generated by the number x, is also $\frac{N_d-1}{2}$.

Let us now determine N_d in cases a) and b). For convenience we write $n := x$.

As it is known, the complete quantity Q of divisors of any number n may be determined by its canonical expansion

$$n = p_1^{s_1} \cdot p_2^{s_2} \cdots \cdot p_q^{s_q}, \tag{2.8}$$

where $p_1, \cdots, p_q$ are the different prime numbers.

Then

$$Q = (s_1+1)(s_2+1)...(s_q+1). \tag{2.9}$$

If n is odd, then from its canonical expansion (2.8) we obtain:

$$n^2 = p_1^{2s_1} \cdot p_2^{2s_2} \cdots \cdot p_q^{2s_q}, \tag{2.10}$$

and

$$N_d = (2s_1+1)(2s_2+1)...(2s_q+1). \tag{2.11}$$

Hence the complete quantity of Pythagorean triples is:

$$N_{tr} = \frac{(2s_1+1)(2s_2+1)...(2s_q+1)-1}{2} \tag{2.12}$$

If n is even, then $p_1 = 2$. Writing down n as $n = 2^{s_1+1} \cdot p_2^{s_2} \cdots p_q^{s_q}$, $s_1 \geq 0$, we obtain:

$$\frac{n}{2} = 2^{s_1} \cdot p_2^{s_2} \cdots p_q^{s_q}, \qquad (\frac{n}{2})^2 = 2^{2s_1} \cdot p_2^{2s_2} \cdots p_q^{2s_q} \tag{2.13}$$

Then the complete quantity of triples is given by the same expression (2.12), but $s_1, s_2, \cdots, s_q$ are taken from the canonical expansion (2.13) of $\frac{n}{2}$: Thus we can formulate the following result:

Every integer $n > 2$ generates N_{tr} Pythagorean triples having the form:

(a) for odd n:

$$\left\{n, \frac{1}{2}\left(\frac{n^2}{d_i} - d_i\right), \frac{1}{2}\left(\frac{n^2}{d_i} + d_i\right)\right\}, \tag{2.14}$$

where d_i are the divisors of n^2 which are less than n, and N_{tr} is determined by expression (2.12) with $s_1, s_2, \cdots, s_q$ taken from canonical expansion of n;

(b) for even n:

$$\left\{n, \frac{(\frac{n}{2})^2}{d_i} - d_i, \frac{(\frac{n}{2})^2}{d_i} + d_i\right\}, \tag{2.15}$$

where d_i are the divisors of $(\frac{n}{2})^2$ which are less than $\frac{n}{2}$, and N_{tr} is determined by expression (2.12) with $s_1, s_2, \cdots, s_q$ taken from canonical expansion of $\frac{n}{2}$.

As an illustration let us consider the following example: $n = 120 = 2^3 \cdot 3 \cdot 5$; $\frac{n}{2} = 2^2 \cdot 3 \cdot 5 = 60$; $(\frac{n}{2})^2 = 3600 = 2^4 \cdot 3^2 \cdot 5^2$; $N_d = 45$; $N_{tr} = 22$, including 4 primitives. The Euclidean method (1.2) (with integers k and l) gives only 6 triples, including 4 primitives.

It follows from (2.14) and (2.15) that for given n there are maximal and minimal values of other two elements of Pythagorean triples, namely:

(a) for odd n:
maximal: $\frac{n^2-1}{2}$ and $\frac{n^2+1}{2}$, at $d_i = 1$
minimal: $\frac{1}{2}(\frac{n^2}{d_{max}} - d_{max})$ and $\frac{1}{2}(\frac{n^2}{d_{max}} + d_{max})$, where d_{max} is the divisor of n^2, most near to n;

(b) for even n:
maximal: $\left(\frac{n}{2}\right)^2 - 1$ and $\left(\frac{n}{2}\right)^2 + 1$, at $d_i = 1$
minimal: $\frac{\left(\frac{n}{2}\right)^2}{d_{max}} - d_{max}$ and $\frac{\left(\frac{n}{2}\right)^2}{d_{max}} + d_{max}$, where d_{max} is the divisor of $(\frac{n}{2})^2$, most near to $\frac{n}{2}$.

3 Determination of the quantity of primitive triples

Now we find the quantity of primitive triples included in N_{tr}

3.1 The case of n being odd

Again we use the canonical expansion (2.10). As a divisor d_i in (2.14) let us take multipliers contained in (2.10), which are less than n and represent the products of numbers $p_j^{2s_j}$ between themselves in all possible quantities and combinations (single, double, triple, and so on); i.e., the numbers of form

$$p_j^{2s_j}, p_j^{2s_j} \cdot p_m^{2s_m}, p_j^{2s_j} \cdot p_m^{2s_m} \cdot p_j^{2s_r}, ... < n \qquad (3.16)$$
$$j, m, r... = 1, 2, ..., q$$

When dividing the number n^2, written in the form (2.10), by the divisors d_i of the type (3.16), all appropriate $p_j, ..., p_r$ will be canceled in the first term of bracket in expressions $\frac{1}{2}\left(\frac{n^2}{d_i} \mp d_i\right)$, whereas the second term contains only these $p_j, ..., p_r$. Therefore these two terms will not contain the same $p_j, ..., p_r$ and cannot divide by any divisors of n. Then the elements of such triples will be coprime and these triples will be primitive.

Therefore the d_i giving the primitive triples consist of indivisible $p_j^{2s_j}$ numbers which can be considered as if they are prime, to the power 1. Then writing (2.10) as

$$n^2 = \left(p_1^{2s_1}\right)^1 \cdot .... \cdot \left(p_q^{2s_q}\right)^1$$

and using (2.9), we obtain that the quantity of divisors of type (3.16) is equal to $\frac{(1+1)^q}{2} = 2^{q-1}$. Hence, we obtain the following result:

The quantity of primitive triples generated by the odd number n, is:

$$N_p = 2^{q-1}, \qquad (3.17)$$

where q is the quantity of prime numbers p_j in the canonical expansion of n.

Thus, N_p does not depend on numbers p_j, s_j and is determined only by the number q.

3.2 The case of n being even

If $\frac{n}{2}$ is odd (i.e. n indivisible by 4), then one can see from (2.15) that y and z are also both even by all divisors d_i. Therefore such n cannot generate the primitive triples. If $\frac{n}{2}$ is even, then the primitive triples are obtained from (2.15) by d_i, having the form (3.16) in the canonical expansion (2.13) of $\left(\frac{n}{2}\right)^2$. Hence, we obtain the following combined expression for quantity of primitive triples generated by even n:

$$N_p = 2^{q-2}\big[1 + (-1)^{n/2}\big], \tag{3.18}$$

where q is the quantity of prime numbers p_j in canonical expansion of $\frac{n}{2}$.

Note that N_p always includes the triple, obtained by $d_i = 1$.

For illustration we consider the example:

$$n = 2220 = 2^2{\cdot}3{\cdot}5{\cdot}37; \frac{n}{2} = 1110 = 2{\cdot}3{\cdot}5{\cdot}37; \left(\frac{n}{2}\right)^2 = 2^2{\cdot}3^2{\cdot}5^2{\cdot}37^2 = 1232100$$

The total quantity of triples $N_{tr} = \frac{(2+1)^4-1}{2} = 40$;

The quantity of primitive triples $N_p = 2^{4-1} = 8$, and they are obtained from (2.15) by the following divisors d_i: $d_1 = 1; d_2 = 2^2; d_3 = 3^2; d_4 = 5^2; d_5 = 2^2 \cdot 3^2; d_6 = 2^2 \cdot 5^2; d_7 = 3^2 \cdot 5^2; d_8 = 2^2 \cdot 3^2 \cdot 5^2$.

In [2, 3] the quantity of primitive Pythagorean triangles with given inradius has been obtained. Here we have determined the complete quantity of Pythagorean triangles with given cathetus.

4 The representation of integers as a difference of the squares of two integer numbers

The formulas (2.14) and (2.15) are proven to be useful for finding all possible representations of integer numbers as a difference of two squares.

4.1 The case of n being an odd number

Let n be any odd number > 1. We want to represent n in form

$$n = k^2 - l^2, \qquad k, l > 0 \tag{4.19}$$

It is clear from identity (1.2) written as

$$(2kl)^2 + n^2 = (k^2 + l^2)^2,$$

that $2kl$ and $k^2 + l^2$ are the elements of Pythagorean triples generated by the odd number n. As we already know, they are given by expressions (2.14).

In particular, we can write:

$$k^2 + l^2 = \frac{1}{2}\left(\frac{n^2}{d_i} + d_i\right) \tag{4.20}$$

From (4.19) and (4.20) we find k and l:

$$k = \frac{n + d_i}{2\sqrt{d_i}}, \qquad l = \frac{n - d_i}{2\sqrt{d_i}} \tag{4.21}$$

From (4.21), it follows that k and l will be integers and positive, if $d_i < n$ and are the squares of integers. Indeed, in this case $\sqrt{d_i}$ is an integer, being the divisor of n and d_i, and, besides, since n and d_i, are both odd, their sum and difference are even. One can see that k and l have opposite parity.

Therefore, the quantity N_r of all representations (4.19) is equal to quantity of divisors of n^2, which are less than n, and are in canonical expansion (2.10) the numbers of the form

$$p_j^{2\alpha_j}, p_j^{2\alpha_j} \cdot p_m^{2\alpha_m}, p_j^{2\alpha_j} \cdot p_m^{2\alpha_m} \cdot p_r^{2\alpha_r}, ... < n \tag{4.22}$$
$$j, m, r... = 1, 2, ..., q, \qquad \alpha_j \leq s_j$$

We find the quantity of such divisors writing (2.10) in form

$$n^2 = \left(p_1^2\right)^{s_1} \cdot \left(p_2^2\right)^{s_2} \cdot ... \cdot \left(p_q^2\right)^{s_q} \tag{4.23}$$

Now, considering the numbers p_j^2 as indivisible and taking into account only $d_j < n$, we find, using (2.9), the complete quantity of representations (4.19) with positive and integer k and l:

$$N_r = \frac{(s_1 + 1)(s_2 + 1)...(s_q + 1)}{2}, \text{if } n \text{ is nonsquare (some } s_j \text{ are odd)} \tag{4.24}$$

and

$$N_r = \frac{(s_1+1)(s_2+1)...(s_q+1)-1}{2}, \text{if } n \text{ is square (all } s_j \text{ are even)} \quad (4.25)$$

Hence, the representations (4.19) are given by expression:

$$n = \left(\frac{n+d_i}{2\sqrt{d_i}}\right)^2 - \left(\frac{n-d_i}{2\sqrt{d_i}}\right)^2, \quad (4.26)$$

where di are the divisors of n^2 which are less than n and are the squares of integers.

For example, if $n = 3465 = 3^2 \cdot 5 \cdot 7 \cdot 11, n^2 = 3^4 \cdot 5^2 \cdot 7^2 \cdot 11^2$, then the quantity of representations (4.19) is equal:

$$N_r = \frac{(2+1)(1+1)^3}{2} = 12,$$

and we may calculate them on (4.26) by d_i equal to:

$$1; 3^2; 5^2; 7^2; 3^4; 11^2; 3^2 \cdot 5^2; 3^2 \cdot 7^2; 3^2 \cdot 11^2; 5^2 \cdot 7^2; 3^4 \cdot 5^2; 5^2 \cdot 11^2$$

The 12 pairs (k, l), in (4.19) are: (1733, 1732); (579, 576); (349, 344); (251, 244); (197, 188); (163,152); (123,108); (93,72); (69,36); (67, 32); (61,16); (59,4).

4.2 The case of even n

Let n be even and we want to represent it as

$$n = k^2 - l^2, \qquad k, l > 0 \quad (4.27)$$

According to (1.2) written as

$$(2kl)^2 + n^2 = (k^2 + l^2)^2,$$

$2kl$ and $k^2 - l^2$ are the elements of Pythagorean triples generated by the even number n. Then, according to (2.15),

$$k^2 + l^2 = \frac{(\frac{n}{2})^2}{d_i} + d_i, \quad (4.28)$$

where d_i are the divisors of $\left(\frac{n}{2}\right)^2$ which are less than $\frac{n}{2}$. From (4.27) and (4.28) we find:

$$k = \frac{n + 2d_i}{2\sqrt{2d_i}}, \qquad l = \frac{n - 2d_i}{2\sqrt{2d_i}} \tag{4.29}$$

It follows from (4.29), that k and l will be positive integers if $d_i < \frac{n}{2}$, and have the form

$$d_i = 2^{2a+1} \cdot (2b + 1)^{2c}, \qquad \text{integers} \quad a, b, c \geq 0 \tag{4.30}$$

Let us find the quantity of such divisors, using the canonical expansion (2.13) of $\left(\frac{n}{2}\right)^2$. Since the maximal value of $2a+1$ is $2s_1-1$, in (4.30) $0 \leq a < s_1$, and $\left(2b+1\right)^{2c}$ are the numbers of the form (4.22), where $j = 2, ...q$. According to (2.13) and (4.23) the quantity of all odd divisors of $\left(\frac{n}{2}\right)^2$ having the form (4.22) is

$$(s_2 + 1) \cdot (s_3 + 1) \cdot ... \cdot (s_q + 1)$$

Then the quantity of all even d_i having the form (4.30) is

$$\frac{1}{2} \cdot 2s_1 \cdot (s_2 + 1) \cdot ... \cdot (s_q + 1) \tag{4.31}$$

Taking into account the fact that for obtaining of all the different representations (4.27) only $d_i < \frac{n}{2}$ must be used, we obtain the following result:

The quantity of the representations (4.27) of the even number n is:

$$N_r = \frac{s_1 \cdot (s_2 + 1) \cdot ... \cdot (s_q + 1)}{2}, \quad \text{if } n \text{ is nonsquare, i.e. either } s_1 \text{ is even or some of } s_2, \cdot ... \cdot, s_q \text{ are odd} \tag{4.32}$$

and

$$N_r = \frac{s_1 \cdot (s_2 + 1) \cdot ... \cdot (s_q + 1) - 1}{2}, \quad \text{if } n \text{ is square, i.e. } s_1 \text{ is odd and all } s_2, \cdot ... \cdot, s_q \text{ are even} \tag{4.33}$$

All these representations are given by the expression:

$$n = \left(\frac{n+2d_i}{2\sqrt{2d_i}}\right)^2 - \left(\frac{n-2d_i}{2\sqrt{2d_i}}\right)^2 \tag{4.34}$$

where d_i are the divisors of $\left(\frac{n}{2}\right)^2$, which are less the $\frac{n}{2}$ and have the form (4.30). One can see that k and l have the same parity.

From (4.32) if $s_1 = 0$, then $N_r = 0$; i.e., the even n not divisible by 4 cannot be represented as the difference of two squares. They also cannot generate the primitive Pythagorean triples.

In [4] the quantities of the representations (4.19) and (4.27) have been obtained by method of factorization of number n into two factors. If we take the designations used there, then our expressions for N_r coincide with [4], with only difference that, in contrast to [4], (4.25) and (4.33) do not include the trivial case $k = \sqrt{n}, l = 0$.

Example. $n = 900 = 2^2 \cdot 3^2 \cdot 5^2; \frac{n}{2} = 2 \cdot 3^2 \cdot 5^2; \left(\frac{n}{2}\right)^2 = 2^2 \cdot 3^4 \cdot 5^4; N_r = \frac{1 \cdot (2+1)^2 - 1}{2} = 4$. The appropriate divisors d_i are $d_1 = 2; d_2 = 2 \cdot 3^2; d_3 = 2 \cdot 3^4; d_4 = 2 \cdot 5^2$, and the corresponding pairs (k, l) are: (226, 224); (78,72); (50,40); (34,16).

Now we return to the Pythagorean triples and find out how to calculate all triples by (1.2).

Let the odd number $n = k^2 - l^2$ be given.

Note that if all $s_j = 1$, then the quantity N_r is equal to quantity of primitive triples, i.e. $N_r = 2^{q-1}$. Generally speaking, $N_r \geq N_p$ and their difference give some quantity of non-primitive triples, but not all non-primitive triples. The remaining $N_{tr} - N_r$ triples are obtained from (2.14) by divisors $d_i < n$, which are not squares of integers. By such d_i expressions (4.21) give the irrational k and l, while $2kl, k^2 - l^2$ and $k^2 + l^2$ remain integer. On the other hand, they are just those non-primitive triples which cannot be obtained from (1.2) with integers k and l. Therefore the complete quantity $N_t r$ can be obtained by (1.2), using the k and l given by expressions (4.21) for all $d_i < n$.

Let the even number $n = 2kl$ be given. Then according to (2.15) we have

$$k^2 + l^2 = \frac{\left(\frac{n}{2}\right)^2}{d_i} + d_i, \qquad k^2 - l^2 = \frac{\left(\frac{n}{2}\right)^2}{d_i} - d_i$$

and

$$k = \frac{n}{2\sqrt{d_i}}, \qquad l = \sqrt{d_i} \tag{4.35}$$

Therefore, all N_{tr} triples generated by the even number n, may be obtained from (1.2) using $k = \frac{n}{2\sqrt{d_i}}$, $l = \sqrt{d_i}$, where d_i are all divisors of $\left(\frac{n}{2}\right)^2$, which are less than $\frac{n}{2}$.

5 Conclusion

In the later methods of generation of the Pythagorean triples the different special representations of generating numbers were used. In particular, the representation by Fibonacci numbers [5], geometrical representations, e.g. Dickson's method [6], and others [7, 8]. Therefore they do not give the universal formulas for finding all triples. In this work we have obtained the general formulas, giving all primitive and non-primitive triples generated by a given number which is represented in the most general form, by its canonical expansion. We have also used this method to find all representations of integers as a difference of two squares and to reveal the relation among quantities of such representations and triples.

We have also shown how one can use the Euclidean formula (1.2) to find all triples, which cannot be obtained by this formula with integers k and l.

All these results are obtained with the common method using formulas (2.14), (2.15) and canonical expansions of the appropriate numbers.

References

[1] E. Asmaryan, On the Pythagorean triples, in: Proceedings Int. Conference on the Fundamental and Applied Problems of Science, Russian, M. RAN, Itogi nauki, issue 26, 2016, 3-11.

[2] N. Robbins, On the number of primitive Pythagorean triangles with a given inradius, Fibonacci Quarterly, **44,** no. 4, 2006, 368-369.

[3] T. Omland, How many Pythagorean triples with a given inradius?, Journal of Number Theory, **170,** Supplement C, 2017, 1–2. http://www.sciencedirect.com/science/article/pii/S0022314X16301603.

[4] M. A. Nyblom, On the representation of the integers as a difference of squares, Fibonacci Quarterly, **40,** no. 3, 2002, 243–246.

[5] A. Horadam, Fibonacci number triples, American Mathematical Monthly, **68,** no. 8, 1961, 751–753.

[6] L. E. Dickson, History of the theory of numbers: Diophantine Analysis, 2^{nd} Edition, Dover Publications, 2005.

[7] R. Amato, A characterization of pythagorean triples, JP Journal of Algebra, Number Theory and Applications, **39,** no. 2, 2017, 221.

[8] T. Roy, F. Jaishmin Sonia, A direct method to generate pythagorean triples and its generalization to pythagorean quadruples and n-tuples, https://arxiv.org/ftp/arxiv/papers/1201/1201.2145.pdf