Известия НАН Армении, Математика, том 55, н. 6, 2020, стр. 68 – 82.

ORIENTATION-DEPENDENT CHORD LENGTH DISTRIBUTION FUNCTION FOR RIGHT PRISMS WITH RECTANGULAR OR RIGHT TRAPEZOIDAL BASES

V. K. OHANYAN, D. M. MARTIROSYAN

Yerevan State University, Yerevan, Armenia[1]
E-mails: *victoohanyan@ysu.am; davit.martirosyan6@ysumail.am*

Abstract. The paper continues the research to reconstruct a convex body in $\mathbb{R}^n$ from the distribution of characteristics of its k-dimensional sections ($k < n$). In this paper we obtain explicit expressions for the covariogram and the orientation dependent chord length distribution of right prisms with rectangular or right trapezoidal bases.

MSC2010 numbers: 60D05; 52A22; 53C65.
Keywords: Stochastic Geometry; convex body; covariogram; random chord length distribution.

1. Introduction

Reconstruction of convex bodies using random cross-sections makes it possible to simplify the calculation because the estimates of probability characteristics can be obtained using the methods of statistics. These type of problems are fundamental in the theory of geometric tomography and stereology and particularly can be applied in medicine (see [1] – [3]). Quantities characterizing random sections of body D carry some information about D. If there is a connection between geometric characteristics of D and probabilistic characteristics of a random cross-section then by a sample (of experiments) we can estimate the geometric characteristics of D. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, $D \subset \mathbb{R}^n$ be a bounded convex body with inner points, S^{n-1} be the $(n-1)$-dimensional unit sphere centered at the origin, and $L_n(\cdot)$ be the n-dimensional Lebesgue measure in $\mathbb{R}^n$. The function

$$C_D(x) = L_n(D \cap \{D + x\}), \qquad x \in \mathbb{R}^n,$$

where $D + x = \{\mathcal{P} + x : \mathcal{P} \in D\}$, is called the covariogram of the body D.
There is a one-to-one correspondence between planar convex bodies and the covariogram (see [9]). Earlier in [2], a conjecture has been formulated by Matheron claiming that such correspondence exists in n-dimensional Euclidean spaces for any

[1]The research of the first author was partially supported by the RA MES State Committee of Science, in frame of the Research, Grant # 18T-1A252 and by the Mathematical Studies Center at Yerevan State University.

n. However, in the case $n \geq 4$ Matheron's hypothesis has received a negative answer (see [1] and [3]). The general 3-dimensional case is still open (see [3]).

This paper continues the research to reconstruct convex bodies in $\mathbb{R}^3$ using covariogram (see [6] – [8], [12]) and the orientation-dependent chord length distribution (see [4], [5], [10], [11], [13], and [14]).

Although the general 3-dimensional case is still open, Matheron's conjecture has been confirmed in the case of bounded convex polyhedrons in $\mathbb{R}^3$. Actually, the covariogram problem was found to be equivalent to the problem of rebuilding a convex domain from the length distribution of its orientation-dependent chords (see [2], [9]).

In the current paper we found explicit expressions for the covariogram and the orientation-dependent chord length distribution of a right parallelepiped with rectangular base. Further, the base is transformed into a right trapezoid with the given acute angle, and the mentioned expressions are obtained for right prisms with right trapezoidal bases.

2. Chord Length Distribution in a Rectangle

Let E be a bounded convex subset of $\mathbb{R}^2$. Consider the vector

$$\phi = (\cos\varphi, \sin\varphi) \in S^1,$$

and let l_φ be the subspace of $\mathbb{R}^2$ spanned by ϕ. By $\phi^\perp$ we denote the orthogonal complement of l_φ. For any $y \in \phi^\perp$, let $l_\varphi + y$ be the line which is parallel to ϕ and passes through y. Denote

$$\chi(l_\varphi + y) = L_1\big((l_\varphi + y) \cap E\big).$$

If the line $l_\varphi + y$ has a common segment with E, then we will say that it makes a chord in E of length $\chi(l_\varphi + y)$.

Let $\Pi_E(\varphi)$ be the orthogonal projection of E onto $\phi^\perp$. Assuming that y is uniformly distributed over $\Pi_E(\varphi)$, the chord length distribution function in direction ϕ for E is defined by

$$F_E(x, \varphi) = \frac{L_1\{y \in \Pi_E(\varphi) : \chi(l_\varphi + y) \leq x\}}{b_E(\varphi)},$$

where $b_E(\varphi) = L_1(\Pi_E(\varphi))$.

When E is a parallelogram, the distribution function $F_E(x, \varphi)$ and the covariogram $C_E(t, \varphi)$ (which is an alternative notation for $C_E(t\phi)$), are explicitly found in [15]. In particular, the following results can be extracted from [15], section 2.

Lemma 2.1. *Let R be the rectangle $[0,b]\times[0,a]\subset\mathbb{R}^2$, where $a\le b$, and let $\pi k-\arctan\frac{a}{b}\le\varphi<\pi(k+1)-\arctan\frac{a}{b}$ for some integer k. Then*

$$F_R(x,\varphi)=\begin{cases}0, & \text{if } \ x\le 0\\ \dfrac{2x|\sin\varphi|\cdot|\cos\varphi|}{a|\cos\varphi|+b|\sin\varphi|}, & \text{if } \ 0<x<x_{\max}(\varphi)\\ 1, & \text{if } \ x\ge x_{\max}(\varphi)\end{cases}\tag{2.1}$$

and

(2.2)
$$C_R(t,\varphi)=\begin{cases}ab-t(a|\cos\varphi|+b|\sin\varphi|)+t^2|\sin\varphi\cos\varphi|, & \text{if } \ 0\le t\le x_{\max}(\varphi)\\ 0, & \text{if } \ t>x_{\max}(\varphi)\end{cases},$$

where

$$x_{\max}(\varphi)=\begin{cases}\dfrac{b}{|\cos\varphi|}, & \text{if } \ -\arctan\frac{a}{b}+\pi k\le\varphi<\arctan\frac{a}{b}+\pi k\\ \dfrac{a}{|\sin\varphi|}, & \text{if } \ \arctan\frac{a}{b}+\pi k\le\varphi<-\arctan\frac{a}{b}+\pi(k+1)\end{cases}.\tag{2.3}$$

Remark 2.1. *$x_{\max}(\varphi)$ represents the length of the maximal chord in R in direction ϕ, that is*

$$x_{\max}(\varphi)=\max_{y\in\Pi_R(\varphi)}\chi(l_\varphi+y).$$

Remark 2.2. *The formula*

$$b_R(\varphi)=a|\cos\varphi|+b|\sin\varphi|\tag{2.4}$$

holds for any real φ.

3. Chord Length Distribution in a Right Rectangular Parallelepiped.

For $\omega\in S^2$, we denote by $\omega^\perp$ the orthogonal complement of $\{t\omega : t\in\mathbb{R}\}$ in $\mathbb{R}^3$. For a bounded convex body $D\subset\mathbb{R}^3$, let $\Pi_D(\omega)$ be the orthogonal projection of D onto the plane $\omega^\perp$.

Let $l_\omega+y$ be the line passing through $y\in\omega^\perp$ with direction vector ω, and $\chi(l_\omega+y)=L_1((l_\omega+y)\cap D)$. Assuming y is uniformly distributed in $\Pi_D(\omega)$, we define the chord length distribution function in direction ω for D by

$$F_D(t,\omega)=\frac{L_2\{y\in\Pi_D(\omega):\chi(l_\omega+y)\le t\}}{b_D(\omega)},\tag{3.1}$$

where $b_D(\omega)=L_2(\Pi_D(\omega))$.

Let D be a cylinder with base B (not necessarily convex) placed on the OXY plane, and height h. If ω is given by its spherical coordinates $(1,\varphi,\theta)$, where 1 is the radius, $\varphi\in[0,2\pi)$ is the azimuthal angle, and $\theta\in[0,\frac{\pi}{2}]$ is the elevation angle, then

$$b_D(\omega)=\|B\|\sin\theta+b_B(\varphi)h\cos\theta,\tag{3.2}$$

where $\|B\|$ is the area of the base. A relation between orientation-dependent chord length distribution functions F_D and F_B is found in [7]:

(3.3)

$$F_D(t,\omega)=\begin{cases}0, & \text{if } \ t<0\\ \frac{b_B(\varphi)\cos\theta}{b_D(\omega)}\left[(h-t\sin\theta)F_B(t\cos\theta,\varphi)+\right. & \\ \left.+t\sin\theta+\sin\theta\int_0^t(1-F_B(u\cos\theta,\varphi))\,du\right], & \text{if } \ 0\le t<x_{\max}(\omega)\\ 1, & \text{if } \ t\ge x_{\max}(\omega)\end{cases}$$

where $x_{\max}(\omega)$ is the length of the maximal chord in D in direction ω.

Theorem 3.1. *Let D be the parallelepiped $[0,b]\times[0,a]\times[0,h]\subset\mathbb{R}^3$ and $F_D(t,\omega)$ be the orientation-dependent chord length distribution function of D in direction $\omega=(\cos\varphi\cos\theta,\sin\varphi\cos\theta,\sin\theta)\in S^2$, where $0\le\theta\le\frac{\pi}{2}$. Then*

$$F_D(t,\omega)=\begin{cases}0, & \text{if } \ t\le 0\\ \frac{\cos\theta}{ab\sin\theta+b_R(\varphi)\cdot h\cos\theta}\cdot\Big(\big(h\cos\theta|\sin2\varphi|+ & \\ +2b_R(\varphi)\sin\theta\big)\cdot t-\frac{3}{4}\sin2\theta|\sin2\varphi|\cdot t^2\Big), & \text{if } \ 0<t<x_{\max}(\omega)\\ 1, & \text{if } \ t\ge x_{\max}(\omega)\end{cases},$$

where $R=[0,b]\times[0,a]$.

Proof. The validity of the formula is obvious when $t\le 0$ or $t\ge x_{\max}(\omega)$, so we assume $0<t<x_{\max}(\omega)$ hereinafter. Since

$$x_{\max}(\omega)=\begin{cases}\frac{x_{\max}(\varphi)}{\cos\theta}, & \text{if } \ 0\le\theta\le\arctan\frac{h}{x_{\max}(\varphi)}\\ \frac{h}{\sin\theta}, & \text{if } \ \arctan\frac{h}{x_{\max}(\varphi)}<\theta\le\frac{\pi}{2}\end{cases}, \tag{3.4}$$

the inequality

$$x_{\max}(\omega)\cos\theta\le x_{\max}(\varphi)$$

holds for any $\theta\in[0,\frac{\pi}{2}]$. Thus, taking into account (2.1), (3.2), and (3.3) we conclude that

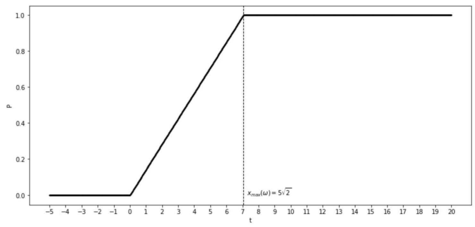
$$F_D(t,\omega)=\frac{b_R(\varphi)\cos\theta}{ab\sin\theta+b_R(\varphi)h\cos\theta}\cdot$$

$$\cdot\left[(h-t\sin\theta)\frac{t\cos\theta|\sin2\varphi|}{b_R(\varphi)}+2t\sin\theta-\sin\theta\int_0^t\frac{u\cos\theta|\sin2\varphi|}{b_R(\varphi)}du\right]=$$

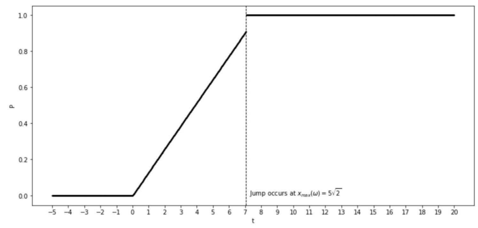
$$=\frac{\cos\theta}{ab\sin\theta+b_R(\varphi)\cdot h\cos\theta}\cdot\left[\big(h\cos\theta|\sin2\varphi|+2b_R(\varphi)\sin\theta\big)\cdot t-\frac{3}{4}\sin2\theta|\sin2\varphi|\cdot t^2\right]$$

□

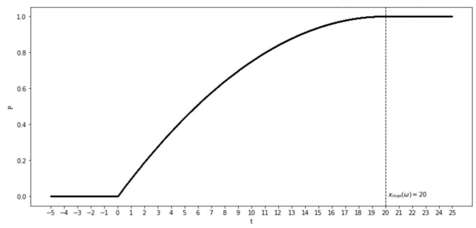
Remark 3.1. *When $\theta=0$ then functions F_D and F_R coincide. If $\theta=\frac{\pi}{2}$ then $x_{\max}(\omega)=h$. In this case F_D coincides with the indicator function of $(-\infty,h]$. For some other special cases the result of Theorem 3.1 is visualized by Figure 1.*



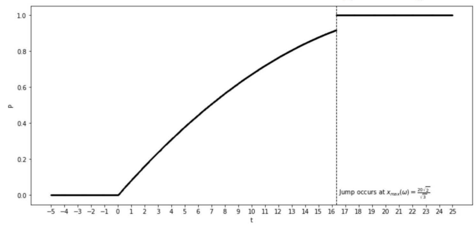
(A) $a = 5, b = 10, h = 5, \varphi = \frac{\pi}{2}, \theta = \frac{\pi}{4}$



(B) $a = 5, b = 10, h = 6, \varphi = \frac{\pi}{2}, \theta = \frac{\pi}{4}$



(C) $a = 10, b = 10, h = 10\sqrt{2}, \varphi = \frac{3\pi}{4}, \theta = \frac{\pi}{4}$



(D) $a = 10, b = 10, h = 10\sqrt{2}, \varphi = \frac{\pi}{4}, \theta = \frac{\pi}{3}$

FIGURE 1. Orientation dependent chord length distribution function F_D for different cases

Remark 3.2. *Direct use of* (3.3) *in the proof of the theorem avoided computation of the covariogram of* D. *The function* $C_D(t\omega)$ *could be found explicitly.*

Indeed, if $0 \le t \le x_{\max}(\omega)$ then $C_D(t\omega) = L_3(D \cap \{D + t\omega\}) =$

$$= L_2(R \cap \ \{R + (t\cos\theta)\phi\}) \cdot (h - t\sin\theta) = (h - t\sin\theta) \cdot C_R(t\cos\theta, \varphi).$$

Taking into account (3.4) and (2.2) we obtain

(3.5)
$$C_D(t\omega) = \begin{cases} (h - t\sin\theta)\big(ab - t\cos\theta \cdot b_R(\varphi) + \frac{t^2}{4}\cos^2\theta |\sin 2\varphi|\big), & \text{if} \quad 0 \le t \le x_{\max}(\varphi) \\ 0, & \text{if} \quad t > x_{\max}(\varphi) \end{cases}.$$

4. Chord Length Distribution in a Right Trapezoid

Let $T \subset \mathbb{R}^2$ be the right trapezoid with the vertices at $O(0,0), A(0,a), C(b - a\cot\psi, a)$, and $B(b,0)$, where $arctan\frac{a}{b} < \psi < \frac{\pi}{2}$. For every right trapezoid one can choose the parameters a, b, and ψ such that it becomes congruent to $OACB$.
In this section we maintain the notations and terminology introduced earlier in Section 2 for any bounded convex set E.

Proposition 4.1. *Let* $\pi k \le \varphi < \pi(k+1)$ *for some integer* k. *Then*

$$b_T(\varphi) = \begin{cases} a|\cos\varphi| + b|\sin\varphi|, & \text{if} \quad \pi k \le \varphi < \frac{\pi}{2} + \pi k \\ b|\sin\varphi|, & \text{if} \quad \frac{\pi}{2} + \pi k \le \varphi < \pi(k+1) - \psi \\ a|\cos\varphi| + (b - a\cot\psi)|\sin\varphi|, & \text{if} \quad \pi(k+1) - \psi \le \varphi < \pi(k+1) \end{cases}.$$

Proof. To reduce the computational burden, from now on we'll use b_1 for the shorter base of T, that is $b_1 = b - a\cot\psi$.
If $\pi k \le \varphi < \frac{\pi}{2} + \pi k$, then $\Pi_T(\varphi) = \Pi_R(\varphi)$. Therefore, due to (2.4), we have

$$b_T(\varphi) = b_R(\varphi) = a|\cos\varphi| + b|\sin\varphi|.$$

Similarly, if $\pi(k+1) - \psi \le \varphi < \pi(k+1)$, then $\Pi_T(\varphi) = \Pi_{[0,b_1]\times[0,a]}(\varphi)$, which implies

$$b_T(\varphi) = L_1\big(\Pi_{[0,b_1]\times[0,a]}(\varphi)\big) = a|\cos\varphi| + b_1|\sin\varphi|.$$

Finally, if $\frac{\pi}{2} + \pi k \le \varphi < \pi(k+1) - \psi$, then

$$b_T(\varphi) = L_1\big(\Pi_{[0,b]\times\{0\}}(\varphi)\big) = b\cos(\varphi - \pi k - \frac{\pi}{2}) = (-1)^k b\sin\varphi = b|\sin\varphi|.$$

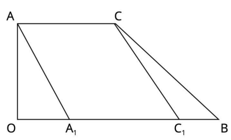
□

Let $\phi_v^\perp$ be the set of vectors $y \in \phi^\perp$ so that the line $l_\varphi + y$ passes through a vertex of trapezoid T and makes a chord of positive Lebesgue measure there. The two quantities introduced below,

$$x_0(\varphi) = \min_{y\in\phi_v^\perp} \chi(l_\varphi + y) \quad \text{and} \quad x_1(\varphi) = \max_{y\in\phi_v^\perp} \chi(l_\varphi + y),$$

will play a crucial role in determination of distribution function F_T. The diagrams shown in Figure 2 facilitate case-by-case computations (see Proposition 4.2) of the above mentioned quantities.

If $\pi k \le \varphi < \frac{\pi}{2} + \pi k$, *then* $|OO_1| = x_1(\varphi), |CC_1| = x_0(\varphi)$



If $\frac{\pi}{2} + \pi k \le \varphi < \pi(k+1) - \psi$, *then* $|AA_1| = |CC_1| = x_1(\varphi) = x_0(\varphi)$

If $\pi(k+1) - \psi \le \varphi < \pi(k+1)$, *then* $|AA_1| = x_0(\varphi), |BB_1| = x_1(\varphi)$

FIGURE 2. Possible dispositions of $x_0(\varphi)$ and $x_1(\varphi)$

Proposition 4.2. *$x_1(\varphi) = x_{\max}(\varphi)$ for any angle φ. Furthermore, if for some $k \in \mathbb{Z}$*

(i) $\pi k \le \varphi < \frac{\pi}{2} + \pi k$, *then*

$$x_0(\varphi) = \begin{cases} \dfrac{b_1}{|\cos\varphi|}, & \textit{if} \quad \pi k \le \varphi < \pi k + \arctan\frac{a}{b_1} \\ \dfrac{a}{|\sin\varphi|}, & \textit{if} \quad \pi k + \arctan\frac{a}{b_1} \le \varphi < \frac{\pi}{2} + \pi k \end{cases}$$

$$x_{\max}(\varphi) = \begin{cases} \dfrac{b\sin\psi}{|\sin(\varphi+\psi)|}, & \textit{if} \quad \pi k \le \varphi < \pi k + \arctan\frac{a}{b_1} \\ \dfrac{a}{|\sin\varphi|}, & \textit{if} \quad \pi k + \arctan\frac{a}{b_1} \le \varphi < \frac{\pi}{2} + \pi k \end{cases}$$

(ii) $\frac{\pi}{2} + \pi k \le \varphi < \pi(k+1) - \psi$, *then*

$$x_0(\varphi) = x_{\max}(\varphi) = \frac{a}{|\sin\varphi|}$$

(iii) $\pi(k+1) - \psi \le \varphi < \pi(k+1)$, *then*

$$x_0(\varphi) = \begin{cases} \dfrac{a}{|\sin\varphi|}, & \textit{if} \quad \pi(k+1) - \psi \le \varphi < \pi(k+1) - \arctan\frac{a}{b} \\ \dfrac{b_1\sin\psi}{|\sin(\varphi+\psi)|}, & \textit{if} \quad \pi(k+1) - \arctan\frac{a}{b} \le \varphi < \pi(k+1) \end{cases}$$

$$x_{\max}(\varphi) = \begin{cases} \dfrac{a}{|\sin\varphi|}, & \textit{if} \quad \pi(k+1) - \psi \le \varphi < \pi(k+1) - \arctan\frac{a}{b} \\ \dfrac{b}{|\cos\varphi|}, & \textit{if} \quad \pi(k+1) - \arctan\frac{a}{b} \le \varphi < \pi(k+1) \end{cases}$$

Proof. A chord of maximal length in a convex polygon with direction φ, also known as φ-diameter of the polygon, is not necessarily unique (this is also seen in Figure 2: the second, third, and fourth cases) but for any given φ there exists a φ-diameter such that at least one endpoint of the chord coincides with a vertex of the given polygon ([16]). Thus,

$$x_{\max}(\varphi) = \max_{y \in \Pi_R(\varphi)} \chi(l_\varphi + y) = \max_{y \in \phi_v^\perp} \chi(l_\varphi + y) = x_1(\varphi).$$

Case (i), sub-case 1 ($\pi k \le \varphi < \pi k + \arctan\frac{a}{b_1}$). Observe the first trapezoid in Figure 2. Here we are given $\angle O_1OB = \varphi - \pi k$. By Sine Rule,

$$x_0(\varphi) = |CC_1| = \frac{b_1}{\cos(\varphi - \pi k)} = \frac{b_1}{|\cos\varphi|};$$

$$x_{\max}(\varphi) = |OO_1| = \frac{b\sin\psi}{\sin(\varphi - \pi k + \psi)} = \frac{b\sin\psi}{|\sin(\varphi+\psi)|}.$$

Case (i), sub-case 2 ($\pi k + \arctan\frac{a}{b_1} \le \varphi < \frac{\pi}{2} + \pi k$). Observe the second trapezoid in Figure 2. Since $\angle AO_1O = \angle O_1OB = \varphi - \pi k$, from the right triangle AO_1O we obtain

$$x_0(\varphi) = x_{\max}(\varphi) = |OO_1| = \frac{a}{\sin(\varphi - \pi k)} = \frac{a}{|\sin\varphi|}.$$

Case (i) is completely proved. The proof of the case (iii) is similar to (i), hence omitted. It remains to discuss the case (ii), where $\frac{\pi}{2} + \pi k \le \varphi < \pi(k+1) - \psi$ and the third (in the middle) trapezoid in Figure 2 is relevant. In this case we have $\angle AA_1B = \varphi - \pi k \Rightarrow \angle AA_1O = \pi(k+1) - \varphi$, which implies

$$x_0(\varphi) = x_{\max}(\varphi) = |AA_1| = \frac{a}{\sin(\pi(k+1) - \varphi)} = \frac{a}{|\sin\varphi|}.$$

□

By definition, for any angle φ, if $x < 0$ then $F_T(x,\varphi) = 0$, and if $x \ge x_{\max}(\varphi)$ then $F_T(x,\varphi) = 1$. The non-trivial case $0 \le x < x_{\max}(\varphi)$ is explored below.

Lemma 4.1. *$F_T(x,\varphi) = 0$ if $x < 0$ and $F_T(x,\varphi) = 1$ if $x \ge x_{\max}(\varphi)$. For $0 \le x < x_{\max}(\varphi), F_T(x,\varphi)$ is represented as follows $(k \in \mathbb{Z})$:*

(i) For $\pi k \le \varphi < \frac{\pi}{2} + \pi k$,

(4.1)
$$F_T(x,\varphi) = \begin{cases} \dfrac{x\sin\varphi[\sin(\varphi+\psi) + \cos\varphi\sin\psi]}{b_T(\varphi)\sin\psi}, & \text{if } \ 0 \le x < x_0(\varphi) \\ \dfrac{x\sin^2(\varphi+\psi) - x_0(\varphi)\sin^2\psi\cos^2\varphi}{b_T(\varphi)\sin\psi\cos\psi}, & \text{if } \ x_0(\varphi) \le x < x_{\max}(\varphi) \end{cases};$$

(ii) if $\frac{\pi}{2} + \pi k \le \varphi < \pi(k+1) - \psi$,

$$F_T(x,\varphi) = \frac{x\sin^2\varphi\cot\psi}{b_T(\varphi)}; \tag{4.2}$$

(iii) if $\pi(k+1) - \psi \le \varphi < \pi(k+1)$,

(4.3)
$$F_T(x,\varphi) = \begin{cases} \dfrac{-x\sin\varphi[\sin(\varphi+\psi) + \cos\varphi\sin\psi]}{b_T(\varphi)\sin\psi}, & \text{if } \ 0 \le x < x_0(\varphi) \\ \dfrac{x\cos^2\varphi\sin^2\psi - x_0(\varphi)\sin^2(\varphi+\psi)}{b_T(\varphi)\sin\psi\cos\psi}, & \text{if } \ x_0(\varphi) \le x < x_{\max}(\varphi) \end{cases}.$$

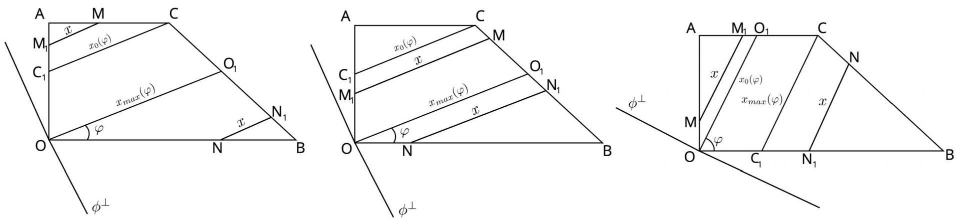
Proof. Let's note that the function $F_T(x,\cdot)$ is π-periodic. This guarantees that generality will not be lost if we restrict the proof to the case $k = 0$.

Case (i), sub-case 1.1: let $0 \le \varphi < \arctan\frac{a}{b_1}$ and $0 \le x < x_0(\varphi)$.
This case is displayed below in Figure 3(A) which shows $|MM_1| = |NN_1| = x < x_0(\varphi) = |CC_1| < |OO_1| = x_{\max}(\varphi)$.

In this case we have $F_T(x,\varphi) = \frac{1}{b_T(\varphi)}\big(b_{\Delta AMM_1}(\varphi) + b_{\Delta BNN_1}(\varphi)\big)$. The quantities $b_{\Delta AMM_1}(\varphi)$ and $b_{\Delta BNN_1}(\varphi)$ are equal to the heights of triangles AMM_1 (with base MM_1) and BNN_1 (with base NN_1), respectively. Those are computed below:

$$b_{\Delta AMM_1}(\varphi) = x\sin\varphi\cos\varphi. \tag{4.4}$$

$$b_{\Delta BNN_1}(\varphi) = \frac{x\sin\varphi\sin(\varphi+\psi)}{\sin\psi}. \tag{4.5}$$



(A) Case (i), sub-case 1.1: $0 \le \varphi < \arctan \frac{a}{b_1}$, $0 \le x < x_0(\varphi)$

(B) Case (i), sub-case 1.2: $0 \le \varphi < \arctan \frac{a}{b_1}$, $x_0(\varphi) \le x < x_{\max}(\varphi)$

(C) Case (i), sub-case 2: $\arctan \frac{a}{b_1} \le \varphi < \frac{\pi}{2}$, $0 \le x < x_0(\varphi) = x_{\max}(\varphi)$

FIGURE 3. Chords in trapezoid in direction ϕ, where $0 \le \varphi < \frac{\pi}{2}$.

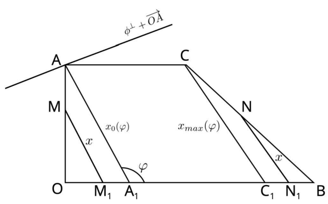
Case (i), sub-case 1.2: let $0 \le \varphi < \arctan \frac{a}{b_1}$ and $x_0(\varphi) \le x < x_{\max}(\varphi)$.
This case is displayed in Figure 3(B), where $x_0(\varphi) = |CC_1| \le x = |MM_1| = |NN_1| < |OO_1| = x_{\max}(\varphi)$. In this case we have $F_T(x, \varphi) = \frac{1}{b_T(\varphi)}\big(b_{ACMM_1}(\varphi) + b_{\Delta BNN_1}(\varphi)\big) = \frac{1}{b_T(\varphi)}\big(b_{\Delta ACC_1}(\varphi) + b_{\Delta BNN_1}(\varphi) + b_{CC_1M_1M}(\varphi)\big) = \frac{1}{b_T(\varphi)}\big(x_0(\varphi)\sin\varphi\cos\varphi + \frac{x\sin\varphi\sin(\varphi+\psi)}{\sin\psi} + b_{CC_1M_1M}(\varphi)\big)$. The quantity

$$b_{CC_1M_1M}(\varphi) = \frac{(x - x_0(\varphi))\cos\varphi\sin(\varphi+\psi)}{\cos\psi} \tag{4.6}$$

is computed from the trapezoid CC_1M_1M with bases CC_1 and MM_1, as a height.
Case (i), sub-case 2: let $\arctan \frac{a}{b_1} \le \varphi < \frac{\pi}{2}$ and $0 \le x < x_0(\varphi) = x_{\max}(\varphi)$.
This case is displayed in Figure 3(C), where $x = |MM_1| = |NN_1| < |OO_1| = |CC_1| = x_0(\varphi) = x_{\max}(\varphi)$. Computation of the chord length distribution function in this case is absolutely identical to the case (i), sub-case 1.1.
Combining (4.4), (4.5), and (4.6), for any $\varphi \in [0, \frac{\pi}{2})$ we obtain $F_T(x, \varphi) =$

$$\begin{cases} \dfrac{x\sin\varphi\cos\varphi + \frac{x\sin\varphi\sin(\varphi+\psi)}{\sin\psi}}{b_T(\varphi)}, & 0 \le x < x_0(\varphi) \\ \dfrac{x_0(\varphi)\sin\varphi\cos\varphi + \frac{x\sin\varphi\sin(\varphi+\psi)}{\sin\psi} + \frac{(x-x_0(\varphi))\cos\varphi\sin(\varphi+\psi)}{\cos\psi}}{b_T(\varphi)}, & x_0(\varphi) \le x < x_{\max}(\varphi) \end{cases},$$

which is equivalent to (4.1).
Case (ii): let $\frac{\pi}{2} \le \varphi < \pi - \psi$ and see Figure 4 below.



Case (ii): $\frac{\pi}{2} \le \varphi < \pi - \psi$, $0 \le x < x_0(\varphi) = x_{\max}(\varphi)$

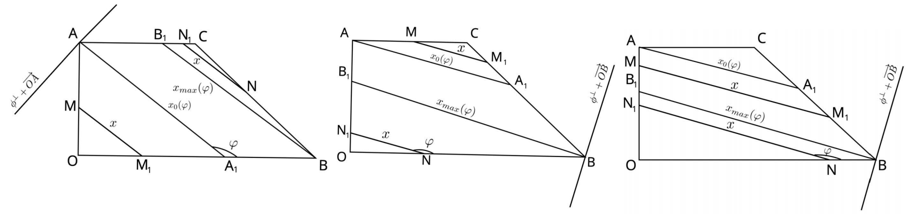
FIGURE 4. Chords in trapezoid in direction ϕ, where $\frac{\pi}{2} \le \varphi < \pi - \psi$.

For $x = |MM_1| = |NN_1| < |AA_1| = |CC_1| = x_0(\varphi) = x_{\max}(\varphi)$ we have

$$F_T(x,\varphi) = \frac{1}{b_T(\varphi)}\big(b_{\Delta OMM_1}(\varphi) + b_{\Delta BNN_1}(\varphi)\big) =$$

$$= \frac{x\sin(\pi-\varphi)\cos(\pi-\varphi) + \frac{x\sin\varphi\sin(\varphi+\psi)}{\sin\psi}}{b_T(\varphi)} = \frac{x\sin^2\varphi\cos\psi}{b_T(\varphi)\sin\psi} = \frac{x\sin^2\varphi\cot\psi}{b_T(\varphi)}.$$

Case (iii): let $\pi - \psi \le \varphi < \pi$. Similar to case (i), the distribution function can be computed by three sub-cases, as shown in Figure 5.



(A) Case (iii), sub-case 1: $\pi - \psi \le \varphi < \pi - \arctan\frac{a}{b}$, $0 \le x < x_0(\varphi) = x_{\max}(\varphi)$

(B) Case (iii), sub-case 2.1: $\pi - \arctan\frac{a}{b} \le \varphi < \pi$, $0 \le x < x_0(\varphi)$

(C) Case (iii), sub-case 2.2: $\pi - \arctan\frac{a}{b} \le \varphi < \pi$, $x_0(\varphi) \le x < x_{\max}(\varphi)$

FIGURE 5. Chords in trapezoid in direction ϕ, where $\pi - \psi \le \varphi < \pi$.

If $\pi - \psi \le \varphi < \pi$ and $0 \le x < x_0(\varphi) \le x_{\max}(\varphi)$, then (sub-cases 1 and 2.1)

$$F_T(x,\varphi) = \frac{x\sin(\pi-\varphi)\cos(\pi-\varphi) - \frac{x\sin\varphi\sin(\varphi+\psi)}{\sin\psi}}{b_T(\varphi)}. \tag{4.7}$$

Inequality $x_0(\varphi) \le x < x_{\max}(\varphi)$ is possible if only $\pi - \arctan\frac{a}{b} \le \varphi < \pi$ (sub-case 2.2). The distribution function in this last case is equal to

(4.8)

$$F_T(x,\varphi) = \frac{x\sin(\pi-\varphi)\cos(\pi-\varphi) - \frac{x_0(\varphi)\sin\varphi\sin(\varphi+\psi)}{\sin\psi} + \frac{(x-x_0(\varphi))\cos\varphi\sin(\varphi+\psi)}{\cos\psi}}{b_T(\varphi)}.$$

By simplifying (4.7) and (4.8), we establish (4.3). □

5. Covariogram of a Right Prism With Right Trapezoidal Base. Computation of Chord Length Distribution Function

Denote by D_T the right prism $\{(x,y,z) : (x,y) \in T,\ 0 \le z \le h\}$.

Due to the Matheron's formula (see [2]) we have

$$\frac{\partial C_T(t,\varphi)}{\partial t} = -L_1\left(\{y \in \phi^{\perp} : L_1\left(T \cap (l_\varphi + y)\right) \ge t\}\right),$$

which can be rewritten in terms of the orientation-dependent chord length distribution function as

$$\frac{\partial C_T(t,\varphi)}{\partial t} = -b_T(\varphi)\cdot[1 - F_T(t,\varphi)].$$

Integration of both parts of the last formula yields

$$C_T(t,\varphi) = C_T(0,\varphi) - b_T(\varphi)\cdot\int_0^t [1 - F_T(u,\varphi)]du,\ t \ge 0. \tag{5.1}$$

Using the formula (5.1) and Lemma 4.1 we immediately come to an explicit formula for $C_T(t,\varphi)$, the covariogram of T. Since $C_T(t,\cdot)$ has period equal to π, it is enough to have it computed for $\varphi \in [0,\pi)$.

$$C_T(t,\varphi) = \frac{a(b+b_1)}{2} - tb_T(\varphi) + b_T(\varphi)\cdot\int_0^t F_T(u,\varphi)du = \frac{a(b+b_1)}{2} - tb_T(\varphi) +$$

$$\begin{cases} \dfrac{t^2\sin\varphi[\sin(\varphi+\psi)+\cos\varphi\sin\psi]}{2\sin\psi}, & \text{if } 0 \le \varphi < \frac{\pi}{2}, \quad 0 \le t < x_0(\varphi) \\ \dfrac{t^2\sin^2(\varphi+\psi)}{2\sin\psi\cos\psi} - \dfrac{tx_0(\varphi)\sin\psi\cos^2\varphi}{\cos\psi}, & \text{if } 0 \le \varphi < \frac{\pi}{2}, \quad x_0(\varphi) \le t < x_{\max}(\varphi) \\ \dfrac{t^2}{2}\sin^2\varphi\cot\psi, & \text{if } \frac{\pi}{2} \le \varphi < \pi-\psi, \quad 0 \le t < x_{\max}(\varphi) \\ -\dfrac{t^2\sin\varphi[\sin(\varphi+\psi)+\cos\varphi\sin\psi]}{2\sin\psi}, & \text{if } \pi-\psi \le \varphi < \pi, \quad 0 \le t < x_0(\varphi) \\ \dfrac{t^2\cos^2\varphi\sin\psi}{2\cos\psi} - \dfrac{tx_0(\varphi)\sin^2(\varphi+\psi)}{\sin\psi\cos\psi}, & \text{if } \pi-\psi \le \varphi < \pi, \quad x_0(\varphi) \le t \le x_{\max}(\varphi) \end{cases}.$$

As a result, for any $t \in [0, x_{\max}(\omega)]$, any $\varphi \in [\pi k, \pi(k+1))$, $k \in \mathbb{Z}$ and any $\theta \in [0,\frac{\pi}{2}]$ the covariogram of D_T is equal to

$$C_{D_T}(t\omega) = L_2(\,T\cap\ \{T+(t\cos\theta)\phi\})\cdot(h-t\sin\theta) = (h-t\sin\theta)\cdot C_T(t\cos\theta,\varphi-\pi k),$$

where $\omega = (\cos\varphi\cos\theta, \sin\varphi\cos\theta, \sin\theta)$ and $x_{\max}(\varphi)$ satisfies (3.4).

Computation of $F_{D_T}(x,\omega)$ requires more workload. Lemma 4.1 shows that if $\pi k \le \varphi < \frac{\pi}{2}+\pi k$ or $\pi(k+1)-\psi \le \varphi < \pi(k+1)$ then the function $F_T(x,\varphi)$ is piecewise linear. To have those pieces written in slope-intercept form let's denote

$$m_0(\varphi) = \frac{\sin\varphi[\sin(\varphi+\psi)+\cos\varphi\sin\psi]}{b_T(\varphi)\sin\psi}, \quad m_1(\varphi) = \frac{\sin^2(\varphi+\psi)}{b_T(\varphi)\sin\psi\cos\psi},$$

$$c(\varphi) = \frac{x_0(\varphi)\tan\psi\cos^2\varphi}{b_T(\varphi)}.$$

Then (4.1) and (4.3) can be represented as

$$(5.2) \qquad F_T(x,\varphi) = \begin{cases} m_0(\varphi)x, & \text{if } 0 \le x < x_0(\varphi) \\ m_1(\varphi)x - c(\varphi), & \text{if } x_0(\varphi) \le x < x_{\max}(\varphi) \end{cases}$$

and

$$(5.3) \qquad F_T(x,\varphi) = \begin{cases} -m_0(\varphi)x, & \text{if } 0 \le x < x_0(\varphi) \\ c(\varphi)x - m_1(\varphi), & \text{if } x_0(\varphi) \le x < x_{\max}(\varphi) \end{cases},$$

respectively.

If $t \le 0$ then $F_{D_T}(t,\omega) = 0$, and if $t \ge x_{\max}(\omega)$ then $F_{D_T}(t,\omega) = 1$. If $0 < t < x_{\max}(\omega)$, then F_{D_T} (unlike F_D) requires several pieces to be written explicitly. The classification of cases is based on two factors:

1) the choice of either of the intervals $[\pi k, \frac{\pi}{2}+\pi k)$, $[\frac{\pi}{2}+\pi k, \pi(k+1)-\psi)$, or $[\pi(k+1)-\psi, \pi(k+1))$ for φ to be taken from;
2) the magnitude of the orthogonal projection of $x_{\max}(\omega)$ onto the base T compared to $x_0(\varphi)$ for the given φ and θ.

The result is formulated below.

Theorem 5.1.

(i) If $\pi k \le \varphi < \frac{\pi}{2}+\pi k$ *and* $x_{\max}(\omega)\cos\theta \le x_0(\varphi)$, *then*

$$F_{D_T}(t,\omega) = \frac{b_T(\varphi)}{b_{D_T}(\omega)}\left(\big(hm_0(\varphi)\cos^2\theta + \sin 2\theta\big)\cdot t - \frac{3}{4}m_0(\varphi)\sin 2\theta\cos\theta\cdot t^2\right),$$

$0 < t < x_{\max}(\omega)$.

(ii) If $\pi k \le \varphi < \frac{\pi}{2}+\pi k$ *and* $x_{\max}(\omega)\cos\theta > x_0(\varphi)$, *then*

$$F_{D_T}(t,\omega) = \frac{b_T(\varphi)}{b_{D_T}(\omega)}\times$$

$$\begin{cases} \big(hm_0(\varphi)\cos^2\theta + \sin 2\theta\big)\cdot t - \frac{3}{4}m_0(\varphi)\sin 2\theta\cos\theta\cdot t^2, & \text{if } \; 0 < t < \frac{x_0(\varphi)}{\cos\theta} \\ -c(\varphi)\big(x_0(\varphi)\sin\theta + h\cos\theta\big) - \frac{m_0(\varphi)-m_1(\varphi)}{2}x_0^2(\varphi)\sin\theta + \\ +\big(hm_1(\varphi)\cos^2\theta + [c(\varphi)+1]\sin 2\theta\big)\cdot t - \\ -\frac{3}{4}m_1(\varphi)\sin 2\theta\cos\theta\cdot t^2, & \text{if } \; \frac{x_0(\varphi)}{\cos\theta} \le t < x_{\max}(\omega) \end{cases}.$$

(iii) If $\frac{\pi}{2}+\pi k \le \varphi < \pi(k+1)-\psi$, *then*

$$F_{D_T}(t,\omega) = \frac{h\cos^2\theta\sin^2\varphi\cot\psi + b_T(\varphi)\sin 2\theta}{b_{D_T}(\omega)}\cdot t - \frac{\sin 2\theta\sin^2\varphi\cot\psi[2\cos\theta+1]}{4b_{D_T}(\omega)}\cdot t^2,$$

$0 < t < x_{\max}(\omega)$.

(iv) If $\pi(k+1)-\psi \le \varphi < \pi(k+1)$ *and* $x_{\max}(\omega)\cos\theta \le x_0(\varphi)$, *then*

$$F_{D_T}(t,\omega) = \frac{b_T(\varphi)}{b_{D_T}(\omega)}\left(\big(\sin 2\theta - hm_0(\varphi)\cos^2\theta\big)\cdot t + \frac{3}{4}m_0(\varphi)\sin 2\theta\cos\theta\cdot t^2\right),$$

$0 < t < x_{\max}(\omega)$.

(v) If $\pi(k+1)-\psi \le \varphi < \pi(k+1)$ *and* $x_{\max}(\omega)\cos\theta > x_0(\varphi)$, *then*

$$F_{D_T}(t,\omega) = \frac{b_T(\varphi)}{b_{D_T}(\omega)}\times$$

$$\begin{cases} \big(\sin 2\theta - hm_0(\varphi)\cos^2\theta\big)\cdot t + \frac{3}{4}m_0(\varphi)\sin 2\theta\cos\theta\cdot t^2, & \text{if } \; 0 < t < \frac{x_0(\varphi)}{\cos\theta} \\ -m_1(\varphi)\big(x_0(\varphi)\sin\theta + h\cos\theta\big) - \frac{m_0(\varphi)-c(\varphi)}{2}x_0^2(\varphi)\sin\theta + \\ +\big(hc(\varphi)\cos^2\theta + [m_1(\varphi)+1]\sin 2\theta\big)\cdot t - \\ -\frac{3}{4}c(\varphi)\sin 2\theta\cos\theta\cdot t^2, & \text{if } \; \frac{x_0(\varphi)}{\cos\theta} \le t < x_{\max}(\omega) \end{cases}.$$

Proof. Case (i): Since $t\cos\theta \le x_0(\varphi)$, (5.2) yields $F_T(\varphi) = tm_0(\varphi)\cos\theta$. Then (3.3) outputs

$$F_{D_T}(t,\omega) = \frac{b_T(\varphi)\cos\theta}{b_{D_T}(\omega)}\times$$

$$\times\left((h - t\sin\theta)\cdot tm_0(\varphi)\cos\theta + 2t\sin\theta - \sin\theta\int_0^t u\cdot m_0(\varphi)\cos\theta du\right) = \tag{5.4}$$

$$= \frac{b_T(\varphi)}{b_{D_T}(\omega)}\left(\big(hm_0(\varphi)\cos^2\theta + \sin 2\theta\big)\cdot t - \frac{3}{4}m_0(\varphi)\sin 2\theta\cos\theta\cdot t^2\right).$$

Case (ii): If $0 < t < \frac{x_0(\varphi)}{\cos\theta}$, then the formula (5.4) still works. But if $\frac{x_0(\varphi)}{\cos\theta} \le t < x_{\max}(\omega)$, then there are two expressions for $F_T(u\cos\theta,\varphi)$ to be used under the integral in (3.3). Due to (5.2), those pieces are

$$F_T(u\cos\theta,\varphi) = \begin{cases} u\cdot m_0(\varphi)\cos\theta, & \text{if } u < \frac{x_0(\varphi)}{\cos\theta} \\ u\cdot m_1(\varphi)\cos\theta - c(\varphi), & \text{if } \frac{x_0(\varphi)}{\cos\theta} \le u \end{cases}. \tag{5.5}$$

Therefore we get

$$F_{D_T}(t,\omega) = \frac{b_T(\varphi)\cos\theta}{b_{D_T}(\omega)}\Bigg[(h - t\sin\theta)(m_1(\varphi)t\cos\theta - c(\varphi)) + 2t\sin\theta -$$

$$-\sin\theta\int_0^{\frac{x_0(\varphi)}{\cos\theta}} m_0(\varphi)\cos\theta u du - \sin\theta\int_{\frac{x_0(\varphi)}{\cos\theta}}^t [m_1(\varphi)\cos\theta u - c(\varphi)]du\Bigg] =$$

$$= \frac{b_T(\varphi)\cos\theta}{b_{D_T}(\omega)}\Bigg[-hc(\varphi) - \frac{m_0(\varphi)\tan\theta}{2}x_0^2(\varphi) + hm_1(\varphi)\cos\theta\cdot t + \sin\theta c(\varphi)t -$$

$$-\sin\theta\cos\theta m_1(\varphi)t^2 + 2t\sin\theta - \sin\theta\cdot\Bigg(\frac{m_1(\varphi)\cos\theta t^2}{2} - c(\varphi)\cdot t - m_1(\varphi)\frac{x_0^2(\varphi)}{2\cos\theta} +$$

$$+ c(\varphi)\frac{x_0(\varphi)}{\cos\theta}\Bigg)\Bigg],$$

and finally

$$\begin{gathered} F_{D_T}(t,\omega) = -c(\varphi)\big(x_0(\varphi)\sin\theta + h\cos\theta\big) - \frac{m_0(\varphi) - m_1(\varphi)}{2}x_0^2(\varphi)\sin\theta + \\ \big(hm_1(\varphi)\cos^2\theta + [c(\varphi)+1]\sin 2\theta\big)\cdot t - \frac{3}{4}m_1(\varphi)\sin 2\theta\cos\theta\cdot t^2. \end{gathered} \tag{5.6}$$

Case (iii): Using (4.2) in (3.3), we obtain

$$F_{D_T}(t,\omega) = \frac{b_T(\varphi)\cos\theta}{b_{D_T}(\omega)}\Bigg[(h - t\sin\theta)\cdot\frac{t\cos\theta\sin^2(\varphi)\cot\psi}{b_T(\varphi)} + 2t\sin\theta - \sin\theta\cdot\frac{t^2}{2}\times$$

$$\times\frac{\sin^2\varphi\cot\psi}{b_T(\varphi)}\Bigg] = \frac{b_T(\varphi)\cos\theta}{b_{D_T}(\omega)}\Bigg[\Bigg(\frac{h\cos\theta\sin^2\varphi\cot\psi}{b_T(\varphi)} + 2\sin\theta\Bigg)\cdot t -$$

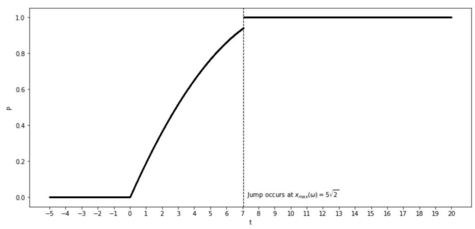
$$-\Bigg(\frac{\sin\theta\cos\theta\sin^2\varphi\cot\psi}{b_T(\varphi)} + \frac{\sin\theta\sin^2\varphi\cot\psi}{2b_T(\varphi)}\Bigg)\cdot t^2\Bigg] =$$

$$= \frac{h\cos^2\theta\sin^2\varphi\cot\psi + b_T(\varphi)\sin 2\theta}{b_{D_T}(\omega)}\cdot t - \frac{\sin 2\theta\sin^2\varphi\cot\psi[2\cos\theta + 1]}{4b_{D_T}(\omega)}\cdot t^2.$$

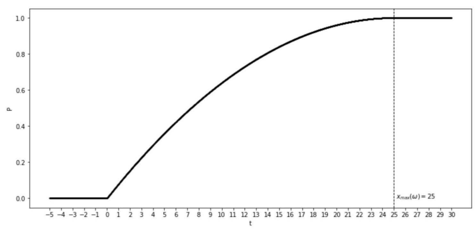
Cases (iv) and (v): Let's notice that replacing $m_0(\varphi)$ by $-m_0(\varphi)$ and interchanging $c(\varphi)$ with $m_1(\varphi)$ in (5.2) will produce (5.3). Applying the mentioned changes to (5.4) and (5.6), we complete the proof of the theorem in cases (iv) and (v), respectively. $\square$

Remark 5.1. *It follows from* (3.2) *that* $b_{D_T}(\omega) = \frac{a(b+b_1)}{2} + b_T(\varphi) h \cos\theta$. *This expansion has not been used in the theorem.*

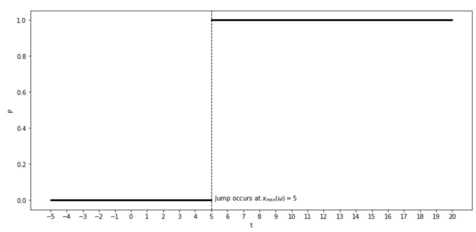
Remark 5.2. *When* $\theta = 0$, *one can check that* $F_{D_T} \equiv F_T$. *If* $\theta = \frac{\pi}{2}$, *then* F_{D_T} *is a step function, which can be seen in Figure 6, example (C). The case (D) illustrates an example where the graph of the distribution function comprises of 4 pieces, 2 horizontal lines and 2 arcs of parabolas.*



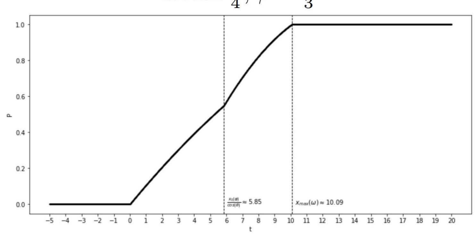
(A) $a = 5, b = 10, h = 5, \varphi = \frac{\pi}{2}, \theta = \frac{\pi}{4}, \psi = \frac{\pi}{4}$



(B) $a = 10\sqrt{3}, b = 20, h = 15, \varphi = \frac{\pi}{3}, \theta = \arctan\frac{3}{4}, \psi = \frac{\pi}{3}$



(C) $a = 5, b = 10, h = 5, \varphi = \frac{\pi}{7}, \theta = \frac{\pi}{2}, \psi = \frac{4\pi}{9}$



(D) $a = 5, b = 10, h = 10, \varphi = \frac{\pi}{20}, \theta = \frac{\pi}{6}, \psi = \frac{\pi}{4}$

Figure 6. Orientation dependent chord length distribution function F_{D_T} for different cases

References

[1] R. J. Gardner, Geometric Tomography, Cambridge Univ. Press, UK, 2nd ed. (2006).

[2] G. Matheron, Random Sets and Integral Geometry, Wiley (1975).

[3] R. Schneider and W. Weil, Stochastic and Integral Geometry, Springer, Berlin (2008).

[4] H. S. Harutyunyan and V. K. Ohanyan, "Chord length distribution function for regular polygons", Advances in Applied Probability (SGSA), **41**, no. 2, 358 – 366 (2009).

[5] N. G. Aharonyan and V. K. Ohanyan, "Chord length distributions for polygons", Journal of Contemporary Math. Anal., **40**, no. 4, 43 – 56 (2005).

[6] Gasparyan A. G., Ohanyan V. K. "Orientation-dependent distribution of the length of a random segment and covariogram", Journal of Contemporary Mathematical Analysis, **50**, no. 2, 90 – 97 (2015).

[7] H. S. Harutyunyan and V. K. Ohanyan, "Covariogram of a cylinder", Journal of Contemporary Mathematical Analysis, **49**, no. 6, 366 – 375 (2014).

[8] N. G. Aharonyan, H.O. Harutyunyan, “Geometric probability calculation for a triangle”, Proceedings of Yerevan State University, Phys. and Math. Sciences, **51** (3), 211 – 216 (2017).
[9] Bianchi G. and Averkov G., “Confirmation of Matheron’s Conjecture on the covariogram of a planar convex body”, Journal of the European Mathematical Society, **11**, no. 6, 1187 – 1202 (2009).
[10] N. G. Aharonyan and V. Khalatyan, “Distribution of the distance between two random points in a body from $\mathbf{R}^n$”, Journal of Contemporary Mathem. Analysis, **55** (2), 3 – 8 (2020).
[11] V. K. Ohanyan and G. L. Adamyan, “Covariogram of a right parallelepiped”, Proceedings of Yerevan State University, Phys. and Math. Sciences, **53**, no. 3, 113 – 120 (2019).
[12] N. G. Aharonyan and V. K. Ohanyan, “Calculation of geomeric probabilities using covariogram of convex bodies”, Journal of Contemporary Mathem. Analysis, **53**, no. 2, 113 – 120 (2018).
[13] N. G. Aharonyan and V. K. Ohanyan, “Pattern recognition by cross sections”, Modeling of Artificial Intelligence, **4**, no. 2, 72 – 77 (2017).
[14] H. S. Harutyunyan and V. K. Ohanyan, “Orientation-dependent section distributions for convex bodies”, Journal of Contemporary Mathematical Analysis, **49**, no. 3, 139 – 156 (2014).
[15] A. G. Gasparyan and V. K. Ohanyan, “Covariogram of a parallelogram”, Journal of Contemporary Mathem. Analysis, **49**, no. 4, 17 – 34 (2014).
[16] D. M. Mount, “The densest double-lattice packing of a convex polygon”, DIMACS, Series in Discrete Mathematics and Theoretical Computer Science, **6**, 245 – 262 (1991).

Поступила 12 мая 2020

После доработки 5 сентября 2020

Принята к публикации 16 сентября 2020