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ON THE DE RHAM THEOREM AND AN APPLICATION TO THE MAXWELL-STOKES TYPE PROBLEM

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Abstract. In this paper, we derive an L^p version of the de Rham theorem. The key is an L^p version of the Nečas inequality. Using this result and the variational method, we show the existence of a solution to the Maxwell-Stokes type system.

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1. Introduction

The final purpose of this paper is to derive the existence of a solution to the Maxwell-Stokes type system.

First, we consider the following quasilinear magneto-static problem:

$$\operatorname{curl}\left[\boldsymbol{G}(x, \operatorname{curl}\boldsymbol{u})\right] = \boldsymbol{f} \text{ in } \Omega, \tag{1.1a}$$

$$\operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega, \tag{1.1b}$$

$$\boldsymbol{u}_T = \boldsymbol{u}_T^0 \text{ on } \Gamma, \tag{1.1c}$$

where Ω is a bounded domain in $\mathbb{R}^3$ with a boundary $\Gamma = \partial\Omega$, $\boldsymbol{u}_T$ denotes the tangent component of $\boldsymbol{u}$, namely, if we write the unit outer normal vector of the boundary by $\boldsymbol{n}$, then $\boldsymbol{u}_T = (\boldsymbol{n} \times \boldsymbol{u}) \times \boldsymbol{n}$, and $\boldsymbol{u}_T^0$ is a given tangential vector field, that is, $\boldsymbol{n} \cdot \boldsymbol{u}_T^0 = 0$ on Γ.

This system is interesting in physics, and may be viewed as the stationary version of the eddy current model, where the relation between the magnetic field $\boldsymbol{H}$ and the magnetic induction $\boldsymbol{B}$ is defined by the nonlinear B-H-curve. For the physical nature of the nonlinear B-H-curve, see Kaltenbacher et al. [14] and Pechstein and Jütter [19]. The eddy-current problem is a quasi-static approximation at very low frequency of the Maxwell equation, and the approximation is obtained by neglecting the displacement current in the Maxwell-Ampère law. Here we want to say that the solvability of (1.1a)-(1.1b) depends on the nonlinearity of a vector function $\boldsymbol{G}(x, \boldsymbol{z})$, the boundary conditions and the shape of the domain Ω with special emphasis, Such system are investigated by many authors, for example, Pan [18], Miranda et al. [15, 16], Yin [21, 23], Yin et al. [22]. If a given function $\boldsymbol{f}$ does not satisfy $\operatorname{div} \boldsymbol{f} = 0$ in Ω, or Ω has holes, then the system (1.1a)-(1.1c) are not nicely posed problem, so we may introduce an unknown scalar function π to the system, which may be called a potential.

To overcome such difficulty, we consider the following Maxwell-Stokes type system:

$$\operatorname{curl}[S_t(x, |\operatorname{curl} \boldsymbol{u}|^2)\operatorname{curl} \boldsymbol{u}] + \nabla\pi = \boldsymbol{f} \text{ in } \Omega, \tag{1.2a}$$

$$\operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega, \tag{1.2b}$$

$$\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma, \tag{1.2c}$$

$$\boldsymbol{u} \cdot \boldsymbol{n} = g \text{ on } \Gamma, \tag{1.2d}$$

where $\boldsymbol{f}$ and g are given functions, and $S(x, t)$ is a Carathéodory function on $\Omega \times [0, \infty)$ satisfying some structure conditions (see section 3). According to the conditions on a function $S(x, t)$, we can see that the equation (1.2a) contains a p-curlcurl equation:

$$\operatorname{curl}[|\operatorname{curl} \boldsymbol{u}|^{p-2}\operatorname{curl} \boldsymbol{u}] + \nabla\pi = \boldsymbol{f} \text{ in } \Omega \ (1 < p < \infty).$$

If we impose the Dirichlet boundary condition to π, then we derived the solvability of the system (1.2a)-(1.2c) in a multi-connected domain without holes in the author's previous paper Aramaki [8]. The de Rham theorem used there was rather restrictive (cf. Aramaki [6]).

However, in the case where Ω has holes, it is necessary to impose the boundary condition (1.2d) for g satisfying some conditions. For this purpose, we have to derive a more general de Rham theorem.

In this paper, we do not impose any boundary condition to the potential, and we derive the existence of solution to the system (1.2a)-(1.2d). To do so, it is necessary to derive an L^p version of de Rham theorem.

The paper is organized as follows. In section 2, we give an L^p version of the de Rham theorem which is ushered by an L^p version of the celebrated Nečas inequality. In section 3, we give some preliminaries for the Maxwell-Stokes type system. Section 4 is devoted to the existence theory of a solution to the Maxwell-Stokes system, using the de Rham theorem given in section 2.

2. A coarse version of the de Rham theorem

In this section, let Ω be a bounded domain which means a bounded, connected open subset of $\mathbb{R}^d$ $(d \geq 2)$ with a Lipschitz boundary Γ, $1 < q < \infty$, and let q' be the conjugate exponent i.e., $(1/q) + (1/q') = 1$.

From now on we use $L^q(\Omega)$, $W^{m,q}(\Omega)$ ($m \geq 0$, integer), $W^{s,q}(\Gamma)$ ($s \in \mathbb{R}$), and so on, for the standard real L^q and Sobolev spaces of real valued functions. For any real Banach space B, we denote B^d by boldface character $\boldsymbol{B}$. Hereafter, we use this character to denote vector and vector-valued functions, and we denote the standard inner product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{R}^d$ by $\boldsymbol{a} \cdot \boldsymbol{b}$. Moreover, for the dual space $\boldsymbol{B'}$, we denote the duality bracket between $\boldsymbol{B'}$ and $\boldsymbol{B}$ by $\langle \cdot, \cdot \rangle_{\boldsymbol{B'},\boldsymbol{B}}$.

We consider a coarse version of the de Rham theorem. In order to do so, we first state the Nečas inequality which takes an important role for the proof of a coarse version of the de Rham theorem.

Theorem 2.1 (Nečas inequality)**.** *Let Ω is a bounded domain of $\mathbb{R}^d$ with a Lipschitz boundary Γ and $1 < q < \infty$. Then the set*

$$\{\pi \in W^{-1,q}(\Omega); \nabla \pi \in \boldsymbol{W}^{-1,q}(\Omega)\}$$

is equal to $L^q(\Omega)$, and there exists a constant $C > 0$ depending only on q and Ω such that

$$\|\pi\|_{L^q(\Omega)} \leq C(\|\pi\|_{W^{-1,q}(\Omega)} + \|\nabla \pi\|_{\boldsymbol{W}^{-1,q}(\Omega)}).$$

For the proof, see Theorem IV.1.1 for $q = 2$ and Remark IV.1.1 for general $1 < q < \infty$ in Boyer and Fabrie [10].

The Nečas inequality now allow the following Poincaré type inequality for the function of $L^q(\Omega)$.

Proposition 2.1. *Let Ω be a bounded domain of $\mathbb{R}^d$ with a Lipschitz boundary Γ and let $1 < q < \infty$. Then there exists a constant $C > 0$ depending only on q and Ω such that*

$$\|\pi\|_{W^{-1,q}(\Omega)} \leq C \left(\frac{1}{|\Omega|} \left| \int_\Omega \pi dx \right| + \|\nabla \pi\|_{\boldsymbol{W}^{-1,q}(\Omega)} \right) \text{ for all } \pi \in L^q(\Omega),$$

where $|\Omega|$ denotes the volume of Ω.

Proof. Assume that the conclusion is false. Then there exists $\{\pi_n\}_{n=1}^\infty \subset L^q(\Omega)$ such that

$$\|\pi_n\|_{W^{-1,q}(\Omega)} \geq n \left(\frac{1}{|\Omega|} \left| \int_\Omega \pi_n dx \right| + \|\nabla \pi_n\|_{\boldsymbol{W}^{-1,q}(\Omega)} \right).$$

By homogeneity, we may assume that $\|\pi_n\|_{W^{-1,q}(\Omega)} = 1$. From Nečas inequality (Theorem 2.1), we can deduce that $\{\pi_n\}$ is bounded in $L^q(\Omega)$. Passing to a subsequence, we may assume that $\pi_n \to \pi$ weakly in $L^q(\Omega)$. Since the embedding $W_0^{1,q'}(\Omega) \hookrightarrow L^{q'}(\Omega)$ is compact and dense, we can see the embedding $L^q(\Omega) \hookrightarrow W^{-1,q}(\Omega)$ is also compact. Therefore, $\pi_n \to \pi$ strongly in $W^{-1,q}(\Omega)$. Since $\|\nabla \pi_n\|_{\boldsymbol{W}^{-1,q}(\Omega)} \to 0$ as

$n \to \infty$, we obtain $\nabla\pi = \mathbf{0}$ in the distribution sense and therefore $\pi = c =$ const. . However, we also have

$$\frac{1}{|\Omega|}\left|\int_\Omega \pi_n dx\right| \le \frac{1}{n}.$$

Since $\pi_n \to \pi = c$ weakly in $L^q(\Omega)$, we obtain $c = 0$, so $\pi = 0$. On the other hand, since $\|\pi\|_{W^{-1,q}(\Omega)} = \lim_{n\to\infty}\|\pi_n\|_{W^{-1,q}(\Omega)} = 1$, this leads to a contradiction. $\square$

Next we derive that the gradient operator from $L^q(\Omega)$ to $\boldsymbol{W}^{-1,q}(\Omega)$ has a closed range.

Proposition 2.2. *Let Ω be a bounded domain of $\mathbb{R}^d$ with a Lipschitz boundary Γ and $1 < q < \infty$. Then the gradient operator* $\mathrm{grad} = \nabla : L^q(\Omega) \to \boldsymbol{W}^{-1.q}(\Omega)$ *has a closed range in $\boldsymbol{W}^{-1,q}(\Omega)$.*

Proof. Let $\pi_n \in L^q(\Omega)$ and $\nabla\pi_n \to \boldsymbol{f}$ in $\boldsymbol{W}^{-1,q}(\Omega)$ as $n \to \infty$. Then we may assume that $\int_\Omega \pi_n dx = 0$ for all $n \in \mathbb{N}$. By Nečas inequality, we have

$$\|\pi_n - \pi_m\|_{L^q(\Omega)} \le C(\|\pi_n - \pi_m\|_{\boldsymbol{W}^{-1,q}(\Omega)} + \|\nabla(\pi_n - \pi_m)\|_{\boldsymbol{W}^{-1,q}(\Omega)}).$$

However, it follows from Proposition 2.1 that we have

$$\|\pi_n - \pi_m\|_{\boldsymbol{W}^{-1,q}(\Omega)} \le C\|\nabla(\pi_n - \pi_m)\|_{\boldsymbol{W}^{-1,q}(\Omega)}).$$

Thus we obtain

$$\|\pi_n - \pi_m\|_{L^q(\Omega)} \le C_1\|\nabla(\pi_n - \pi_m)\|_{\boldsymbol{W}^{-1,q}(\Omega)}).$$

Since $\nabla\pi_n \to \boldsymbol{f}$ in $\boldsymbol{W}^{-1.q}(\Omega)$, $\{\pi_n\}$ is a Cauchy sequence in $L^q(\Omega)$. Therefore, there exists $\pi \in L^q(\Omega)$ such that $\pi_n \to \pi$ in $L^q(\Omega)$. So we have $\boldsymbol{f} = \nabla\pi \in \nabla(L^q(\Omega))$. $\square$

We are in a position to state a coarse version of the de Rham theorem.

Theorem 2.2 (A coarse version of the de Rham theorem)**.** *Let Ω be a bounded domain of $\mathbb{R}^d$ with a Lipschitz boundary Γ, $1 < q < \infty$ and let $\boldsymbol{h} \in \boldsymbol{W}^{-1,q'}(\Omega)$. If $\boldsymbol{h}$ satisfies*

$$\langle \boldsymbol{h}, \boldsymbol{v}\rangle_{\boldsymbol{W}^{-1,q'}(\Omega),\boldsymbol{W}_0^{1,q}(\Omega)} = 0 \quad \text{for all } \boldsymbol{v} \in \boldsymbol{W}_0^{1,q}(\Omega) \text{ satisfying } \operatorname{div}\boldsymbol{v} = 0 \text{ in } \Omega, \tag{2.1}$$

then there exists a function $\pi \in L_0^{q'}(\Omega) := \left\{q \in L^{q'}(\Omega); \int_\Omega q dx = 0\right\}$ such that

$$\boldsymbol{h} = \nabla\pi \text{ in } \Omega. \tag{2.2}$$

Conversely, if (2.2) *holds, then clearly* (2.1) *holds.*

Proof. In general, for any subset A of a normed linear space X, define

$$A^\perp = \{f \in X'; \langle f, x\rangle_{X',X} = 0 \text{ for all } x \in A\},$$

and for any subset A' of the dual space X', define

$$^{\perp}(A') = \{x \in X; \langle f, x\rangle_{X',X} = 0 \text{ for all } f \in A'\}.$$

It is well known that if A is a closed subspace of X, then it holds that $^{\perp}(A^{\perp}) = A$ (cf. Taylor and Lay [20, p. 164]).

Define $X = \boldsymbol{W}^{-1,q'}(\Omega)$ and

$$\boldsymbol{Y}^{q'}(\Omega) = \{\nabla\pi : \pi \in L^{q'}(\Omega)\}.$$

Then it follows from Proposition 2.2 that $\boldsymbol{Y}^{q'}(\Omega)$ is a closed subspace of $X = \boldsymbol{W}^{-1,q'}(\Omega)$, so

$$^{\perp}(\boldsymbol{Y}^{q'}(\Omega)^{\perp}) = \boldsymbol{Y}^{q'}(\Omega). \tag{2.3}$$

If we define

$$\boldsymbol{Z}^{q}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}_0^{1,q}(\Omega); \operatorname{div}\boldsymbol{v} = 0 \text{ in } \Omega\},$$

then $\boldsymbol{Z}^q(\Omega)$ is a closed subspace of $X' = \boldsymbol{W}^{-1,q'}(\Omega)' = \boldsymbol{W}_0^{1,q}(\Omega)$. The theorem clearly means that

$$^{\perp}\boldsymbol{Z}^{q}(\Omega) \subset \boldsymbol{Y}^{q'}(\Omega). \tag{2.4}$$

To derive (2.4), it suffices to show that $\boldsymbol{Y}^{q'}(\Omega)^{\perp} \subset \boldsymbol{Z}^q(\Omega)$ since (2.3) holds. Assume that $\boldsymbol{v} \in \boldsymbol{Y}^{q'}(\Omega)^{\perp} \subset X' = \boldsymbol{W}_0^{1,q}(\Omega)$. Then for any $\pi \in L^{q'}(\Omega)$,

$$-\int_{\Omega} \pi \operatorname{div}\boldsymbol{v}\, dx = \langle \nabla\pi, \boldsymbol{v}\rangle_{\boldsymbol{W}^{-1,q'}(\Omega),\boldsymbol{W}_0^{1,q}(\Omega)} = 0.$$

This implies that

$$\langle \operatorname{div}\boldsymbol{v}, \pi\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = 0 \text{ for all } \pi \in \mathcal{D}(\Omega),$$

where $\mathcal{D}(\Omega)$ is the space of C^{∞} functions with compact supports in Ω, and $\mathcal{D}'(\Omega)$ is the space of distributions in Ω. Therefore, $\operatorname{div}\boldsymbol{v} = 0$ in $\mathcal{D}'(\Omega)$. Since $\operatorname{div}\boldsymbol{v} \in L^q(\Omega)$, we obtain that $\operatorname{div}\boldsymbol{v} = 0$ in $L^q(\Omega)$, so $\boldsymbol{v} \in \boldsymbol{Z}^q(\Omega)$. □

Remark 2.1. *To tell the truth, the Nečas inequality (Theorem 2.1), Proposition 2.1 and a coarse version of the de Rham theorem (Theorem 2.2) are equivalent. For this facts, see Amrouche et al.* [4] *for $q = 2$, and see* [6] *for genaral $1 < q < \infty$*

3. Preliminaries for the Maxwell-Stokes type problem

In this section, we give preliminaries for the Maxwell-Stokes type problem as an application of a coarse version of the de Rham theorem (Theorem 2.2). To do so, we assume that Ω is a bounded domain (connected open subset) of $\mathbb{R}^3$ with a Lipschitz boundary Γ satisfying the following conditions as in Amrouche and Seloula [2] (cf. Amrouche and Seloula [3] and Girault and Raviart [13]).

Assume that Ω is locally situated on one side of Γ. In addition, (O1) Γ has a finite number of connected components $\Gamma_0, \Gamma_1, \dots, \Gamma_I$ with Γ_0 denoting the boundary of the infinite connected component of $\mathbb{R}^3 \setminus \overline{\Omega}$.

(O2) There exist J connected open surfaces Σ_j, $(j = 1, \dots, J)$, called cuts, contained in Ω such that

(a) Σ_j is an open subset of a smooth manifold $\mathcal{M}_j$.

(b) $\partial\Sigma_j \subset \Gamma$ $(j = 1, \dots, J)$, where $\partial\Sigma_j$ denotes the boundary of Σ_j, and Σ_j is non-tangential to Γ.

(c) $\overline{\Sigma_j} \cap \overline{\Sigma_k} = \emptyset\,(j \neq k)$.

(d) The open set $\dot{\Omega} = \Omega \setminus (\cup_{j=1}^J \Sigma_j)$ is simply connected and of Lipschitz class.

The number J is called the first Betti number which is equal to the number of handles of Ω, and I is called the second Betti number which is equal to the number of holes. We say that if $J = 0$, Ω is simply connected, and if $I = 0$, Ω has no holes.

Define two spaces by

$$\begin{aligned} \mathbb{K}_N^p(\Omega) &= \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega); \operatorname{div} \boldsymbol{v} = 0, \operatorname{curl} \boldsymbol{v} = \boldsymbol{0} \text{ in } \Omega, \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\}, \\ \mathbb{K}_T^p(\Omega) &= \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega); \operatorname{div} \boldsymbol{v} = 0, \operatorname{curl} \boldsymbol{v} = \boldsymbol{0} \text{ in } \Omega, \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}. \end{aligned}$$

Then it is well known that $\dim \mathbb{K}_T^p(\Omega) = J$ and $\dim \mathbb{K}_N^p(\Omega) = I$.

We assume that a Carathéodory function $S(x,t)$ satisfies the following conditions: There exist $1 < p < \infty$ and positive constants $0 < \lambda \le \Lambda < \infty$ such that for a.e. $x \in \Omega$, $S(x,\cdot) \in C^2((0,\infty)) \cap C^0([0,\infty))$ as a function of t, and $S(x,t)$ satisfies that

$$S(x,0) = 0 \text{ and } \lambda t^{(p-2)/2} \le S_t(x,t) \le \Lambda t^{(p-2)/2} \text{ for } t > 0. \tag{3.1a}$$

$$\lambda t^{(p-2)/2} \le S_t(x,t) + 2tS_{tt}(x,t) \le \Lambda t^{(p-2)/2} \text{ for } t > 0. \tag{3.1b}$$

$$\text{If } 1 < p < 2, S_{tt}(x,t) < 0, \text{ and if } p \ge 2, S_{tt}(x,t) \ge 0 \text{ for } t > 0. \tag{3.1c}$$

Here $S_t = \partial S/\partial t, S_{tt} = \partial^2 S/\partial t^2$. We note that from (3.1a), we have

$$\frac{2}{p}\lambda t^{p/2} \le S(x,t) \le \frac{2}{p}\Lambda t^{p/2} \text{ for } t \ge 0 \text{ and a.e. } x \in \Omega. \tag{3.2}$$

Example 3.1. *If $S(x,t) = \nu(x)t^{p/2}$, where ν is a measurable function in Ω and satisfies $0 < \nu_* \le \nu(x) \le \nu^* < \infty$, then it follows from elementary calculations that* (3.1a)-(3.1c) *hold.*

We have the following lemma with respect to the monotonicity of S_t.

Lemma 3.1. *There exists a constant $c > 0$ such that for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$,*

$$\big(S_t(x,|\boldsymbol{a}|^2)\boldsymbol{a} - S_t(x,|\boldsymbol{b}|^2)\boldsymbol{b}\big) \cdot (\boldsymbol{a} - \boldsymbol{b}) \ge \begin{cases} c|\boldsymbol{a} - \boldsymbol{b}|^p & \text{if } p \ge 2, \\ c(|\boldsymbol{a}| + |\boldsymbol{b}|)^{p-2}|\boldsymbol{a} - \boldsymbol{b}|^2 & \text{if } 1 < p < 2. \end{cases}$$

In particular, S_t is strictly monotone, that is,

$$\big(S_t(x,|\boldsymbol{a}|^2)\boldsymbol{a} - S_t(x,|\boldsymbol{b}|^2)\boldsymbol{b}\big)\cdot(\boldsymbol{a}-\boldsymbol{b}) > 0 \textit{ if } \boldsymbol{a}\neq\boldsymbol{b}.$$

For the proof, see Aramaki [7, Lemma 3.5].

Next, we show that $S(x,|\boldsymbol{a}|^2)$ is strictly convex with respect to $\boldsymbol{a}\in\mathbb{R}^3$.

Lemma 3.2. *For a.e. $x\in\Omega$, $S(x,|\boldsymbol{a}|^2)$ is strictly convex with respect to $\boldsymbol{a}\in\mathbb{R}^3$, that is, for $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^3$ and $0\le\theta\le 1$,*

$$S(x,|\theta\boldsymbol{a}+(1-\theta)\boldsymbol{b}|^2) \le \theta S(x,|\boldsymbol{a}|^2)+(1-\theta)S(x,|\boldsymbol{b}|^2), \tag{3.3}$$

and in particular, if $\boldsymbol{a}\neq\boldsymbol{b}$ and $0<\theta<1$, then we have

$$S(x,|\theta\boldsymbol{a}+(1-\theta)\boldsymbol{b}|^2) < \theta S(x,|\boldsymbol{a}|^2)+(1-\theta)S(x,|\boldsymbol{b}|^2). \tag{3.4}$$

Proof. For brevity of notation, we put $F(x,t)=S(x,t^2)$. Since

$$F_t(x,t)=2tS_t(x,t^2)\ge 2\lambda t^{p-1}>0 \text{ a.e. } x\in\Omega \text{ and } t>0$$

from (3.1a), and

$$F_{tt}(x,t)=2(S_t(x,t^2)+2t^2S_{tt}(x,t^2))\ge 2\lambda t^{p-2}>0 \text{ a.e. } x\in\Omega \text{ and } t>0$$

from (3.1b), we see that for a.e. $x\in\Omega$, $F(x,t)$ is strictly increasing and strictly convex as a function ot $t\in[0,\infty)$. Therefore, for a.e. $x\in\Omega$, $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^3$ and $0\le\theta\le 1$, we have

$$F(x,|\theta\boldsymbol{a}+(1-\theta)\boldsymbol{b}|)\le F(x,\theta|\boldsymbol{a}|+(1-\theta)|\boldsymbol{b}|)\le\theta F(x,|\boldsymbol{a}|)+(1-\theta)F(x,|\boldsymbol{b}|).$$

Thus $F(x,|\boldsymbol{a}|)=S(x,|\boldsymbol{a}|^2)$ is a convex function of $\boldsymbol{a}\in\mathbb{R}^3$.

Let $\boldsymbol{a}\neq\boldsymbol{b}$ and $0<\theta<1$, Without loss of generality, we may assume $\boldsymbol{a}\neq\boldsymbol{0}$. If $|\theta\boldsymbol{a}+(1-\theta)\boldsymbol{b}|<\theta|\boldsymbol{a}|+(1-\theta)|\boldsymbol{b}|$, then we have

$$F(x,|\theta\boldsymbol{a}+(1-\theta)\boldsymbol{b}|)< F(x,\theta|\boldsymbol{a}|+(1-\theta)|\boldsymbol{b}|)\le\theta F(x,|\boldsymbol{a}|)+(1-\theta)F(x,|\boldsymbol{b}|).$$

If $|\theta\boldsymbol{a}+(1-\theta)\boldsymbol{b}|=\theta|\boldsymbol{a}|+(1-\theta)|\boldsymbol{b}|$, then this implies that $\boldsymbol{a}\cdot\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}|$. By the Schwarz inequality, $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly dependent, so we can write $\boldsymbol{b}=c\boldsymbol{a}$, where $c\ge 0$ and $c\neq 1$. This implies $|\boldsymbol{a}|\neq|\boldsymbol{b}|$. Thus it follow from the strict convexity of $F(x,t)$ as a function ot t that

$$F(x,|\theta\boldsymbol{a}+(1-\theta)\boldsymbol{b}|)= F(x,\theta|\boldsymbol{a}|+(1-\theta)|\boldsymbol{b}|)<\theta F(x,|\boldsymbol{a}|)+(1-\theta)F(x,|\boldsymbol{b}|).$$

□

4. Existence of a solution to the Maxwell-Stokes type system

In this section, we derive the existence of a solution to the Maxwell-Stokes type system. Let Ω be a bounded domain in $\mathbb{R}^3$ satisfying (O1) and (O2), $1 < p < \infty$ and let a Carathéodory function $S(x,t)$ satisfy (3.1a)-(3.1c).

We consider the following Maxwell-Stokes type problem: for given $\boldsymbol{f}$ and g, find $(\boldsymbol{u}, \pi)$ such that

$$\operatorname{curl}[S_t(x, |\operatorname{curl}\boldsymbol{u}|^2)\operatorname{curl}\boldsymbol{u}] + \nabla\pi = \boldsymbol{f} \text{ in } \Omega, \tag{4.1a}$$

$$\operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega, \tag{4.1b}$$

$$\boldsymbol{u} = g\boldsymbol{n} \text{ on } \Gamma. \tag{4.1c}$$

Define a space

$$\mathbb{V}^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega); \operatorname{div}\boldsymbol{v} = 0 \text{ in } \Omega, \boldsymbol{v}\times\boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma, \langle \boldsymbol{v}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0 \text{ for } i = 1, \dots, I\}.$$

Then we can see that $\mathbb{V}^p(\Omega)$ is a separable, reflexive Banach space equipped with the semi-norm

$$\|\boldsymbol{v}\|_{\mathbb{V}^p(\Omega)} = \|\operatorname{curl}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.2}$$

By [3, p. 40], since we have

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \le C(p,\Omega)\|\operatorname{curl}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \text{ for all } \boldsymbol{v} \in \mathbb{V}^p(\Omega),$$

the definition (4.2) is, in fact, the norm and $\|\boldsymbol{v}\|_{\mathbb{V}^p(\Omega)}$ and $\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}$ are equivalent (cf. [15, 16]).

Assume that a given function $\boldsymbol{f}$ satisfies that $\boldsymbol{f} \in \mathbb{V}^p(\Omega)' \cap \boldsymbol{W}^{-1,p'}(\Omega)$, where p' denotes the conjugate exponent of p, and

$$\boldsymbol{f}\big|_{\mathbb{V}^p(\Omega)\cap\boldsymbol{W}_0^{1,p}(\Omega)} \text{ in } \boldsymbol{W}^{-1,p'}(\Omega) = \boldsymbol{f}\big|_{\mathbb{V}^p(\Omega)\cap\boldsymbol{W}_0^{1,p}(\Omega)} \text{ in } \mathbb{V}^p(\Omega)',$$

that is,

$$\langle \boldsymbol{f}, \boldsymbol{v}\rangle_{\mathbb{V}^p(\Omega)',\mathbb{V}^p(\Omega)} = \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{\boldsymbol{W}^{-1,p'}(\Omega)',\boldsymbol{W}_0^{1,p}(\Omega)} \tag{4.3}$$

for all $\boldsymbol{v} \in \mathbb{V}^p(\Omega) \cap \boldsymbol{W}_0^{1,p}(\Omega)$.

Moreover, we assume that $g \in W^{1-1/p,p}(\Gamma)$ and satisfies that

$$\int_{\Gamma_i} g\,d\sigma = 0 \text{ for every } i = 0, 1, \dots, I, \tag{4.4}$$

where $d\sigma$ denotes the surface area of Γ. We define a space

$$\boldsymbol{U}_g^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega); \operatorname{div}\boldsymbol{v} = 0 \text{ in } \Omega, \boldsymbol{v} = g\boldsymbol{n} \text{ on } \Gamma\}.$$

If $\boldsymbol{v} \in \boldsymbol{U}_g^p(\Omega)$, then $\boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0}$ and $\langle \boldsymbol{v} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = \int_{\Gamma_i} g d\sigma = 0$, and so we can easily see that $\boldsymbol{U}_g^p(\Omega)$ is a closed, convex subset of $\mathbb{V}^p(\Omega)$. If we put $\boldsymbol{g} = g\boldsymbol{n}$ on Γ, then from (4.4), we have

$$\int_\Gamma \boldsymbol{g} \cdot \boldsymbol{n} d\sigma = \int_\Gamma g d\sigma = \sum_{i=0}^{I} \int_{\Gamma_i} g d\sigma = 0.$$

Therefore, it follows from Amrouche and Girault [1, Lemma 3.3] that there exists $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ such that $\operatorname{div} \boldsymbol{v} = 0$ in Ω and $\boldsymbol{v} = \boldsymbol{g} = g\boldsymbol{n}$ on Γ. Thus $\boldsymbol{v} \in \boldsymbol{U}_g^p(\Omega)$, so $\boldsymbol{U}_g^p(\Omega)$ is a non-empty set. Since $\boldsymbol{U}_g^p(\Omega)$ is a closed and convex subset of $\mathbb{V}^p(\Omega)$, we can see that $\boldsymbol{U}_g^p(\Omega)$ is sequentially weakly closed subset of $\mathbb{V}^p(\Omega)$ (cf. Ciarlet [12, Theorem 5.13-1]).

We are in a position to state a main theorem.

Theorem 4.1. *Assume that Ω is a bounded domain of $\mathbb{R}^3$ with a Lipschitz boundary Γ and satisfies (O1) and (O2), $1 < p < \infty$ and a Carathéodory function $S(x,t)$ satisfies* (3.1a)-(3.1c). *Moreover, assume that $\boldsymbol{f} \in \mathbb{V}^p(\Omega)' \cap \boldsymbol{W}^{-1,p'}(\Omega)$ satisfies* (4.3) *and $g \in W^{1-1/p,p}(\Gamma)$ satisfies* (4.4). *Then the Maxwell-Stokes type system* (4.1a)-(4.1c) *has a unique weak solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^{p'}(\Omega)/\mathbb{R}$, and there exists a constant $C > 0$ dependent only on p and Ω such that*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}^p + \|\pi\|_{L^{p'}(\Omega)/\mathbb{R}}^{p'} \le C(\|\boldsymbol{f}\|_{\boldsymbol{W}^{-1,p'}(\Omega)}^{p'} + \|\boldsymbol{f}\|_{\mathbb{V}^p(\Omega)'}^{p'} + \|g\|_{W^{1-1/p,p}(\Gamma)}^p). \tag{4.5}$$

Proof. We prove this theorem using the direct method of calculus of variation and using a coarse version of the de Rham theorem. In order to do so, we consider a functional

$$J[\boldsymbol{v}] = \frac{1}{2} \int_\Omega S(x, |\operatorname{curl} \boldsymbol{v}|^2) dx - \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\mathbb{V}^p(\Omega)', \mathbb{V}^p(\Omega)} \text{ for } \boldsymbol{v} \in \boldsymbol{U}_g^p(\Omega). \tag{4.6}$$

Step 1. J has a unique minimizer $\boldsymbol{u} \in \boldsymbol{U}_g^p(\Omega)$, that is,

$$J(\boldsymbol{u}) = \inf_{\boldsymbol{v} \in \boldsymbol{U}_g^p(\Omega)} J(\boldsymbol{v}).$$

From (3.2), the duality and the Young inequality, for any $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that for all $\boldsymbol{v} \in \boldsymbol{U}_g^p(\Omega)$,

$$\begin{aligned} J[\boldsymbol{v}] &\ge \frac{\lambda}{p} \|\operatorname{curl} \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}^p - \|\boldsymbol{f}\|_{\mathbb{V}^p(\Omega)'} \|\boldsymbol{v}\|_{\mathbb{V}^p(\Omega)} \\ &\ge \frac{\lambda}{p} \|\boldsymbol{v}\|_{\mathbb{V}^p(\Omega)}^p - C(\varepsilon) \|\boldsymbol{f}\|_{\mathbb{V}^p(\Omega)'}^{p'} - \varepsilon \|\boldsymbol{v}\|_{\mathbb{V}^p(\Omega)}^p. \end{aligned}$$

If we put $\varepsilon = \lambda/2p$, we have

$$J[\boldsymbol{v}] \ge \frac{\lambda}{2p} \|\boldsymbol{v}\|_{\mathbb{V}^p(\Omega)}^p - C\|\boldsymbol{f}\|_{\mathbb{V}^p(\Omega)'}^{p'} > -\infty \text{ for } \boldsymbol{v} \in \boldsymbol{U}_g^p(\Omega). \tag{4.7}$$

Thus we see that J is coercive on $\boldsymbol{U}_g^p(\Omega)$.

Next we show that $J : \boldsymbol{U}_g^p(\Omega) \to \mathbb{R}$ is sequentially weakly lower semi-continuous. Let $\boldsymbol{v}_j \to \boldsymbol{v}$ weakly in $\mathbb{V}^p(\Omega)$. Since curl $: \mathbb{V}^p(\Omega) \to \boldsymbol{L}^p(\Omega)$ is linear and bounded, it is clear that $\operatorname{curl} \boldsymbol{v}_j \to \operatorname{curl} \boldsymbol{v}$ weakly in $\boldsymbol{L}^p(\Omega)$.

Then it follows from Aramaki [5] that

$$\int_\Omega S(x, |\operatorname{curl} \boldsymbol{v}|^2)dx \le \liminf_{j\to\infty} \int_\Omega S(x, |\operatorname{curl} \boldsymbol{v}_j|^2)dx.$$

On the other hand, since $\boldsymbol{f} \in \mathbb{V}^p(\Omega)'$ and $\boldsymbol{v}_j \to \boldsymbol{v}$ weakly in $\mathbb{V}^p(\Omega)$, we have

$$\langle \boldsymbol{f}, \boldsymbol{v}\rangle_{\mathbb{V}^p(\Omega)', \mathbb{V}^p(\Omega)} = \lim_{j\to\infty} \langle \boldsymbol{f}, \boldsymbol{v}_j\rangle_{\mathbb{V}^p(\Omega)', \mathbb{V}^p(\Omega)}.$$

Thus we have

$$J(\boldsymbol{v}) \le \liminf_{j\to\infty} J(\boldsymbol{v}_j),$$

that is, $J : \boldsymbol{U}_g^p(\Omega) \to \mathbb{R}$ is sequentially weakly lower semi-continuous. Therefore, J has a minimizer $\boldsymbol{u} \in \boldsymbol{U}_g^p(\Omega)$. See, for example, Ciarlet [12, Theorem 9.3-1].

Using (3.3), we can easily see that J is a convex functional on $\boldsymbol{U}_g^p(\Omega)$. Moreover, since $\|\boldsymbol{v}\|_{\mathbb{V}^p(\Omega)}$ and $\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}$ are equivalent for $\boldsymbol{v} \in \mathbb{V}^p(\Omega)$, it follows that if $\boldsymbol{u} \ne \boldsymbol{v}$ in $\boldsymbol{U}_g^p(\Omega)$, then $\operatorname{curl} \boldsymbol{u} \ne \operatorname{curl} \boldsymbol{v}$ in $\boldsymbol{L}^p(\Omega)$. From Lemma 3.2, we see that J is strictly convex on $\boldsymbol{U}_g^p(\Omega)$. Thus the minimizer is unique.

Step 2. Let $\boldsymbol{u} \in \boldsymbol{U}_g^p(\Omega)$ be a unique minimizer of J on $\boldsymbol{U}_g^p(\Omega)$. For any

$$\boldsymbol{w} \in \boldsymbol{Z}^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega); \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\},$$

we have $\boldsymbol{u} + \tau\boldsymbol{w} \in \boldsymbol{U}_g^p(\Omega)$ for all $\tau \in \mathbb{R}$, and so $J(\boldsymbol{u}) \le J(\boldsymbol{u} + \tau\boldsymbol{w})$. By the Euler-Lagrange equation and assumption (4.3), we have

$$\begin{aligned} 0 = \frac{d}{d\tau} J[\boldsymbol{u} + \tau\boldsymbol{w}]\Big|_{\tau=0} &= \int_\Omega S_t(x, |\operatorname{curl} \boldsymbol{u}|^2)\operatorname{curl} \boldsymbol{u} \cdot \operatorname{curl} \boldsymbol{w} dx - \langle \boldsymbol{f}, \boldsymbol{w}\rangle_{\mathbb{V}^p(\Omega)', \mathbb{V}^p(\Omega)} \\ &= \int_\Omega S_t(x, |\operatorname{curl} \boldsymbol{u}|^2)\operatorname{curl} \boldsymbol{u} \cdot \operatorname{curl} \boldsymbol{w} dx - \langle \boldsymbol{f}, \boldsymbol{w}\rangle_{\boldsymbol{W}^{-1,p'}(\Omega)', \boldsymbol{W}_0^{1,p}(\Omega)}. \end{aligned}$$

From a coarse version of the de Rham theorem (Theorem 2.2), we can derive that there exists a function $\pi \in L_0^{p'}(\Omega)$ such that

$$\operatorname{curl}[S_t(x, |\operatorname{curl} \boldsymbol{u}|^2)\operatorname{curl} \boldsymbol{u}] + \nabla\pi = \boldsymbol{f} \text{ in } \boldsymbol{W}^{-1.p'}(\Omega). \tag{4.8}$$

Thus $(\boldsymbol{u}, \pi)$ is a solution of the system (4.1a)-(4.1c).

Step 3. We show the uniqueness of a solution of (4.1a)-(4.1c). Let $(\boldsymbol{u}_1, \pi_1), (\boldsymbol{u}_2, \pi_2) \in \boldsymbol{W}^{1,p}(\Omega) \times L^{p'}(\Omega)/\mathbb{R}$ be two solutions of (4.1a)-(4.1c). Since $\boldsymbol{u}_i \cdot \boldsymbol{n} = g$ and $\boldsymbol{u}_i \times \boldsymbol{n} = \boldsymbol{0}$ for $i = 1, 2$, we see that $\boldsymbol{u}_1 - \boldsymbol{u}_2 = \boldsymbol{0}$ on Γ, so we have $\boldsymbol{u}_1 - \boldsymbol{u}_2 \in \boldsymbol{Z}^p(\Omega)$. Thus we have

$$\langle \nabla \pi_i, \boldsymbol{u}_1 - \boldsymbol{u}_2 \rangle_{\boldsymbol{W}^{-1,p'}(\Omega), \boldsymbol{W}_0^{1,p}(\Omega)} = 0 \text{ for } i = 1, 2.$$

Therefore, if we take the inner product of (4.1a) and $\boldsymbol{u}_1 - \boldsymbol{u}_2$, and then integrate over Ω, we have

$$\int_\Omega \big(S_t(x, |\mathrm{curl}\, \boldsymbol{u}_1|^2)\mathrm{curl}\, \boldsymbol{u}_1 - S_t(x, |\mathrm{curl}\, \boldsymbol{u}_2|^2)\mathrm{curl}\, \boldsymbol{u}_2\big) \cdot \mathrm{curl}\,(\boldsymbol{u}_1 - \boldsymbol{u}_2) dx = 0.$$

From this equality and the strictly monotonicity of S_t (Lemma 3.1), we have $\mathrm{curl}\, \boldsymbol{u}_1 = \mathrm{curl}\, \boldsymbol{u}_2$ in Ω. This implies $\boldsymbol{u}_1 = \boldsymbol{u}_2$. From (4.1a), we have $\nabla \pi_1 = \nabla \pi_2$ in $\boldsymbol{W}^{-1,p'}(\Omega)$, so in the distribution sense. Since Ω is connected, we have $\pi_1 - \pi_2$ is equal to a constant, so $\pi_1 = \pi_2$ in $L^{p'}(\Omega)/\mathbb{R}$.

Step 4. We derive the estimate (4.5). Let $\boldsymbol{u} \in \boldsymbol{U}_g^p(\Omega)$ be the minimizer of J. Then for any $\boldsymbol{v} \in \boldsymbol{U}_g^p(\Omega)$ and for $0 < \theta < 1$, since

$$J(\boldsymbol{u}) \le J((1-\theta)\boldsymbol{u} + \theta \boldsymbol{v}) = J(\boldsymbol{u} + \theta(\boldsymbol{v} - \boldsymbol{u})),$$

we have

$$\begin{aligned} 0 &\le \frac{d}{d\theta} J(\boldsymbol{u} + \theta(\boldsymbol{v} - \boldsymbol{u}))\big|_{\theta=0+} \\ &= \int_\Omega S_t(x, |\mathrm{curl}\, \boldsymbol{u}|^2)\mathrm{curl}\, \boldsymbol{u} \cdot \mathrm{curl}\,(\boldsymbol{v} - \boldsymbol{u}) dx - \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u} \rangle_{\mathbb{V}^p(\Omega)', \mathbb{V}^p(\Omega)}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \text{(4.9)} \quad & \int_\Omega S_t(x, |\mathrm{curl}\, \boldsymbol{u}|^2)\mathrm{curl}\, \boldsymbol{u} \cdot \mathrm{curl}\, \boldsymbol{u} dx - \langle \boldsymbol{f}, \boldsymbol{u} \rangle_{\mathbb{V}^p(\Omega)', \mathbb{V}^p(\Omega)} \\ & \qquad \le \int_\Omega S_t(x, |\mathrm{curl}\, \boldsymbol{u}|^2)\mathrm{curl}\, \boldsymbol{u} \cdot \mathrm{curl}\, \boldsymbol{v} dx - \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\mathbb{V}^p(\Omega)', \mathbb{V}^p(\Omega)} \end{aligned}$$

for all $\boldsymbol{v} \in \boldsymbol{U}_g^p(\Omega)$. If we put $\boldsymbol{g} = g\boldsymbol{n} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$, it follows from [1, Lemma 3.3] that there exists $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ such that $\mathrm{div}\, \boldsymbol{v} = 0$ in Ω and $\boldsymbol{v} = \boldsymbol{g} = g\boldsymbol{n}$ on Γ, so $\boldsymbol{v} \in \boldsymbol{U}_g^p(\Omega)$, and there exists a constant $C > 0$ dependent only on p and Ω such that

$$\inf_{\boldsymbol{w} \in \boldsymbol{Z}^p(\Omega)} \|\boldsymbol{v} + \boldsymbol{w}\|_{\boldsymbol{W}^{1,p}(\Omega)} \le \|g\|_{W^{1-1/p,p}(\Gamma)}.$$

Here we can easily show that $\inf_{\boldsymbol{w} \in \boldsymbol{Z}^p(\Omega)} \|\boldsymbol{v} + \boldsymbol{w}\|_{\boldsymbol{W}^{1,p}(\Omega)}$ is achieved. Hence there exists $\boldsymbol{v}_0 \in \boldsymbol{U}_g^p(\Omega)$, and there exists a constant $C > 0$ depending only on p and Ω such that

$$\text{(4.10)} \qquad \|\boldsymbol{v}_0\|_{\boldsymbol{W}^{1,p}(\Omega)} \le C\|g\|_{W^{1-1/p,p}(\Omega)}.$$

We estimate (4.9) with $\boldsymbol{v}=\boldsymbol{v}_0$. From (3.1a), for any $\varepsilon>0$, there exists a constant $C(\varepsilon)>0$ such that

$$\begin{aligned}&\int_\Omega S_t(x,|\mathrm{curl}\,\boldsymbol{u}|^2)\mathrm{curl}\,\boldsymbol{u}\cdot\mathrm{curl}\,\boldsymbol{u}dx-\langle \boldsymbol{f},\boldsymbol{u}\rangle_{\mathbb{V}^p(\Omega)',\mathbb{V}^p(\Omega)}\\ &\quad\geq\lambda\int_\Omega|\mathrm{curl}\,\boldsymbol{u}|^pdx-\|\boldsymbol{f}\|_{\mathbb{V}^p(\Omega)'}\|\boldsymbol{u}\|_{\mathbb{V}^p(\Omega)}\\ &\quad\geq\lambda\|\boldsymbol{u}\|^p_{\mathbb{V}^p(\Omega)}-C(\varepsilon)\|\boldsymbol{f}\|^{p'}_{\mathbb{V}^p(\Omega)'}-\varepsilon\|\boldsymbol{u}\|^p_{\mathbb{V}^p(\Omega)}.\end{aligned}$$

On the other hand, using (3.1a), Hölder inequality and (4.10), we have

$$\begin{aligned}&\int_\Omega S_t(x,|\mathrm{curl}\,\boldsymbol{u}|^2)\mathrm{curl}\,\boldsymbol{u}\cdot\mathrm{curl}\,\boldsymbol{v}_0dx-\langle \boldsymbol{f},\boldsymbol{v}_0\rangle_{\mathbb{V}^p(\Omega)',\mathbb{V}^p(\Omega)}\\ &\quad\leq\Lambda\int_\Omega|\mathrm{curl}\,\boldsymbol{u}|^{p-1}|\mathrm{curl}\,\boldsymbol{v}_0|dx+C\|\boldsymbol{f}\|_{\mathbb{V}^p(\Omega)'}\|\boldsymbol{v}_0\|_{\boldsymbol{W}^{1,p}(\Omega)}\\ &\quad\leq\Lambda\|\boldsymbol{u}\|^{p-1}_{\mathbb{V}^p(\Omega)}\|\mathrm{curl}\,\boldsymbol{v}_0\|_{\boldsymbol{L}^p(\Omega)}+C\|\boldsymbol{f}\|^{p'}_{\mathbb{V}^p(\Omega)'}+C\|\boldsymbol{v}_0\|^p_{\mathbb{V}^p(\Omega)}\\ &\quad\leq\varepsilon\|\boldsymbol{u}\|^p_{\mathbb{V}^p(\Omega)}+C(\varepsilon)\|\boldsymbol{v}_0\|^p_{\boldsymbol{W}^{1,p}(\Omega)}+C\|\boldsymbol{f}\|^{p'}_{\mathbb{V}^p(\Omega)'}\\ &\quad\leq\varepsilon\|\boldsymbol{u}\|^p_{\mathbb{V}^p(\Omega)}+C_1(\varepsilon)\|g\|^p_{W^{1-1/p,p}(\Gamma)}+C\|\boldsymbol{f}\|^{p'}_{\mathbb{V}^p(\Omega)'}.\end{aligned}$$

Therefore, if we choose $\varepsilon>0$ small enough, then there exists a constant $C>0$ depending only on p and Ω such that

$$\|\boldsymbol{u}\|^p_{\mathbb{V}^p(\Omega)}\leq C(\|\boldsymbol{f}\|^{p'}_{\mathbb{V}^p(\Omega)'}+\|g\|^p_{W^{1-1/p,p}(\Gamma)}). \tag{4.11}$$

Since $L^{p'}(\Omega)\subset W^{-1,p'}(\Omega)$, taking the Nečas inequality (Theorem 2.1 and the Poincaré inequality (Proposition 2.1) into consideration, we have

$$\begin{aligned}\|\pi\|_{L^{p'}(\Omega)}&\leq C(\|\pi\|_{W^{-1,p'}(\Omega)}+\|\nabla\pi\|_{\boldsymbol{W}^{-1,p'}(\Omega)})\\ &\leq C_1\left(\frac{1}{|\Omega|}\left|\int_\Omega\pi dx\right|+\|\nabla\pi\|_{\boldsymbol{W}^{-1,p'}(\Omega)}\right)\end{aligned}$$

for all $\pi\in L^{p'}(\Omega)$. Since $\mathrm{curl}\,[S_t(x,|\mathrm{curl}\,\boldsymbol{u}|^2)\mathrm{curl}\,\boldsymbol{u}]+\nabla\pi=\boldsymbol{f}$ in $\boldsymbol{W}^{-1,p'}(\Omega)$, we have

$$\begin{aligned}\|\pi\|_{L^{p'}(\Omega)/\mathbb{R}}&\leq\left\|\pi-\int_\Omega\pi dx\right\|_{L^{p'}(\Omega)}\\ &\leq C\|\boldsymbol{f}\|_{\boldsymbol{W}^{-1,p'}(\Omega)}+\|\mathrm{curl}\,[S_t(x,|\mathrm{curl}\,\boldsymbol{u}|^2)\mathrm{curl}\,\boldsymbol{u}]\|_{\boldsymbol{W}^{-1,p'}(\Omega)}.\end{aligned}$$

Here, if we note that for all $\boldsymbol{v}\in\boldsymbol{W}^{1,p}_0(\Omega)$,

$$\begin{aligned}&|\langle\mathrm{curl}\,[S_t(x,|\mathrm{curl}\,\boldsymbol{u}|^2)\mathrm{curl}\,\boldsymbol{u}],\boldsymbol{v}\rangle_{\boldsymbol{W}^{-1,p'}(\Omega),\boldsymbol{W}^{1,p}_0(\Omega)}|\\ &\quad=\left|\int_\Omega S_t(x,|\mathrm{curl}\,\boldsymbol{u}|^2)\mathrm{curl}\,\boldsymbol{u}\cdot\mathrm{curl}\,\boldsymbol{v}dx\right|\\ &\leq C\|\boldsymbol{u}\|^{p-1}_{\mathbb{V}^p(\Omega}\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)},\end{aligned}$$

then we obtain

$$\|\mathrm{curl}\,[S_t(x,|\mathrm{curl}\,\boldsymbol{u}|^2)\mathrm{curl}\,\boldsymbol{u}]\|_{\boldsymbol{W}^{-1,p'}(\Omega)} \le C\|\boldsymbol{u}\|^{p-1}_{\mathbb{V}^p(\Omega)}.$$

Thus we have

$$\begin{aligned}(4.12)\quad \|\pi\|^{p'}_{L^{p'}(\Omega)/\mathbb{R}} &\le C(\|\boldsymbol{f}\|^{p'}_{\boldsymbol{W}^{-1,p'}(\Omega)} + \|\boldsymbol{u}\|^{p}_{\mathbb{V}^p(\Omega)})\\ &\le C(\|\boldsymbol{f}\|^{p'}_{\mathbb{V}^p(\Omega)'} + \|\boldsymbol{f}\|^{p'}_{\boldsymbol{W}^{-1,p'}(\Omega)} + \|\boldsymbol{u}\|^{p}_{\mathbb{V}^p(\Omega)}.\end{aligned}$$

Summing (4.11) and (4.12), we get the estimate (4.5). □

Remark 4.1. *When $p = 2$ and $S(x,t) = t$, the equation* (4.1a) *reduces to the Stokes equation*

$$-\Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{F}.$$

For such the Stokes system, there exist many articles, for example, see Cattabriga [11] *and Amrouche and Girault* [1] *and the references therein.*

As a concluding remark, we are sure of a potential application of the developed theory of this paper for solving more general Maxwell problem in L^p setting, in particualr for inclusion and evolutionary variational inequalities (obstacle problems), and mention recent results in this direction: Azevedo et al. [9], Yousept [24, 25] and Miranda et al. [17].

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