Известия НАН Армении, Математика, том 55, н. 6, 2020, стр. 10 – 22.

ON CONTINUITY OF BUFFON FUNCTIONALS IN THE SPACE OF PLANES IN $\mathbb{R}^3$

R. V. AMBARTZUMIAN

Institute of Mathematics of NAS RA
E-mail: *rouben@instmath.sci.am*

Abstract. The paper considers measures in the space $\mathbb{E}$ of planes in $\mathbb{R}^3$, and combinatorial decompositions for their values on "Buffon sets" in $\mathbb{E}$. These decompositions, written in terms of a "wedge function" depending on the measure, have been known since long in Combinatorial Integral Geometry, yet their explicit expressions have been well established only for "non-degenerate" Buffon sets. Theorem 1 removes this gap and presents a decomposition algorithm valid with no similar restriction. Theorem 2 presents a result in a direction converse to Theorem 1. Starting from the decomposition algorithm, a combinatorial valuation Ψ_F is defined that depends on "general" continuous additive wedge function $F(W)$. The question is: when Ψ_F becomes a measure in the space $\mathbb{E}$? Theorem 2 points at special "tetrahedral inequalities", the analogues of triangular inequalities of the planar theory. If Ψ_F satisfies these "tetrahedral inequalities", then Ψ_F becomes a measure and the corresponding $F(W)$ is called a "wedge metric" (to stress the connection of the paper's topic with Hilbert's Fourth Problem).

MSC2010 numbers: 53C65; 53C60; 31A10.

Keywords: integral geometry; Buffon functional; wedge metric; Hilbert's Fourth Problem.

1. Introduction

The paper considers measures in the space $\mathbb{E}$ of planes in $\mathbb{R}^3$, and combinatorial decompositions for their values on Buffon sets in $\mathbb{E}$ (i.e. members of Buffon rings in $\mathbb{E}$). The existence of similar decompositions in the spaces of Integral Geometry was first discovered in [5], they together with first applications have been discussed in the books [6] and [8] (see also [18]). Although later on further applications have been found (in convexity theory, see [7], [11]-[13], [15], Hilbert's Fourth problem, see [4], [13], [14], [17], [19], [20] , tomography, see [9]), in the basic theory the initial effort left many gaps. The present paper fills some of the gaps by presenting new results, Theorems 1, 2.

One of the basics of the combinatorial theory for $\mathbb{E}$ known already in [1], [2], [3] was the so-called "four indicator rule" valid for "non-degenerate" Buffon sets. In Theorem 1 we give its extension for quite general Buffon sets in $\mathbb{E}$. (The corresponding result for the space of lines in the plane have been recently announced in [10].)

The general combinatorial algorithm given in Theorem 1 permits construction of the combinatorial valuation Ψ_F that depends on a continuous additive "wedge function" $F(W)$. The valuation Ψ_F lives on subsets of $\mathbb{E}$ that make up a special set ring $\mathbf{U_3}$ of Buffon sets, including the degenerate cases. After brief discussion of the key properties of Ψ_F, follows demonstration of the role of special "tetrahedral inequalities" in the generation of measures by Ψ_F (Theorem 2). In the author's earlier paper [4] a similar theorem was proved about generation of measures in the space of lines on the plane by linearly additive pseudo-metrics. That theorem was called in [17] "the most elegant and natural solution" of Hilbert's Fourth Problem. By analogy, an additive $F(W)$ satisfying the tetrahedral inequalities we call a "wedge metric". The construction of Ψ_F on the basis of Theorem 1 permits to essentially simplify the proof of Theorem 2, as compared with the proof of the corresponding planar theorem given in [4] that was based on the planar version of the "four indicator rule".

2. Wedge combinatorics

2.1. **Wedges in $\mathbb{R}^3$.** The tool of wedges in $\mathbb{R}^3$ have been shown in [5] and the books [6], [8] to be a rather effective in the theory of measures in the space

$$\mathbb{E} = \text{the space of planes in Euclidean 3-space } \mathbb{R}^3, \;\; e \in \mathbb{E}.$$

We consider the spaces

$\mathbf{S}$ = the space of directions in $\mathbb{R}^3$, (elliptical plane), $\omega \in \mathbf{S}$,

$\mathbf{s}_\Omega$ = the circle of spatial directions $\omega \in \mathbf{S}$ orthogonal to some $\Omega \in \mathbf{S}$,

and use the notation

γ = a line in $\mathbb{R}^3$,

ν = a segment of a line $\gamma \subset \mathbb{R}^3$.

Given a line $\gamma \subset \mathbb{R}^3$ of direction $\Omega \in \mathbf{S}$ or a needle $\nu \subset \gamma$, instead of $\mathbf{s}_\Omega$ we may write $\mathbf{s}_\gamma$ or $\mathbf{s}_\nu$.

A flag f in $\mathbb{R}^3$ is a triad

$$f = (P, \gamma, \Phi)$$

consisting of a point $P \in \mathbb{R}^3$, a line γ containing P, and $\Phi \in \mathbf{s}_\gamma$.

Denote by $\mathcal{C}_\gamma$ the following family of flags depending on a line $\gamma \subset \mathbb{R}^3$:

$$\mathcal{C}_\gamma = \{f = (P, \gamma, \Phi) : P \in \gamma \text{ and } \Phi \in \mathbf{s}_\gamma\},$$

$\mathcal{C}_\gamma$ can be identified with unit radius circular cylinder with axis γ. A wedge is defined to be a subset of $\mathcal{C}_\gamma$ having the product form

$$W = \{P \in \nu\} \times \{\Phi \in \lambda\} = \nu \times \lambda \subset \mathcal{C}_\gamma,$$

where $\nu \subset \gamma$ is a needle (= finite open segment of a line in $\mathbb{R}^3$) and $\lambda \subset \mathbf{s}_\gamma$ is an arc (of length not exceeding π). By $e_\nu(\Phi)$ we denote the plane containing the needle ν and the direction $\Phi \in \mathbf{s}_\nu$. For every edge $W = \nu \times \lambda$ the dihedral region V is defined to be

$$V = \cup_{\Phi\in\lambda}\, e_\nu(\Phi). \tag{2.1}$$

2.2. **Wedges associated with $\{P_i\}$.** Let a finite set of points $\{P_i\}$ be given in $\mathbb{R}^3$. For a 2-subset $\{P_i, P_j\}$ from that set, by $e_{ij}(\Phi)$ we denote the plane containing the needle $\nu = \{P_i\, P_j\}$ and the direction $\Phi \in \mathbf{s}_\nu$. The values of Φ for which the plane $e_{ij}(\Phi)$ contains points from $\{P_i\}$ outside the line carrying P_i and P_j, split $\mathbf{s}_\nu$ into pairwise disjoint open arcs

$$\lambda_1, ..., \lambda_l \subset \mathbf{s}_\nu.$$

(they "belong" to $\{P_i\, P_j\}$). Each $\{P_i, P_j\} = \nu$ together with one of the belonging arcs $\lambda_r = \lambda$ determines a wedge $W_s = (\nu, \lambda)$. In this writing, the index s "codes" $(\{ij\}, r)$, i.e. there is a one-to-one correspondence

$$s \mapsto (\{ij\}, r). \tag{2.2}$$

All the wedges W_s obtained in this way form the system of wedges *associated* with our finite set $\{P_i\}$. By the definition of λ_r, for every associated wedge W_s its dihedral region

$$V_s = \cup_{\Phi\in\lambda_r} e_{ij}(\Phi)$$

does not contain points from $\{P_i\}$ in its interior, while each of the two planes bounding V_s necessarily contain points from $\{P_i\}$ other then those on their intersection line.

2.3. **Buffon rings and sets.** Let a finite set of points $\{P_i\}$ be given in $\mathbb{R}^3$. Two planes which avoid any of the points P_i we call equivalent if they induce the same partition of the set $\{P_i\}$. An equivalence class Υ (a maximal set of equivalent planes) is always a connected set in the topology of $\mathbb{E}$, but its closure will not be compact if for each plane $e \in \Upsilon$ the total $\{P_i\}$ lies in one of the two half-spaces separated by e. All other equivalence classes have compact closures: these we call *atoms*. By $Br\{P_i\}$ we denote the minimal ring of subsets of $\mathbb{E}$ which contains all atoms (they become atoms of the ring $Br\{P_i\}$ in the usual sense). We call $Br\{P_i\}$ the Buffon ring that corresponds to the set $\{P_i\}$. A set $A \subset \mathbb{E}$ we call Buffon if

$A \in Br\{P_i\}$ for some point set $\{P_i\} \subset \mathbb{R}^3$. An element $\mathbf{A} \in Br\{P_i\}$ necessarily has the form $\mathbf{A} = \bigcup a_i$, where a_i are some of the atoms of $Br\{P_i\}$.

For the time being, we assume that the number of points in the finite point set $\{P_i\}$ exceeds 2.

Let W_s be a wedge form the system of wedges *associated* with $\{P_i\}$, $s = (\{ij\}, r)$ in the sense of the map of (2.2), γ_{ij} = the line through P_i and P_j. Below always $\Phi \in \lambda_r$.

If γ_{ij} contains no points from $\{P_i\}$ except P_i and P_j, then there exist exactly four different equivalence classes Υ for which we have $e_{ij}(\Phi) \in \partial\Upsilon$. They do not depend on the choice of $\Phi \in \lambda_r$ and we denote them as $\Upsilon_s(++)$, $\Upsilon_s(--)$, $\Upsilon_s(+-)$ and $\Upsilon_s(-+)$. The sign rule is as follows (see Figure 1):

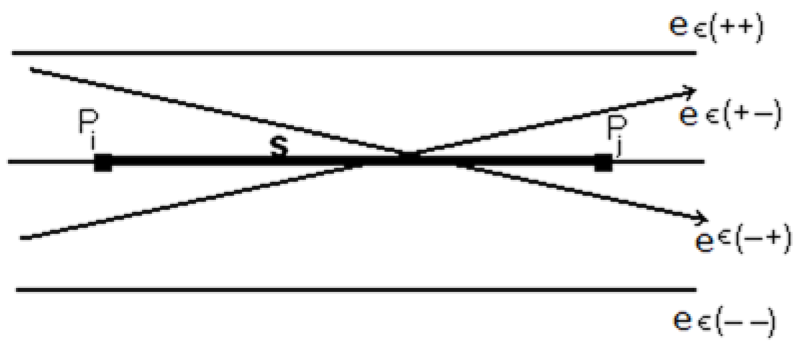

Figure 1

every plane $e \in \Upsilon_s(++)$ or $e \in \Upsilon_s(--)$ leaves P_i and P_j in one half-space,

every plane $e \in \Upsilon_s(+-)$ or $e \in \Upsilon_s(-+)$ leaves P_i and P_j in different half-spaces.

Given $\mathbf{A} \in Br\{P_i\}$, the values of the indicator function

$$I_{\mathbf{A}}(e) = \begin{cases} 1, & \text{if } e \in \mathbf{A}, \\ 0, & \text{otherwise} \end{cases}$$

on the planes from the above four sets we denote correspondingly as

$$I_{\mathbf{A}}(s,+-) \equiv I_{\mathbf{A}}(e) \text{ for } e \in \Upsilon_s(+-), \quad I_{\mathbf{A}}(s,-+) \equiv I_{\mathbf{A}}(e) \text{ for } e \in \Upsilon_s(-+),$$

$$I_{\mathbf{A}}(s,++) \equiv I_{\mathbf{A}}(e) \text{ for } e \in \Upsilon_s(++), \quad I_{\mathbf{A}}(s,--) \equiv I_{\mathbf{A}}(e) \text{ for } e \in \Upsilon_s(--).$$

2.4. **No collinear triads case.** This subsection describes the state of the art in [5], [6] and [8].

Let M be some locally finite measure in $\mathbb{E}$ that vanishes on every bundle of planes (bundle= the set of planes that contain some point $P \in \mathbb{R}^3$). Let a point set $\{P_i\}$

with no three points on a line be given. Then for every Buffon set $A \in Br\{P_i\}$ the following decomposition is valid:

$$M(A) \,=\, \frac{1}{2} \sum u_s(A)\, F(W_s), \tag{2.3}$$

where the summation is over the system of wedges W_s associated with $\{P_i\}$. The coefficients $u_s(\mathbf{A})$ are integers that do not depend on the choice of measure m and are given by "four indicator formula"

$$u_s(\mathbf{A}) \,=\, I_{\mathbf{A}}(s,+-) \,+\, I_{\mathbf{A}}(s,-+) \,-\, I_{\mathbf{A}}(s,++) \,-\, I_{\mathbf{A}}(s,--). \tag{2.4}$$

For the "wedge function" $F(W)$ the following representation was proposed in [6]:

$$F(W) \,=(2\pi)^{-1} \int_{e \text{ hits } \nu} |W \cap e|\, M(de), \tag{2.5}$$

where $W \cap e$ is the angular trace left by the wedge on the plane e, that is, see (2.1)

$$W \cap e \,=\, e \bigcap V,$$

while $|...|$ stands for the usual angular measure on $\mathbf{s}_\nu$.

2.5. **Euclidean motions invariant measure.** In the space $\mathbb{E}$ there exists [18] unique up to a constant factor Euclidean motions invariant locally finite measure; we denote it as μ. We assume that the constant factor is chosen in a way to ensure

$$\mu(\text{ planes that hit the unit ball in } \mathbb{R}^3) \,=\, 2\pi.$$

For $M \,=\, \mu$ the wedge function $F(W)$ given by (2.5) reduces to the product of length of ν and the angular measure of λ:

$$F(W) = |\nu|\, |\lambda|. \tag{2.6}$$

2.6. **More remarks.** If the number of points in $\{P_i\}$ equals 2, i.e. $\{P_i\} = \{P_1, P_2\}$ then $Br\{P_i\}$ contains only one element $\mathbf{A}$ = the planes that separate P_1 from P_2, and their is only one wedge $W_1 \,=\, \nu \times \mathbf{s}_\nu$ with ν = the needle joining P_1 and P_2. Yet (2.3) remains valid since formally $I_{\mathbf{A}}(1,++) \,=\, I_{\mathbf{A}}(1,--) = 0$ and we get $u_1(\mathbf{A}) = 2$. If the number of points in $\{P_i\}$ equals 1, then the corresponding Buffon ring is empty.

In case the point set $\{P_i\}$ is confined to some plane in $\mathbb{R}^3$, every wedge associated with $\{P_i\}$ gets the form $W \,=\, \nu \times \mathbf{s}_\nu$, hence always $F(W_s) \,=\, \pi\, |\nu_s|$. The equation (2.3) reduces to the four indicator rule for lines in the plane.

The book [6] starts with derivation of decomposition (2.3) for $M \,=\, \mu$ by direct analytical "Invariant Imbedding" method. For general M (2.3) was derived in [2] basing on the planar decomposition for projection of $\{P_i\}$ on the plane.

2.7. **Collinear triads permitted.** We are going to formulate Theorem 1, which removes the restrictions on the point set $\{P_i\}$ present in the formulation of the four indicators rule. Theorem 1 is instrumental in the construction of the functional Ψ in the Section 4 below.

For the case where the lines γ_{ij} may contain points from $\{P_i\}$ other from P_i and P_j, a decomposition similar to (2.3) survives. However the sets $\Upsilon_s(+-)$, $\Upsilon_s(-+)$, $\Upsilon_s(++)$ and $\Upsilon_s(--)$ are now no longer well defined, hence the algorithm (2.4) requires modification.
Let W_s be a wedge form the system of wedges *associated* with $\{P_i\}$, and $s = (\{ij\}, r)$ in the sense of (2.2).

The class $(+)$*:* We say that W_s belongs to the class (+) if the interior of the needle with endpoints P_i, P_j does not contain any points from $\{P_i\}$. For every $W_s \in (+)$,we define the equivalence classes $\Upsilon_s(+-)$ and $\Upsilon_s(-+)$ in the same way as above. (That definition is no longer consistent for W_s outside (+)).

The class $(-)$*:* We say that W_s belongs to the class (-) if the interior of complement of the needle with endpoints P_i, P_j contains no other points from $\{P_i\}$. For every $W_s \in (-)$ we define the equivalence classes $\Upsilon_s(++)$ and $\Upsilon_s(--)$ in the same way as above. (That definition is no longer consistent for W_s outside (-)).

Let $\mathbf{A} \in Br\{P_i\}$ be a Buffon set. For W_s from the class (+) we denote by

$$u_s^+(\mathbf{A}) \,=\, I_{\mathbf{A}}(s,+-) \,+\, I_{\mathbf{A}}(s,-+),$$

and for W_s from the class (-) we denote by

$$u_s^-(\mathbf{A}) \,=\, I_{\mathbf{A}}(s,++) \,+\, I_{\mathbf{A}}(s,--).$$

Theorem 1. *Let M be some locally finite measure in $\mathbb{E}$ that vanishes on every bundle of planes. For any point set $\{P_i\}$ and every Buffon set $A \in Br\{P_i\}$ the following decomposition is valid:*

$$M(\mathbf{A}) \,=\, \frac{1}{2} \sum_{W_s \in (+)} u_s^+(\mathbf{A})\, F(W_s) \,-\, \frac{1}{2} \sum_{W_s \in (-)} u_s^-(\mathbf{A})\, F(W_s), \tag{2.7}$$

where the wedge function $F(W)$ is given by the integral (2.5).

The proof follows by a simple check of (2.7).

2.8. **An example.** Let Π be a bounded convex polygon in some plane $e_0 \subset \mathbb{R}^3$ with vertices $v_1, ..., v_n$. Let Q be a point outside e_0. The pair (Q, Π) corresponds to a *pyramid* K with *apex* Q and *base* Π. We put $\{P_i\} = \{Q, v_1, ..., v_n\}$; then the set

$$B = \{e \in \mathbb{E} \;\; e \text{ separates } Q \text{ from } \Pi\}$$

belongs to $Br\{P_i\}$.

For any $W_s = (\nu, V)$ from the system of associated wedges, see (2.1), the needle ν is always an edge of K. An edge of K we call *lateral* if it is of Q, v_i type and *basal* if it is of $v_i v_j$ type. A wedge W_s we call a *support* wedge if $V_s \cap \operatorname{int} K = \emptyset$, and a *covering* wedge if $\operatorname{int} K \subset V_s$. We write $W_s \in I$ if W_s is a support wedge on a lateral edge and $W_s \in II$ if W_s is a covering wedge on a basal edge. We have

$u_s(B) \;=\; 1$ if $W_s \in I$,

$u_s(B) \;=\; -1$ if $W_s \in II$, and

$u_s(B) = 0$ for all other cases.

We remark, that if we assume that position of the apex Q changes so that Q tends to some limiting position $Q_0 \in$ interior of τ, then the ratio $M(B)[\mu(B)]^{-1}$ would tend to the value of the density of the measure M on the plane containing τ.

3. The valuation Ψ_F

Below, we use both (equivalent) definitions of a wedge:

$$W = \nu \times \lambda \text{ a product set on the unit cylinder } \mathcal{C}_\gamma \text{ and}$$

$$W = (\nu, V), \text{ definition of } V \text{ is given in (2.1).}$$

The wedges from the family $\{W : W \subset \mathcal{C}_\gamma\}$ can be described as $(P_1, P_2, \omega_1, \omega_2)$, where $P_1, P_2 \in \gamma$ are the endpoints of ν, while ω_1, ω_2 are the spatial directions normal to $\mathcal{C}_\gamma$ at the endpoints of the arc λ. This provides a topology in the space of wedges; so we can speak about continuous "wedge functions" $F(W)$ (an $F(W)$ maps the space of wedges onto the numerical axis). Within each class $\{W : W \subset C_\gamma\}$ the notion of additivity of an $F(W)$ in both ν and in λ is well defined as usual. In fact the functions $F(W)$ generated by means of (2.5) actually generate measures on the cylinders $\mathcal{C}_\gamma$.

Let $\{P_i\}_1$ and $\{P_i\}_2$ be two finite point sets in $\mathbb{R}^3$. Two sets $B_1 \in Br\{P_i\}_1$ and $B_2 \in Br\{P_i\}_2$ we call equivalent if their closures coincide.

We define $\mathbf{U_3}$ to be the set of equivalence classes within the $\bigcup Br\{P_i\}$, where the union is taken over all possible choices of finite sets $\{P_i\} \subset \mathbb{R}^3$. For elements $A, B \in \mathbf{U_3}$ usual set theoretic operations $\cup$ and $\cap$ can be defined. For A, a finite point set $\{P_i\} \subset \mathbb{R}^3$ can be found, such that (up to equivalence) $A \in Br\{P_i\}$.

Similarly, $B \in Br\{Q_i\}$ for some finite set of points $\{Q_i\} \subset \mathbb{R}^3$. Then up to equivalence

$$A \cup B \in Br[\{P_i\} \cup \{Q_i\}] \text{ and } A \cap B \in Br[\{P_i\} \cup \{Q_i\}].$$

Given a continuous wedge function $F(W)$, for any $\{P_i\}$ and any $A \in Br\{P_i\}$ we define a functional

$$\Psi_F(A; \{P_i\}) = \frac{1}{2} \sum u_s^+(A) F(W_s) - \frac{1}{2} \sum u_s^-(A) F(W_s), \tag{3.1}$$

where $u_s(A)$ are calculated according to the rules of (2.7), both sums are over the system of wedges associated with $\{P_i\}$.

Lemma 1. *If $F(W)$ is additive (both in ν and λ) on every cylinder $\mathcal{C}_\gamma$, then the value $\Psi_F(A; \{P_i\})$ does not depend on the choice of $\{P_i\}$, as long as $A \in Br\{P_i\}$ holds. This means that*

$$\Psi_F(A) \equiv \Psi_F(A; \{P_i\}),$$

consistently defines an additive functional Ψ_F that lives on $\mathbf{U_3}$.

The proof of Lemma 1 follows from the "stability" of the sums (see (2.7))

$$\sum_{\nu \subset \gamma} u_s^+(A) F(W_s) - \sum_{\nu \subset \gamma} u_s^-(A) F(W_s),$$

where each sum is over all wedges w_s that have ν on the same line γ (the later contains at least two points from $\{P_i\}$). Stability means no dependence on the presence of "non-essential" points in $\{P_i\}$: a point P_i is "non-essential" for A if the bundle of planes through P_i is disjoint from ∂A.

Next we formulate a continuity property of Ψ_F to be used in the measure construction below.

Let s_1, s_2, s_3 be three linear segments in $\mathbb{R}^3$, while

$$s_i^{(n)} \subset s_i, \;\; i = 1, 2, 3$$

be a sequence of needles which approximates ν_i in the sense of endpoint convergence:

$$\lim \nu_i^{(n)} = \nu_i, \;\; i = 1, 2, 3.$$

In $\mathbb{E}$ we consider the sets

$$A = \cap[s_i] \text{ and , } A_n = \cap[s_i^{(n)}].$$

Lemma 2. *If a wedge function F is continuous and additive and the functional Ψ_F is nonnegative, i.e.*

$$\Psi_F \geq 0 \quad on \quad \mathbf{U_3},$$

then the limit of $\Psi_F(A_n)$ *exists and*

$$\lim \Psi_F(A_n) = \Psi_F(A).$$

Proof. By additivity of Ψ_F

$$\Psi_F(A) - \Psi_F(A_n) = \sum_r \Psi_F(B_r),$$

where each of the sets B_r is necessarily of the form $[\tau] \cap C$, with τ = a needle component of some set differences $s_k \setminus s_k^{(n)}$, while $C \in \mathbf{U_3}$. Hence by assumed nonnegativity of Ψ_F we have

$$\Psi_F([\tau_1] \cap C) \leq \Psi_F([\tau_1] \cap C) + \Psi_F([\tau_1] \cap C^c) = \Psi_F([\tau_1]) \to 0, \quad \text{where } C = [\nu_1^{(n)}] \cap [\nu_2^{(n)}].$$

Hence for each r, $\lim \Psi_F(B_r) = 0$ as $n \to \infty$ and the lemma is proved.

3.1. **Tiling in the elliptical 3-space.** The space $\mathbb{E}$ of planes in $\mathbb{R}^3$ is homeomorphic to $\mathcal{E}_3 \setminus N$, the three dimensional elliptical space with a point N deleted. Recall that $\mathcal{E}_3$ has the interpretation of the space of diameters of the unit sphere in $\mathbb{R}^4$. We consider a standard map $\mathbb{E} \Rightarrow \mathcal{E}_3 \setminus N$ under which the images of bundles and pencils (a pencil is the set of planes through a line in R^3) are the "planes" and "lines" in the elliptical geometry of $\mathcal{E}_3$

Assume a finite set $\{P_i\}$ is given in $\mathbb{R}^3$. The corresponding "planes" produce a partition of $\mathcal{E}_3$ into convex polyhedrons. Except for the cell that contains the point N, these cells are the images of the atoms of the ring $Br\{P_i\}$.To a general $A \in Br\{P_i\}$ corresponds a union of cells.

Let in $\mathbb{R}^3$ we have a tetrahedron Θ with vertices P_1, P_2, P_3, P_4. The number of atoms in

$$Br\{P_1, P_2, P_3, P_4\} = Br(\Theta)$$

is seven: an atom of $Br(\Theta)$ can be either of 1-3 type (separation of one vertex from three others) or of 2-2 type (separation of two vertices from two others). The four bundles

$[P_i]$ = planes through the point P_i, $[P_i] \subset \mathbb{E}$,

split $\mathcal{E}_3$ in eight components; each of the eight is a tetrahedron $\theta \subset \mathcal{E}_3$ (each θ is bounded by four "planes" in $\mathcal{E}_3$). Except for the one which contains N , these θ-s are images of the atoms of $Br(\Theta)$. Those four θ-s which correspond to the atoms of 1-3 type have a two–dimensional face in common with the cell containing N, while those three θ-s which correspond to the atoms of 2-2 type, each have two one–dimensional edges in common with the letter cell.

Given a tetrahedron $\theta \subset \mathcal{E}_3 \setminus N$, we write $\theta \in \mathbf{U_3}$ if the $\mathbb{E}$-image of θ belongs to $\mathbf{U_3}$. In fact $\theta \in \mathbf{U_3}$ occurs whenever among the four planes in $\mathbb{R}^3$ that correspond to the vertices of θ there are no parallel pairs. So $\theta \in \mathbf{U_3}$ determines a tetrahedron $\Theta \subset \mathbb{R}^3$, while θ itself is identified with an atom of $Br(\Theta)$.

Assume $\sigma \subset \mathcal{E}_3 \setminus N$ corresponds to an atom of some $Br\{P_i\}$. It is always possible ("tiling") to represent σ as a union of pairwise disjoint tetrahedral cells $\theta_s \subset \mathcal{E}_3 \setminus N$ from the class $\mathbf{U_3}$:

$$\sigma = \bigcup \theta_s.$$

It follows that for every $A \in \mathbf{U_3}$ a representation

$$A = \bigcup A_s \tag{3.2}$$

is valid, where each $A_s \in \mathbf{U_3}$ is an atom of some $Br(\Theta_s)$. In terms of the functional Ψ_F this rewrites as $\Psi_F(A) = \sum \Psi_F(A_s)$. Hence the condition

$$\Psi_F(A) \geq 0 \quad \text{for any tetrahedron } \Theta \text{ and every atom } A \in Br(\Theta) \tag{3.3}$$

guarantees that $\Psi_F(A) \geq 0$ for any $A \in \mathbf{U_3}$. The actual expression of $\Psi_F(A)$ in (3.3) depends on the type (1-3 or 2-2) of the atom A.

3.2. **Tetrahedral inequalities.** Given a tetrahedron Θ with vertices $P_1, P_2, P_3, P_4 \subset \mathbb{R}^3$, we denote

$]P_1[$ = the set of planes that separate P_1 from P_2, P_3, P_4

$]P_1, P_2[$ = the set of planes that separate P_1, P_2 from P_3, P_4

In order to explicitly put down $\Psi_F(]P_1[)$ and $\Psi_F(]P_1, P_2[)$, we define the following groups of wedges associated with P_1, P_2, P_3, P_4:

$$I = \{w = (\nu, V) : \nu \text{ is "lateral"; } \Theta \cap V = \emptyset\},$$

$$II = \{w = (\nu, V) : \nu \text{ is "basal"; } \Theta \subset V\},$$

$$III = \{w = (\nu, V) : \nu \text{ is "pure"; } \Theta \cap V = \emptyset\},$$

$$IV = \{w = (\nu, V) : \nu \text{ is "mixed"; } \Theta \subset V\},$$

where

ν is lateral means that $\nu = P_1, P_2,\ P_1, P_3$ or P_1, P_4;

ν is basal means that $\nu = P_2, P_3,\ P_3, P_4$ or P_4, P_1;

ν is pure means that $\nu = P_1, P_2$ or P_3, P_4

ν is mixed means that $\nu = P_1, P_3$ or $P_1, P_4,\ P_2, P_3$ or P_2, P_4.

The first kind tetrahedral inequality writes:

$$\Psi_F(]P_1[) = \sum_I F(w_s) - \sum_{II} F(w_s) \geq 0. \tag{3.4}$$

The second kind tetrahedral inequality writes:

$$\Psi_F(]P_1, P_2[) = \sum_{III} F(w_s) - \sum_{IV} F(w_s) \geq 0. \tag{3.5}$$

We came to the following result.

Lemma 3. *If a continuous and additive wedge function F satisfies the tetrahedral inequalities* (3.4) *and* (3.5) *for any tetrahedron $P_1, P_2, P_3, P_4 \subset \mathbb{R}^3$ and any numeration of its vertices, then Ψ_F the combinatorial valuation is nonnegative on $\mathbf{U_3}$.*

3.3. **Measure generation.** We are now ready to outline the proof of a theorem, whose role in $\mathbb{E}$ compares with that of the theorem on planar pseudo-metrics proved in [4]. The continuous and additive wedge functions we consider are "general", i.e. they are not supposed to possess any special representation like (2.5).

Theorem 2. *Let F be a continuous and additive wedge function that satisfies the tetrahedral inequalities* (3.4) *and* (3.5) *. Then there exists a unique (nonnegative) measure M in $\mathbb{E}$ whose value on any set $A \in \mathbf{U_3}$ can be calculated as*

$$M(A) = \Psi_F(A).$$

Let F_1 be another wedge function possessing the same properties as F, and M_1 let be the corresponding measure in $\mathbb{E}$. If for some tetrahedron $\Theta \subset \mathbb{R}^3$ one has

$$F_1(W) = F(W) \quad \text{on wedges } W = (\nu, V) \text{ with endpoints of } \nu \text{ on the edges of } \Theta,$$

then the restrictions of M and M_1 to the set $[\Theta]$ = planes that hit Θ coincide.

Proof. Let $\Theta = \{P_1, P_2, P_3, P_4\}$ be a tetrahedron in $\mathbb{R}^3$ with (open) edges $\nu_k, k = 1, ..., 6$. Given an atom $]P_i, P_j[$ of $Br\{P_i\}$ (a 2-2 tetrahedral set), we choose from the corresponding collection of "mixed" edges a triad $\nu_k,\ \nu_m,\ \nu_r$. Also, there is a natural correspondence $]P_i[\to \nu_k, \nu_l, \nu_r$ where ν_k, ν_l, ν_r are the three edges of Θ that meet at P_i. So for atoms $A \in Br(\Theta)$ we get a map

$$A \to (\nu_k, \nu_m, \nu_r). \tag{3.6}$$

Now each plane e that hits Θ but avoids any P_i can be described by the points l_k, l_m, l_r of intersection of e with corresponding $\nu_k,\ \nu_m,\ \nu_r$. For each atom we consider the usual semi-algebra of subsets of the corresponding product $\nu_k \times \nu_l \times \nu_r$ consisting of the products

$$I_1 \times I_2 \times I_3 \quad \text{with} \quad I_1 \subseteq \nu_k,\ I_2 \subseteq \nu_m,\ I_3 \subseteq \nu_r,$$

where I_1, I_2, I_3 can be open, semi-open or closed intervals. The sets of the type

$[I_1]\cap[I_2]\cap[I_3]$ = the image of $I_1\times I_2\times I_3$ form a semi-algebra in $[\Theta]$. By (3.6) and Lemma 3, the valuation Ψ_F is nonnegative, and so is each value $\Psi_F([I_1]\cap[I_2]\cap[I_3])$. By Lemma 2, the latter value can be obtained as a limit of values of Ψ_F on the sets belonging to the "compact class"

$$\{[I_1]\cap[I_2]\cap[I_3]: \quad I_1, I_2, I_3 \text{ are closed intervals}\}.$$

By a standard criterion of measure theory this implies that Ψ_F is (can be extended to) a measure M_Θ on $[\Theta]$, after we additionally put $M_\Theta([P_i])=0, \quad i=1,2,3,4$. The next (and final) step consists in proving that the family of measures M_Θ is consistent: for any tetrahedron $\Theta_1\subset\Theta$

$$\text{(3.7)} \qquad M_{\Theta_1} \quad \text{is the restriction of } M_\Theta \text{ on the set } [\Theta_1]=\{e\in\mathbb{E}:\ e \text{ hits } \Theta_1\}.$$

To prove (3.7)) we take three intervals I_1, I_2, I_3 from ν_k,ν_l,ν_r = edges of Θ, and three intervals J_1, J_2, J_3 from a triad ν_k',ν_l',ν_r' = edges of Θ_1. It is enough to show that for the sets

$$A_1=[I_1]\cap[I_2]\cap[I_3] \quad \text{and} \quad A_2=[J_1]\cap[J_2]\cap[J_3]$$

we have

$$A_1\cap A_2=\emptyset \quad \text{implies} \quad \Psi_F(A_1\cap A_2)=0 \text{ and}$$

$$A_1\subset A_2 \quad \text{implies} \quad \Psi_F(A_1\cap A_2)=\Psi_F(A_1).$$

The last two implications can be seen directly from the algorithm (3.1) as applied to $A_1, A_2\in Br\{P_i\}$, where $\{P_i\}$ is the set of endpoints of the intervals I_1, I_2, I_3 and J_1, J_2, J_3.

We note, that this consistency proof implies, that the measure M_Θ does not depend on the map (3.6). Consistency of the measures M_Θ implies the existence of some (unique) measure μ on $\mathbb{E}$ such that M_Θ is the restriction of M on the set $[\Theta]$. The second assertion of the theorem follows from our construction of the measure M_Θ and the uniqueness of the measure extension. The proof is complete.

3.4. **A uniqueness problem.** Assume that we have some wedge function $F_0(W)$ that satisfies the conditions of Theorem 3. Let $M(de)$ be the measure in the space of planes guaranteed by the Theorem. Using that $M(de)$, we construct the function $F(W)$ as given by the integral (2.5). Is it true, that always

$$F(W) \;=\; F_0(W)\ ?$$

In other words, can a measure in the space $\mathbb{E}$ be generated, according to Theorem 2, by two different wedge functions? This seems to be the basic unsolved problem in the theory of wedge metrics.

References

[1] R. Alexander, Book review on "Combinatorial Integral Geometry", Bulletin (New Series) of the American Math. Society , **10**, No. 2 (1984).
[2] R. Alexander, "Planes for which the lines are the shortest paths between points", Illinoice J. Math. **22**, 177 – 190 (1978).
[3] R. V. Ambartzumian, "The Solution of the Buffon-Sylvester Problem in R^3", Z. Wahrscheinlichkeitstheorie verw. Geb., **27**, 53 – 74 (1973).
[4] R. V. Ambartzumian, "A note on pseudometrics on the plane", Z. Wahrscheinlichkeitstheorie verw. Geb., **37** (2), 145 – 155 (1976).
[5] R. V. Ambartzumian, "Stochastic Geometry from the standpoint of integral geometry" , Adv. Appl. Prob. **9**, 792 – 823 (1977).
[6] R. V. Ambartzumian, Combinatorial Integral Geometry with Applications to Mathematical Stereology, J.Wiley, Chichester (1982).
[7] R. V. Ambartzumian, Combinatorial Integral Geometry, Metrics and Zonoids, Acta Applicandae Mathematicae, **9**, Nos. 1 - 2 (1987).
[8] R. V. Ambartzumian, Factorization Calculus and Geomertic Probability, Cambridge University Press (1990).
[9] R. V. Ambartzumian, "Parallel X-ray tomography of convex domains as a search problem in two dimensions", Izv. AN Armenii. Matematika [English translation: Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences)], **48** (1), 23 – 35 (2013).
[10] R. V. Ambartzumian, "Sevan methodologies revisited: Random line processes", Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences)], **48** (1), 4 – 22 (2013).
[11] R. Aramyan, "Convex bodies and measures in the space of planes", Izv. AN Armenii. Matematika [English translation: Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences)], **47** (2), 19 – 30 (2012).
[12] R. Aramyan, "Reconstruction of measures in the space of planes", Lobachevskii Journal of Mathematics, **32** (4), 241 – 246 (2011).
[13] A. J. Baddeley, "Combinatorial foundations of stochastic geometry", Proc. London Math. Soc., **XLII**, 151 – 177 (1980).
[14] H. Busemann, "Herbert geometries in which the planes minimize area", Ann. Mat. Pura Appl., **55** (4), 171 – 189 (1961).
[15] G. Panina, "Many-dimensional combinatorial Ambartzumian's formulae", Math. Nachr., **159**, 271 – 277 (1992).
[16] Stochastic Geometry(1974), edited by E.F. Harding and D.G.Kendall, John Wiley and Sons Ltd.
[17] J. C. Alvarez Paiva, "Hilbert's Fourth Problem in Two Dimensions I. Mass Selecta: Teaching and Learning Advanced Undergraduate Mathematics. S. Katok, A. Sossinsky, and S. Tabachnikov (eds.), Amer. Math. Soc., Rhode Island, 165 – 183 (2003).
[18] L. A. Santalo, Integral Geometry and Geometric Probability. Addison-Wesley Publishing Company (1976).
[19] Rolf Schneider, Crofton Measures in Projective Finsler spaces. Proc. Wuhan August 4 (2005).
[20] Z. I. Szabo, "Hilbert's Fourth Problem, I", Advances in Mathematics **59**, 185 – 301 (1986).

Поступила 26 февраля 2020

После доработки 20 августа 2020

Принята к публикации 16 сентября 2020