Известия НАН Армении, Математика, том 55, н. 6, 2020, стр. 3 – 9.

DISTRIBUTION OF THE DISTANCE BETWEEN TWO RANDOM POINTS IN A BODY FROM R^n

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Abstract. In the present paper a formula for calculation of the density function $f_\rho(x)$ of the distance between two independent points randomly and uniformly chosen in a bounded convex body D is given. The formula permits to find an explicit form of density function $f_\rho(x)$ for body with known chord length distributions. In particular, we obtain an explicit expression for $f_\rho(x)$ in the case of a ball of diameter d. A simulation model is suggested to calculate empirically the cumulative distribution function of the distance between two points in a body from R^n, where explicit form of the function is hard to obtain. In particular, simulation is performed for balls and ellipsoids in R^n.

MSC2010 numbers: 60D05; 52A22; 53C65.
Keywords: chord length distribution; bounded convex body.

1. Introduction

In the present paper we continue investigations of [10] and [11]. In the previous two papers a similar problem was considered. In this paper we generalize those results for the bodies in n-dimensional Euclidean space.

Let D be a bounded, convex body in n-dimensional Euclidean space, with the volume $V(D)$ and the surface area $S(D)$. Let P_1 and P_2 be two points chosen at random, independently and with uniform distribution in D. Firstly, we are going to find the probability that the distance $\rho(P_1, P_2)$ between P_1 and P_2 is less or equal to x, that is we would like to find the distribution function $F_\rho(x)$ of $\rho(P_1, P_2)$. By definition, we have

$$F_\rho(x) = P(P_1, P_2 \in D : \rho(P_1, P_2) \le x) = \frac{\iint\limits_{\{(P_1,P_2)\,:\,\rho(P_1,P_2)\le x\}} dP_1 dP_2}{\iint\limits_{\{P_1,P_2\in D\}} dP_1 dP_2}, \tag{1.1}$$

where dP_i, $i = 1, 2$ is an element of Lebesgue measure in R^n. As

$$\iint\limits_{\{P_1,\ P_2\in D\}} dP_1 dP_2 = V^2(D)$$

[1]The research of the first author was supported by RA MES State committee of Science, Grant # 18T-1A252

(here we use that points P_1 and P_2 select independently in D) we get

$$F_\rho(x) = \frac{1}{V^2(D)} \iint\limits_{\{P_1,P_2)\,:\,\rho(P_1,P_2)\le x\}} dP_1 dP_2. \tag{1.2}$$

From the expression of the volume element in spherical coordinate where as the origin we select the point P_1, we have

$$\begin{cases} x_1 = r\cos\psi_1 \\ x_2 = r\sin\psi_1\cos\psi_2 \\ x_3 = r\sin\psi_1\sin\psi_2\cos\psi_3 \\ x_{n-1} = r\sin\psi_1 ... \sin\psi_{n-2}\cos\psi_{n-1} \\ x_n = r\sin\psi_1 ... \sin\psi_{n-2}\sin\psi_{n-1}, \end{cases}$$

where r is the distance between P_1 and P_2. Thus, using transformation from the Cartesian coordinate system to spherical coordinate system, we obtain

$$dP_2 = dx_{21}dx_{22}....dx_{2n} = r^{n-1}\sin^{n-2}\psi_1\sin^{n-3}\psi_2....\sin\psi_{n-2}drd\psi_1 d\psi_2....d\psi_{n-1}. \tag{1.3}$$

Using (1.3) expression we have

$$dP_1dP_2 = r^{n-1}\sin^{n-2}\psi_1\sin^{n-3}\psi_2....\sin\psi_{n-2}dP_1drd\psi_1 d\psi_2....d\psi_{n-1}, \tag{1.4}$$

where dK is an element of kinematic measure in R^n.

The kinematic density in Euclidean space was first introduced by Poincare. In modern terminology it is the Haar measure of the group of motions (translations and rotations) which acts on R^n. Let R^n be the Euclidean n-space, and let dK be the kinematic density. We know that

$$dK = \sin^{n-2}\psi_1\sin^{n-3}\psi_2....\sin\psi_{n-2}dP_1 d\psi_1 d\psi_2....d\psi_{n-1}. \tag{1.5}$$

Using (1.4) and (1.5) we can rewrite (1.2) in the following form:

$$F_\rho(x) = \frac{1}{V^2(D)}\int_0^x r^{n-1}K(D,r)dr,$$

where $K(D,r)$ is the kinematic measure of all oriented segments of length r that lie inside D. Therefore, we obtain a relationship between the density function $f_\rho(x)$ of $\rho(P_1,P_2)$ and the kinematic measure $K(D,x)$:

$$f_\rho(x) = \frac{x^{n-1}K(D,x)}{V^2(D)}. \tag{1.6}$$

It should be noted that we can calculate the kinematic measure of all the unoriented segments that lie inside D and then multiply the result by 2.

Let $S_1 = MS$ be the image of segment S under an Euclidean motion. M is the group of all Eculidean motions in the space R^n. For the locally compact group M, there is a locally finite Haar measure, i.e. a locally finite, non identically zero Borel measure, invariant both from the left and the right. Segment S_1 can be defined by means of the two coordinates (γ, t), where $\gamma \in J$ (J is the space of all straight lines in R^n) contains segment S_1 , and t is the one dimensional coordinate of the center of the segment S_1 on the line γ. In the space M, we define a measure by its element in the following way:

$$K(dS_1) = d\gamma\, dt,$$

where $d\gamma$ is an element in a locally finite measure in the space J, which is invariant with respect to the group M and dt is the one-dimensional Lebesgue measure on γ. The measure $K(\cdot,\cdot)$ is said to be a kinematic measure on the group M.

2. The main formula

This section gives a main formula for calculating the kinematic measure $K(D, r)$ in terms of chord length distribution function of body D. Obviously,

$$K(D, r) = 0, \quad \text{if } r \geq diam(D)$$

where $diam(D)$ is the diameter of D, i.e. $diam(D) = max\{\rho(x, y) : x, y \in D\}$, where $\rho(x, y)$ is the distance between the points x and y . Therefore, only the case $0 \leq r \leq diam(D)$ is considered in the paper. It is evident that in the mentioned case

$$K(D, r) = \int_{[D]} \int_{t \in (\chi(\gamma) - r)} d\gamma\, dt = \int_{[D]} (\chi(\gamma) - r)^+ d\gamma, \tag{2.1}$$

where $[D] = \{\gamma \in J : \gamma \cap D \neq \varnothing\}$ is the set of lines in R^n intersecting body D, $\chi(\gamma) = \gamma \cap D$ is a chord in D, while

$$x^+ = \begin{cases} 0, & \text{if } \ x \leq 0 \\ x, & \text{if } \ x \geq 0. \end{cases}$$

Let O_n be the surface area of the n-dimensional unit sphere. O_n is defined [1]

$$O_n = \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})},$$

where Γ is the gamma function which satisfies the recursion formula,

$$\Gamma(n + 1) = n\Gamma(n)$$

especially, $\Gamma(n + 1) = n!$ and $\Gamma(1/2) = \sqrt{\pi}$.

Consequently,

$$K(D, r) = \int_{\chi(\gamma) > r} \chi(\gamma) d\gamma - r \int_{\chi(\gamma) > r} d\gamma =$$

$$= \frac{O_{n-1}}{2} V(D) - G(r) - r \frac{O_{n-2}}{2(n-1)} S(D)[1 - F_D(r)], \tag{2.2}$$

where

$$G(x) = \int_{\chi(\gamma)\le x} \chi(\gamma)\, d\gamma$$

and $F_D(\cdot)$ is the chord length distribution function of body D, defined as

$$F_D(y) = \frac{2(n-1)}{O_{n-2}} \int_{\chi(\gamma)\le y} d\gamma$$

(since $\int_{[D]} d\gamma = \frac{O_{n-2}}{2(n-1)} \cdot S(D)$).

Now we will prove the following formula:

$$G(x) = \frac{O_{n-2}}{2(n-1)} S(D) \int_0^x u f_D(u) du,$$

where $f_D(x)$ is the chord length density function of body D, i.e. $f_D(x) = F'_D(x)$ is the first derivative of the distribution function. Afterwards, for calculating the derivative of the function $G(x)$ we observe that

$$\frac{G(x+\Delta x) - G(x)}{\Delta x} = \frac{1}{\Delta x} \int_{x<\chi(\gamma)\le x+\Delta x} \chi(\gamma)\gamma =$$

$$= (x+\theta\Delta x)\frac{O_{n-2}}{2(n-1)} S(D) \frac{F_D(x+\Delta x) - F_D(x)}{\Delta x}.$$

Then, assuming that the distribution function $F_D(x)$ possesses the density $f_D(x)$, when $\Delta x \to 0$, we get $G'(x) = \frac{O_{n-2}}{2(n-1)} S(D) x f_D(x)$ which implies

$$G(x) = G(0) + \frac{O_{n-2}}{2(n-1)} S(D) \int_0^x u f_D(u) du = \frac{O_{n-2}}{2(n-1)} S(D) \int_0^x u f_D(u) du, \tag{2.3}$$

since $G(0) = \int_{\chi(\gamma)\le 0} \chi(\gamma)\, d\gamma = 0$. Now, we transform formula (2.3) by means of integration by parts:

$$G(x) = \frac{O_{n-2}}{2(n-1)} S(D) \int_0^x u f_D(u) du = -\frac{O_{n-2}}{2(n-1)} S(D) \int_0^x u d[1 - F_D(u)] =$$

$$= -x \frac{O_{n-2}}{2(n-1)} S(D)[1 - F_D(x)] + \frac{O_{n-2}}{2(n-1)} S(D) \int_0^x [1 - F_D(u)] du. \tag{2.4}$$

At last, substituting (2.4) into formula (2.2) we come to the main formula of this section:

$$K(D,r) = \frac{O_{n-1}}{2} V(D) - \frac{O_{n-2}}{2(n-1)} S(D) \int_0^r [1 - F_D(u)] du.$$

Theorem 2.1. *For any body D in R^n*

$$K(D,r) = \frac{O_{n-1}}{2} V(D) - \frac{O_{n-2}}{2(n-1)} S(D) \int_0^r [1 - F_D(u)] du. \tag{2.5}$$

Thus, if the explicit form of the function $K(D,r)$ is given, then we can derive the explicit expression for the density function by means of (2.5). Formula (2.5) has been obtained for unoriented segments. For oriented segments this formula should be multiplied by 2. Substituting (2.5) into (1.6) (and multiplying by 2) we obtain the main formula of the present paper:

$$f_\rho(x) = \frac{1}{V^2(D)}\Big(x^{n-1}O_{n-1}V(D) - \frac{x^{n-1}O_{n-2}}{(n-1)}S(D)\int_0^x [1-F_D(u)]du\Big). \tag{2.6}$$

If n=2 then

$$f_\rho(x) = \frac{1}{V^2(D)}\Big(O_1 xV(D) - xO_0 S(D)\int_0^x [1-F_D(u)]du\Big),$$

where $O_0 = 2$ and $O_1 = 2\pi$. This result is proved in [9].

If n=3 then

$$f_\rho(x) = \frac{1}{V^2(D)}\Big(x^2 O_2 V(D) - \frac{x^2 O_1 S(D)}{2}\int_0^x [1-F_D(u)]du\Big),$$

where $O_1 = 2\pi$ and $O_2 = 4\pi$. This result is proved in [11].

3. The case of a ball in R^n

In case of the ball $D = B_d$ with diameter d, the chord length distribution function has the following form

$$F_{B_d}(y) = \begin{cases} 0, & \text{if} \quad y \le 0 \\ 1 - \left[1 - \left(\frac{y}{d}\right)^2\right]^{\frac{n-1}{2}}, & \text{if} \quad 0 \le y \le d \\ 1, & \text{if} \quad r \ge d. \end{cases} \tag{3.1}$$

Consequently, substituting (3.1) into (2.5) we obtain

$$K(D,r) = \frac{O_{n-1}}{2}V(D) - \frac{O_{n-2}}{2(n-1)}S(D)\int_0^r [1 - \left(\frac{u}{d}\right)^2\Big]^{\frac{n-1}{2}} du.$$

Substituting this result in (1.6) or (2.6) we obtain the density function of the distance between two points chosen independently in the ball of diameter d

$$f_\rho(x) = \frac{1}{V^2(D)}\Big(x^{n-1}O_{n-1}V(D) - \frac{x^{n-1}O_{n-2}}{(n-1)}S(D)\int_0^x [1 - \left(\frac{u}{d}\right)^2\Big]^{\frac{n-1}{2}} du\Big).$$

This formula for n=2 and n=3 was obtained in [10] and [11].

4. Moments of distance between two points in R^n

One of the simplest applications of the formulae (2.6) is the calculation of the k-th moment between two points randomly and independently distributed on the bounded convex domain. To find the k-th moment between points (we denote it by M_k^n, where n is the dimension of space) we need to calculate the following integral

$$M_k^n = \int_0^d x^k f_\rho^n(x)dx. \tag{4.1}$$

Using (2.6) we rewrite the last equation in the following form:

$$M_k^n = \int_0^d x^k f_\rho^n(x)dx = \frac{O_{n-1}}{V(D)}\int_0^d x^{n+k-1}dx -$$

$$-\frac{O_{n-2}S(D)}{(n-1)V^2(D)}\int_0^d x^{n+k-1}dx\int_0^x[1-F_D(u)]du = \tag{4.2}$$

$$=\frac{O_{n-1}d^{n+k}}{V(D)(n+k)}-\frac{O_{n-2}S(D)}{(n-1)(n+k)V^2(D)}\int_0^d dx^{n+k}\int_0^x[1-F_D(u)]du=$$

$$=\frac{O_{n-1}d^{n+k}}{V(D)(n+k)}-\frac{O_{n-2}S(D)}{(n-1)(n+k)V^2(D)}\int_0^d dx^{n+k}\int_0^x[1-F_D(u)]du=$$

$$=\frac{O_{n-1}d^{n+k}}{V(D)(n+k)}-\frac{O_{n-2}S(D)}{(n-1)(n+k)V^2(D)}\times$$

$$\Big[d^{n+k}\int_0^d[1-F_D(u)]du-\int_0^d x^{n+k}(1-F_D(u)du)\Big].$$

In (4.2) we can calculate the integral $\int_0^d[1-F_D(u)]du$. Consider the value of G(x) function at point $x=d$. Since $G(d)=\frac{O_{n-1}V(D)}{2}$, we get

$$G(d)=\frac{O_{n-2}S(D)}{2(n-1)}\int_0^d uf_D(u)du=-\frac{O_{n-2}S(D)}{2(n-1)}\int_0^d ud(1-F_D(u))=$$

$$=-\frac{O_{n-2}S(D)}{2(n-1)}(d(1-F_D(d))-\int_0^d(1-F_D(u)du)=\frac{O_{n-2}S(D)}{2(n-1)}\int_0^d(1-F_D(u)du$$

therefore

$$\int_0^d[1-F_D(u)]du=\frac{O_{n-1}V(D)(n-1)}{O_{n-2}S(D)} \tag{4.3}$$

Putting (4.3) in (4.2) we obtain

$$M_k^n=\frac{O_{n-2}S(D)}{(n-1)(n+k)V^2(D)}\int_0^d x^{n+k}(1-F_D(u)du). \tag{4.4}$$

5. Mean distance between two points in a domain and the case of ball in R^n

Using (4.4) for k=1 we obtain a formula for calculating the mean distance between two points uniformly and independently distributed in a bounded convex domain :

$$M_1^n=\frac{O_{n-2}S(D)}{(n-1)(n+1)V^2(D)}\int_0^d x^{n+1}(1-F_D(u))du. \tag{5.1}$$

In case of the ball $D=B_d$ with diameter d, putting (3.1) in (5.1) we obtain

$$M_1^n=\frac{O_{n-2}S(D)}{(n-1)(n+1)V^2(D)}\int_0^d x^{n+1}\Big[1-\Big(\frac{x}{d}\Big)^2\Big]^{\frac{n-1}{2}}dx.$$

If n=2 then

$$M_1^2=\frac{4\pi r}{3\pi^2r^4}\int_0^d x^3\Big[1-\Big(\frac{x}{d}\Big)^2\Big]^{\frac{1}{2}}dx=\frac{64}{45}d.$$

This is the result from [9].

If $n=3$ then

$$M_1^3=\frac{9}{16r^4}\int_0^d x^4\Big[1-\Big(\frac{x}{d}\Big)^2\Big]dx=\frac{18}{35}d.$$

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Поступила 30 мая 2020

После доработки 16 августа 2020

Принята к публикации 16 сентября 2020