АСТРОФИЗИКА

ТОМ 63 МАЙ, 2020 ВЫПУСК 2

NEW SOLUTIONS GENERATING TECHNIQUE TO GENERALIZED KOMPANEETS EQUATION AND THE CORRESPONDING HEUN FUNCTIONS

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Received 15 August 2019
Accepted 11 March 2020

In the present paper, we are developing an analytical method for solving the time-dependent Kompaneets equation in its generalized form. The technique is generalizing the Dubinov and Kitayev method. In the particular case of a low photon number density, for the corresponding linear equation, the solutions are expressed in terms of Heun functions.

Keywords: *Compton scatterings: Kompaneets equation: Heun functions*

1. *Introduction*. As it is known, the original Kompaneets equation [1],

$$\frac{\partial n}{\partial t}=\frac{kT}{mc^2}\frac{1}{x^2}\frac{\partial}{\partial x}\left[\sigma_T Ncx^4\left(\frac{\partial n}{\partial x}+n+n^2\right)\right], \tag{1}$$

describes the time rate of change of the photon occupation number, n, of an isotropic radiation due to Compton scatterings by a non-relativistic Maxwellian electron gas.

In the above relation, x is defined by $x=h\nu/(kT)$, with $h\nu$ representing the photon energy and T the electron temperature, N is the electron number density and σ_T is the Thomson cross section. With the dimensionless Comptonization parameter

$$y=\frac{kT}{mc^2}\tau,$$

where $\tau=\sigma_T Nct$ is the optical depth, the equation (1) gets the simpler expression

$$\frac{\partial n}{\partial y}=\frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left(\frac{\partial n}{\partial x}+n+n^2\right)\right], \tag{2}$$

where the three terms in the right hand side correspond to the diffusion of photons due to the Doppler effect and the transfer of energy from electrons to the radiation, the Compton effect and the induced Compton scatterings.

In the last sixty years, there have been a lot of attempts of finding analytical solutions to the nonlinear equation (2) and only truncated or the time stationary cases have been successfully worked out [2-4].

As an example, in the approximation $n^2 >> n$, the well-known solution obtained by Ibragimov [2] has been recently extended to wider classes of time-dependent exact solutions arising from "non-classical symmetries" [5], each of these solutions being expressed in terms of elementary functions.

In spite of the general conclusion that the nonlinear equation (2) has no time-depending analytic solution, Dubinov and Kitayev have developed a method for solving the equation (2), by separation of variables [6]. Even though the proposed procedure is elegant and original, in the present paper we discuss its applicability to more general forms of the Kompaneets equation, of interest in astrophysics and cosmology, and the additional constraints that should be imposed for extracting the actual solutions from the larger class of possible ones.

2. *The non-linear generalized Kompaneets equation and the solution-generating technique*. Since the equation (2) is valid for $h\nu << kT << mc^2$ and it fails to describe the down-Comptonization of high energy photons in hard X-ray or γ-ray astronomy, generalized forms of the original Kompaneets equation have been proposed [7,8].

In order to include a wide range of possibilities, let us start with the general differential equation

$$\frac{\partial n}{\partial y} = \frac{1}{x^2}\frac{\partial}{\partial x}\left\{x^4\left[g\frac{\partial n}{\partial x} + hn(n+1)\right]\right\}, \tag{3}$$

where g and h are functions of x alone. One may notice that the terms n and n^2 are multiplied by the same function, $h(x)$, and this can be physically motivated once one imposes the current of the general form

$$j(x) = f_1(x)\left[\frac{\partial n}{\partial x} + f_2(n, x)\right]$$

to vanish for the well-known equilibrium solution

$$n_* = \left[ke^x - 1\right]^{-1}.$$

For the extended expression of (3), i.e.

$$\begin{aligned}\frac{\partial n}{\partial y} &= x^2 g\frac{\partial^2 n}{\partial x^2} + \frac{\partial n}{\partial x}\left[x^2(g' + h) + 4xg\right] \\ &+ n\left[x^2 h' + 4xh\right] + n^2\left[x^2 h' + 4xh\right] + 2x^2 hn\frac{\partial n}{\partial x},\end{aligned} \tag{4}$$

by looking for a solution in the multiplicative form

$$n(y, x) = f(x)T(y), \tag{5}$$

one has to deal with the general expression

$$f\dot{T} = T\left\{x^2 g f'' + \left[x^2(g'+h)+4xg\right]f' + \left[x^2 h' + 4xh\right]f\right\} + T^2\left\{\left[x^2 h' + 4xh\right]f^2 + 2x^2 h f f'\right\}, \quad (6)$$

where "dot" and "prime" mean the derivatives with respect to y and x.

Considering the fully quadratic time-dependent contribution, i.e. T^2, it is clear that one can divide the above equation termwise by the last term and obtains [6]

$$\frac{\dot{T}}{T^2}\frac{f}{\left[x^2 h' + 4xh\right]f^2 + 2x^2 h f f'} = \frac{1}{T}\frac{x^2 g f'' + \left[x^2(g'+h)+4xg\right]f' + \left[x^2 h' + 4xh\right]f}{\left[x^2 h' + 4xh\right]f^2 + 2x^2 h f f'} + 1.$$

Next, by taking the y-derivative, one gets the following result:

$$\frac{\frac{d}{dy}\left(\frac{\dot{T}}{T^2}\right)}{\frac{d}{dy}\left(\frac{1}{T}\right)} = \frac{x^2 g f'' + \left[x^2(g'+h)+4xg\right]f' + \left[x^2 h' + 4xh\right]f}{f} = -k, \quad (7)$$

where, the variables being separated, one can impose each side to be equal to the same constant, k. Let us notice that if one is dividing the equation (6) by one of the other two terms, the main results would be almost the same.

For the time-depending part, the differential equation

$$\frac{d}{dy}\left(\frac{\dot{T}}{T^2}\right) = k\frac{\dot{T}}{T^2} \quad (8)$$

is satisfied by the Bose-Einstein distribution

$$T(y) = \left[e^{\eta + ky} - 1\right]^{-1}, \quad (9)$$

where η is an integration constant. If k is positive, the function (9) is decreasing from $\left[e^{\eta} - 1\right]^{-1}$ (for $x \to 0$) to zero, when x goes to infinity. In the opposite case of negative k, the function $T(y)$ is increasing, for y in the physically allowed range $0 < y < y_{max} = \eta/|k|$.

The equation for the function f, coming from (7), i.e.

$$x^2 g f'' + \left[x^2(g'+h)+4xg\right]f' + \left[x^2 h' + 4xh + k\right]f = 0, \quad (10)$$

is a little bit more involved and its explicit form depends on the choice of the functions $g(x)$ and $h(x)$.

At this stage, let us notice that the above procedure, proposed in [6] for $g = h = 1$, does not depend on the explicit form of the paranthesis multiplying the function T^2 in (6). Moreover, by replacing the function (9) in (6), a simple calculation leads to the conclusion that not only the paranthesis multiplying T should be equal to $-kf$, but also the one multiplying T^2 and this leads to the additional constraint

$$\left[x^2 h' + 4xh\right]f^2 + 2x^2 h f f' = -kf. \quad (11)$$

As an example, let us consider the case corresponding to $h(x) = 1$ in the equation (3). The relation (11) becomes

$$2x^2 f' + 4xf = -k\,, \tag{12}$$

being satisfied by

$$f(x) = -\frac{k}{2x}\,, \tag{13}$$

which, replaced in (10), leads to the following form of the function $g(x)$,

$$g(x) = \frac{C}{x^2} + x + \frac{k}{2}\,. \tag{14}$$

Thus, we have been able to construct a solution to the general Kompaneets equation (3), for $h = 1$ and the function $g(x)$ given in (14), and this has the form

$$n(y, x) = \frac{k}{2x\left(1 - e^{\eta + ky}\right)}\,. \tag{15}$$

A similar expression, i.e.

$$(n, x) = \frac{4g}{x\left(1 - e^{\eta + 2gy}\right)}\,,$$

can be obtained for $h = 1/4$ and g an arbitrary constant.

In the case analyzed in [6], corresponding to $g = h = 1$, once we impose the condition (12), one may easily check that the solution (13) does not satisfy the equation (10), with $g = h = 1$, i.e.

$$x^2 f'' + \left[x^2 + 4x\right] f' + (4x + k) f = 0\,.$$

What happens is the fact that the function $f(x)$ obtained by the method proposed in [6], which is ignoring the constraint (11), is practically the solution to the truncated Kompaneets equation, without the term n^2, i.e.

$$\frac{\partial n}{\partial y} = \frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left(g\frac{\partial n}{\partial x} + hn\right)\right]. \tag{16}$$

Indeed, its extended expression

$$\frac{\partial n}{\partial y} = x^2 g \frac{\partial^2 n}{\partial x^2} + \frac{\partial n}{\partial x}\left[x^2(g' + h) + 4xg\right] + n\left[x^2 h' + 4xh\right],$$

with the variables separation (5), has the form

$$f\dot{T} = T\left\{x^2 g f'' + \left[x^2(g' + h) + 4xg\right] f' + \left[x^2 h' + 4xh\right] f\right\}$$

leading to the relation

$$\frac{\dot{T}}{T} = \frac{x^2 g f'' + \left[x^2(g' + h) + 4xg\right] f' + \left[x^2 h' + 4xh\right] f}{f} = -k\,.$$

The solution of the time-depending part is

$$T(y) = Ce^{-ky}, \tag{17}$$

while the function $f(x)$ is satisfying precisely the equation (10). This has been solved in [6], for the case $g = h = 1$, the authors suggesting that their solution is satisfying the whole original Kompaneets equation (2).

In the other particular case corresponding to dominantly induced Compton scatterings ($n^2 >> n$), the extended equation

$$\frac{\partial n}{\partial y} = x^2 g \frac{\partial^2 n}{\partial x^2} + \frac{\partial n}{\partial x}\left[x^2 g' + 4xg\right] + n^2\left[x^2 h' + 4xh\right] + 2x^2 hn\frac{\partial n}{\partial x}$$

leads to the relation

$$f\dot{T} = T\left\{x^2 gf'' + \left[x^2 g' + 4xg\right]f'\right\} + T^2\left\{\left[x^2 h' + 4xh\right]f^2 + 2x^2 hff'\right\},$$

and one may apply the same variable separation technique described above.

For $T(y)$ given in (9), the essential relations (10) and (11) turn into the simpler forms

$$x^2 gf'' + \left[x^2 g' + 4xg\right]f' + kf = 0;$$
$$x^2 h'f + 2x^2 hf' + 4xhf + k = 0.$$

Unlike the previous case, for $k = 2$, one is able now to find a solution for the particular choice $g = h = 1$, and this is the Ibragimov solution [2]

$$n(x, y) = \frac{1}{x\left(1 - Ce^{2y}\right)}, \tag{18}$$

which is also similar to our expression (15).

For arbitrary constants $h = h_0$ and $g = g_0$, the solution

$$f(x) = -\frac{g_0}{h_0 x} + \frac{C}{x^2},$$

agrees (for $k = 2g_0$) with the form obtained in [5].

3. *Linearly generalized Kompaneets equations and their Heun solutions*. Let us focus now on the general equation (10), coming from the linear Kompaneets equation (16) with the time-depending part given in (17).

In the simplest case corresponding to $g = h = 1$, it becomes

$$x^2 f'' + \left(x^2 + 4x\right)f' + (4x + k)f = 0, \tag{19}$$

its solutions being expressed in terms of the generalized Laguerre polynomials as

$$f(x) = C_{1,2}e^{-x}x^{-\frac{3}{2}\pm\frac{\mu}{2}}L^{\pm\mu}_{\frac{3}{2}\mp\frac{\mu}{2}}(x), \tag{20}$$

where $\mu = \sqrt{9 - 4k}$ and $C_{1,2}$ are integration constants. For the essential value $k = 2$, leading to $\mu = 1$, the above functions are turning into the same expression

$$f(x)=C_1 e^{-x}\left[1-\frac{2}{x}\right],$$

and $T(y)=Ce^{-2y}$.

Following [9], let us move to the physically important case $kT << h\nu << mc^2$ by taking into account the contribution $ax^2 n'$, with

$$a=\frac{7}{10}\frac{kT}{mc^2}, \tag{21}$$

which plays a significant role for highly energetic photons.

Thus, the starting equation being now

$$\frac{\partial n}{\partial y}=\frac{1}{x^2}\frac{\partial}{\partial x}\left\{x^4\left[\left(1+ax^2\right)\frac{dn}{dx}+n\right]\right\}, \tag{22}$$

one may identify the functions *g* and *h* as being: $g=1+ax^2$ and $h=1$. The time depending part in the photon density is again (17), while the equation for the function *f*(*x*) reads:

$$x^2\left(1+ax^2\right)f''+\left(6ax^3+x^2+4x\right)f'+\left(4x+k\right)f=0. \tag{23}$$

Up to the normalization constants, the solutions are given in terms of Heun general functions [10,11] as

$$\begin{aligned} f(x)=C\left(1-|a|x^2\right)^{1/4\sqrt{|a|}}\, x^{-\frac{3}{2}\pm\frac{\mu}{2}} \exp\left[-\frac{\operatorname{arctanh}\left(\sqrt{|a|}\,x\right)}{2\sqrt{|a|}}\right] \\ \times \operatorname{Heun}G\left[a,q,\alpha,\beta,\gamma,\delta,-\sqrt{|a|}\,x\right], \end{aligned} \tag{24}$$

where the parameters are:

$$\begin{aligned} a=-1,\quad q=-\frac{2}{\sqrt{|a|}},\quad \alpha=\frac{7}{2}+\frac{1}{2\sqrt{|a|}}\pm\frac{\mu}{2},\quad \beta=-\frac{3}{2}+\frac{1}{2\sqrt{|a|}}\pm\frac{\mu}{2}, \\ \gamma=1\pm\mu,\quad \delta=1+\frac{1}{2\sqrt{|a|}}, \end{aligned} \tag{25}$$

with the same notation $\mu=\sqrt{9-4k}$. As it can be noticed, the argument of the exponential implies automatically that $0\le\sqrt{|a|}\,x\le 1$. Inserting here the explicit form of the parameter *a*, it yields that

$$0\le x=\frac{h\nu}{kT}\le\sqrt{\frac{10}{7}\frac{T_*}{T}},$$

where $T_*=mc^2/k$ stands for threshold temperature.

The Heun equation in its canonical form given in literature [10,11] has regular singularities at $z=0$, $z=1$, $z=a$ and $z=\infty$. The expansion of the Heun general functions $\operatorname{Heun}G[a,q,\alpha,\beta,\gamma,\delta,z]$ around $z=0$ is given by

$$\mathrm{Heun}\,G \approx 1 + \frac{q}{\gamma\, a} z + O\left(z^2\right),$$

i.e.

$$\mathrm{Heun}\,G \approx 1 - \frac{2x}{1 \pm \mu} + O\left(x^2\right),$$

so that, for small x-values and $k = 2$, the photon number density has the simple expression

$$n(x, y) \approx C\left(\frac{1}{x} - \frac{3}{2}\right) e^{-2y}.$$

Finally, for describing more general Compton scattering processes in the non-relativistic energy regime ($h\nu << mc^2$ and $kT << mc^2$) and with no comparison between $h\nu$ and kT, the original Kompaneets equation has been generalized by Zhang and Chen to the new form [12]

$$\frac{\partial n}{\partial y} = \frac{1}{x^2}\frac{\partial}{\partial x}\left\{x^4 (1 + bx)\left[\frac{dn}{dx} + n(n+1)\right]\right\}, \tag{26}$$

where

$$b = \frac{14}{5}\frac{kT}{mc^2}. \tag{27}$$

The general equation (26) has no analytic solution since, for $g(x) = h(x) = 1 + bx$, we could not find any function to satisfy both the equation (10) and the constraint (11). However, in the approximation $n >> n^2$, one has to deal with the equation (10) alone, which becomes

$$x^2 (1 + bx) f'' + \left(bx^3 + 5bx^2 + x^2 + 4x\right) f' + \left(5bx^2 + 4x + k\right) f = 0. \tag{28}$$

Its solutions,

$$\begin{aligned} f_1(x) &= C_1 e^{-x} x^{-\frac{3}{2}+\frac{\mu}{2}}\ \mathrm{Heun}\,C[\alpha, +\beta, \gamma, \delta, -bx], \\ f_2(x) &= C_2 e^{-x} x^{-\frac{3}{2}-\frac{\mu}{2}}\ \mathrm{Heun}\,C[\alpha, -\beta, \gamma, \delta, -bx], \end{aligned} \tag{29}$$

are expressed in terms of Heun Confluent functions of parameters

$$\alpha = \frac{1}{b},\quad \beta = \mu,\quad \gamma = 0,\quad \delta = -\frac{5}{2b},\quad \eta = \frac{2}{b} - k - \frac{3}{2}, \tag{30}$$

with $\mu = \sqrt{9 - 4k}$.

A polynomial form of the Heun Confluent functions can be achieve once we impose the condition

$$\delta = -\alpha\left[n + \frac{\beta + \gamma + 2}{2}\right],$$

which, in our case, means the same condition as the one for the Laguerre functions in (20), namely

$$\frac{3}{2} = n \pm \frac{\mu}{2}.$$

For $\mu = 1$, leading to $k = 2$, the corresponding first degree polynomial is

$$\text{Heun}\,C \approx 1 + \frac{3b-1}{2}x + O\left(x^2\right),$$

while $n > 3$ imposes a negative value of k.

Thus, for small x values, the solution to the linear Kompaneets equation coming from (26) is given by the simple function

$$n\left(y, x\right) \approx C\left[\frac{1}{x} - \frac{3}{2}\left(1-b\right) + \frac{1-3b}{2}x\right]e^{-2y}. \tag{31}$$

4. *Conclusions*. Even though intensive studies have been conducted onto the features and properties of the Kompaneets equation (2), closed-form solutions are rarely found in literature.

A method for building solutions in the multiplicative form (5), to the generalized Kompaneets equation (3), in its explicit form (6), is discussed in the present paper.

For the time-evolving part, we have found the quasi-Bose-Einstein distribution (9), while for the differential equation depending on the photon energy, whose general form is (10), one has to impose the additional constraint (11). This approach is generalizing the procedure proposed in [6].

Thus, one may conclude by saying that some solutions to the timedepending Kompaneets equation (3) can be found as the product between the function $T(y)$ given in (9) and the function $f(x)$ which, together with the functions $g(x)$ and $h(x)$ should satisfy both the equation (10) and the constraint (11).

In the case of a low photon number density, the spontaneous scattering is dominant over the induced one and, by neglecting the term n^2, we have arrived to the linear equation (16).

For highly energetic photons, it turns out that the solutions are expressed in terms of Heun functions in their general or confluent forms. In the last two decades, these have been been intensively worked out and there is a raising number of articles on the Heun functions and their applications in theoretical and applied science [13-17].

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НОВЫЙ МЕТОД ГЕНЕРАЦИИ РЕШЕНИЙ ДЛЯ ОБОБЩЕННОГО УРАВНЕНИЯ КОМПАНЕЙЦА И СООТВЕТСТВУЮЩИХ ФУНКЦИЙ ГОЙНА

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В настоящей статье мы разрабатываем аналитический метод решения нестационарного уравнения Компанейца в его обобщенном виде. Методика обобщает метод Дубинова и Китаева. В частном случае низкой плотности числа фотонов для соответствующего линейного уравнения решения выражаются через функции Гойна.

Ключевые слова: *комптоновское рассеяние: уравнение Компанейца: функции Гойна*

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