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Scalar Field Modes in de Sitter Spacetime with Negative Curvature Spatial Foliation

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Abstract. The mode functions for a massive scalar field with general curvature parameter are determined in background of $(D+1)$ -dimensional de Sitter spacetime with neagtive curvature spatial foliation. The general case of a maximally symmetric vacuum state is considered. It is shown that the adiabatic and conformal vacua coincide and the corresponding mode functions are specified. Limiting cases are discussed including the flat spacetime limit. The latter corresponds to the Milne universe. For a conformally coupled massless field the problem under consideration is conformally related to the one in static spacetime with a constant negative curvature space. It is shown that in this special case the modes are reduced to those previously discussed in the literature. The mode sum for the positive-frequency Wightman function is constructed.

Keywords: de Sitter spacetime, scalar field, negative constant curvature space

1. Introduction

De Sitter (dS) spacetime is the maximally symmetric solution of the Einstein equations with a positive cosmological constant in the absence of other sources. The interest to that geometry is motiveated by several reasons. In inflationary models of the Early Universe (for reviews see [1,2]) the cosmological expansion is described by an approximately dS spacetime. The short phase with this type of accelerated expansion provides a natural solution for a number of problems in Standard Cosmology. In particular, the latter include the horizon, flatness and topological defects problems. In addition, the quantum fluctuations of the inflaton field during the inflation generate seeds for cosmological inhomogeneities on which the large scale structures are formed in the postinflationary era. This mechanism of the large scale structure formation is in good agreement with recent observational data on temperature anisotropies of the cosmic microwave background radiation. Another important conclusion from those data, in combination with the observations of galaxy clusters and high-redshift Type Ia supernovae, is that the expansion of the Universe at recent epoch is accelerating [3-5]. The expansion is well approximated by a model where the mean energy density of the Universe is dominated by a positive cosmological constant. With this source as a dark energy, the dS spacetime appears as a future attractor for the large scale geometry of the Universe. Consequently, the investigation of physical effects in dS background geometry is of interest for understanding both the early Universe and its future. As another motivation we can mention here the holographic duality between quantum gravity on dS spacetime and a quantum field theory living on its timelike infinity [6,7].

The dS spacetime is maximally symmetric and a large number of physical problems is exactly solvable on its background. In particular, the investigation of quantum field theoretical effects has attracted a great deal of attention. Among the most important points in the canonical procedure of the quantization on curved background geometries is the determination of a complete set of the solutions to the classical field equations (mode functions for the field). On the base of those mode functions a Fock space of states is constructed [8-10]. In particular, the vacuum state is defined as the state of a quantum field annihilated by the annihilation operator. The mode functions of fields depend on the choice of the spacetime coordinate system. This is already the case in quantum field theories on flat spacetime. The different coordinate systems, in generall, lead to different sets of Fock states, in particular, to inequivalent vacuum states. Examples are the inertial and the Fulling-Rindler vacua in Minkowski spacetime. The Fulling-Rindler vacuum is the state of a quantum field annihilated by the annihilation operator constructed on the base of the mode functions corresponding to a uniformly accelerated observer in the Minkowksi bulk. In curved spacetimes the natural choice of the coordinate system in the quantization procedure depends on the problem under consideration. For example, in inflationary models of the cosmological expansion the planar coordinates are used to describe the dS background (for different coordinate systems for dS spacetime see, for example, [11,12]). In considering thermal effects and entropy in dS spacetime, the appropriate coordinates are the static ones. Both the inflationary and static coordinates cover a part of dS spacetime. The whole spacetime is covered by global coordinates. In those coordinates the spatial sections of $(D+1)$-dimensional dS spacetime are D-dimensional spheres, S^D.

In the present paper we consider the mode functions for a massive scalar field in $(D+1)$-dimensional dS spacetime described in the coordinate system with negative curvature spatial sections. The mode functions realizing the adiabatic vacuum in $D=3$ spatial dimensions have been discussed in [13]. The coordinate system with negative curvature spatial foliation is widely used recently in considerations of the entanglement entropy in dS spacetime (see [14-20] and references therein). The paper is organized as follows. In the next section we consider the normalized mode functions for the general case of the vacuum state. In section 3, the limiting case of large curvature radius is considered and it is shown that the mode functions for the Milne universe are obtained. The special case of a conformally coupled scalar field is discussed in section 4. The corresponding modes are conformally related to the modes in static spacetime with negative constant curvature space. In section 5 we consider the adiabatic vacuum and show that it coincides with the conformal vacuum. An integral representation for the Wightman function for general vacuum is provided in section 6. The main results are summarized in section 7.

2. Scalar field modes for general vacua

We consider $(D+1)$-dimensional dS spacetime. The line element will be written in the Friedmann-Robertson-Walker form with a negative spatial curvature (we use the units with $c=\hbar=1$):

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} = dt^2 - \alpha^2 \sinh^2(t/\alpha)(dr^2 + \sinh^2 r d\Omega_{D-1}^2), \tag{1}$$

where α is the curvature radius and $d\Omega_{D-1}^2$ is the line element on a sphere S^{D-1} with unit radius. The spatial part of the line element is written in terms of the hyperspherical coordinates $(r,\vartheta,\phi) \equiv (r,\theta_1,\theta_2,\ldots\theta_n,\phi)$, $n=D-2$. Note that the radial coordinate r is dimensionless. For the Ricci tensor and Ricci scalar one has

$$R_{\mu\nu} = \frac{D}{\alpha^2} g_{\mu\nu},\, R = \frac{D(D+1)}{\alpha^2}, \tag{2}$$

with the metric tensor determined from (1). The dS spacetime is maximally symmetric and the Ricci scalar is constant.

For a scalar field $\varphi(x)$ with the curvature coupling parameter ξ the field equation has the form

$$\left(g^{\mu\nu}\nabla_{\mu}\nabla_{\nu} + m^2 + \xi R\right)\varphi = 0, \tag{3}$$

where ∇_{μ} is the covariant derivative operator. For the special cases of minimally and conformally coupled fields one has $\xi=0$ and $\xi=\xi_D=(D-1)/(4D)$. We present the solution to the field equation in the factorized form

$$\varphi\left(t,r,\vartheta,\phi\right) = f(t)g(r)Y(m_p;\vartheta,\phi), \tag{4}$$

where the angular dependence is given by the spherical harmonic of degree l (for the properties of the spherical harmonics in arbitratry spatial dimensions see [21]), $Y(m_p;\vartheta,\phi)$. Here, $m_p=(m_0\equiv l, m_1,\ldots,m_n)$, $l=0,1,2,\ldots$, and $m_1,m_2,\ldots,m_n$ are integers obeying the conditions $0\le m_i \le m_{i-1}$, $i=1,2,\ldots,n-1$, $-m_{n-1}\leqslant m_n \leqslant m_{n-1}$. The spherical harmonics obey the equation

$$\Delta_{(\vartheta,\phi)}Y(m_p;\vartheta,\phi) = -l(l+n)Y(m_p;\vartheta,\phi), \tag{5}$$

where $n=D-2$ and $\Delta_{(\vartheta,\phi)}$ is the Laplace operator on the sphere S^{D-1} with unit radius.

Substituting (4) into (3) we get the following equations for the radial and temporal functions

$$\begin{gathered}\frac{\partial_r\left[\sinh^{D-1} r\partial_r g(r)\right]}{\sinh^{D-1} r}+\left[\beta_1-\frac{l(l+n)}{\sinh^2 r}\right]g(r)=0,\\ \frac{\partial_y\left[\sinh^{D} y\partial_y f(t)\right]}{\sinh^{D} y}+\left[m^2\alpha^2+\xi D(D+1)+\frac{\beta_1}{\sinh^2 y}\right]f(t)=0,\end{gathered}\tag{6}$$

where $y=t/\alpha$ and β_1 is the separation constant. As seen, both these equations have the same structure. From the second equation in (6) the following relation is obtained:

$$f(t)\partial_y f^*(t)-\left(\partial_y f(t)\right)f^*(t)=\mathrm{const}\cdot\sinh^{-D} y,\tag{7}$$

with the star corresponding to the complex conjugate. The solutions of the equations (6) are expressed in terms of the associated Legendre functions of the first and second kinds, $P_\rho^\chi(x)$ and $Q_\rho^\chi(x)$, respectively (here the associated Legendre functions are defined in accordance with [22]). Alternatively, as independent solutions we can take $P_\rho^\chi(x)$ and $P_\rho^{-\chi}(x)$. The solutions are presented in the form

$$f(t)=\frac{X_\nu^{iz}(\cosh y)}{\sinh^{(D-1)/2} y},\ g(r)=\frac{Y_{iz-1/2}^{-\mu}(\cosh r)}{\sinh^{D/2-1} r},\ y=t/\alpha,\tag{8}$$

where

$$\begin{aligned}X_\nu^{iz}(u) &= b_1P_{\nu-1/2}^{iz}(u)+b_2P_{\nu-1/2}^{-iz}(u),\\ Y_{iz-1/2}^{-\mu}(u) &= c_1P_{iz-1/2}^{-\mu}(u)+c_2Q_{iz-1/2}^{-\mu}(u),\end{aligned}\tag{9}$$

b_i and c_i, $i=1,2$, are constants. In (8), we have introduced the notations

$$\mu=l+D/2-1,\ \nu=\sqrt{D^2/4-m^2\alpha^2-\xi D(D+1)},\tag{10}$$

and z is expressed in terms of the separation constant as $z^2=\beta_1-(D-1)^2/4$. The parameter ν can be either positive or purely imaginary. Note that in both these cases

$$\left[P_{\nu-1/2}^{\pm iz}(u)\right]^*=P_{\nu-1/2}^{\mp iz}(u).\tag{11}$$

Near the origin, $r \to 0$, one has $P^{-\mu}_{iz-1/2}(\cosh r) \propto \sinh^{\mu} r$, $Q^{-\mu}_{iz-1/2}(\cosh r) \propto \sinh^{-\mu} r$, and, hence, the part of the radial function with the associated Legendre functions of the second kind is irregular: it diverges as $1/r^{l+D-2}$. In the normalization condition for the mode functions of a scalar field the radial integral appears in the form $\int_1^\infty du Y^{-\mu}_{iz-1/2}(u)\left[Y^{-\mu}_{iz'-1/2}(u)\right]^*$ with $u = \cosh r$. In the part of this integral with the product $Q^{-\mu}_{iz-1/2}(u)\left[Q^{-\mu}_{iz'-1/2}(u)\right]^*$, near the lower limit the integrand behaves as $1/(u-1)^{2l+D-2}$ and the integral diverges. This shows that the mode functions with $c_{2g} \neq 0$ are not normalizable. Hence, from the normalizability condition it follows that $c_{2g} = 0$.

From the discussion above it follows that the normalizable modes, specified by the set of quantum numbers $\sigma = (z, m_p)$, are given by the expression

$$\varphi_\sigma(x) = c_1 \frac{X^{iz}_\nu(\cosh(t/\alpha))}{\sinh^{(D-1)/2}(t/\alpha)} \frac{P^{-\mu}_{iz-1/2}(\cosh r)}{\sinh^{D/2-1} r} Y(m_p; \vartheta, \phi). \tag{12}$$

The latter contains two independent constants. The one of them is determined by the choice of the vacuum state and the second one - from the normalization condition. In quantum field theory on curved backgrounds the choice of the vacuum state, in general, is not unique. Physically motivated choice is an important point in constructing a quantum field theory in a fixed classical gravitational background. The dS spacetime is maximally symmetric and it is natural to choose a vacuum state with the same symmetry. This requirement does not fix the vacuum state uniquely: there is a one-parameter family of maximally symmetric vacua (see, for instance, [23,24]).

The orthonormalization condition for the modes (12) has the form [8]

$$\int d^D x \sqrt{|g|}\left[\varphi_\sigma(x)\partial_t \varphi^*_{\sigma'}(x) - \left(\partial_t \varphi_\sigma(x)\right)\varphi^*_{\sigma'}(x)\right] = i\delta_{m_p m'_p}\delta(z-z'), \tag{13}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$. The integration over the angular coordinates is done by using the orthonormalization relation for the spherical harmonics:

$$\int Y\left(m_p; \vartheta, \phi\right) Y^*\left(m'_p; \vartheta, \phi\right) d\Omega = N(m_p)\delta_{m_p m'_p}, \tag{14}$$

where the explicit form for $N(m_k)$ can be found in [21]. For the radial part of the integral we use the orthonormalization relation for the the associated Legendre function:

$$\int_1^\infty du P^{-\mu}_{iz-1/2}(u) P^{-\mu}_{iz'-1/2}(u) = \frac{\pi}{z\sinh(\pi z)} \frac{\delta(z-z')}{|\Gamma(\mu+1/2+iz)|^2}, \tag{15}$$

with the Euler gamma function $\Gamma(x)$. In what follows we will assume that the function $f(t) = X_\nu^{iz}(\cosh y)\sinh^{(1-D)/2} y$ is normalized by the condition (7) with $\mathrm{const} = i$. Under this condition, by using (14) and (15), from (13) we get

$$|c_1|^2 = \frac{z\sinh(\pi z)}{\pi N(m_p)} \frac{|\Gamma(\mu + 1/2 + iz)|^2}{\alpha^{D-1}}. \tag{16}$$

In terms of the function $X_\nu^{iz}(u)$, the normalization condition for $f(t)$ takes the form

$$W\{X_\nu^{iz}(u), \left[X_\nu^{iz}(u)\right]^*\} = X_\nu^{iz}(u)\partial_u \left[X_\nu^{iz}(u)\right]^* - \left[X_\nu^{iz}(u)\right]^* \partial_u X_\nu^{iz}(u) = \frac{i}{u^2 - 1}, \tag{17}$$

where $u = \cosh y$ and $W\{F_1, F_2\}$ is the Wronskian between the functions F_1 and F_2. This leads to a relation connecting two constants b_i in (9). By using the Wronskian relation

$$W\left\{P_{\nu-1/2}^{iz}(u), P_{\nu-1/2}^{-iz}(u)\right\} = \frac{2i\sinh(z\pi)}{\pi(u^2 - 1)}, \tag{18}$$

one finds

$$|b_1|^2 - |b_2|^2 = \frac{\pi}{2\sinh(z\pi)}. \tag{19}$$

As it has been mentioned before, one of the coefficients is determined by the choice of the vacuum state.

By making use of the relation

$$P_{\nu-1/2}^{-iz}(u) = \frac{\Gamma(\nu + 1/2 - iz)}{\Gamma(\nu + 1/2 + iz)}\left[P_{\nu-1/2}^{iz}(u) - \frac{2i}{\pi} e^{z\pi} \sinh(z\pi) Q_{\nu-1/2}^{iz}(u)\right], \tag{20}$$

we can write the function $X_\nu^{iz}(u)$ as

$$X_\nu^{iz}(u) = d_1 P_{\nu-1/2}^{iz}(u) + d_2 Q_{\nu-1/2}^{iz}(u), \tag{21}$$

where

$$
\begin{aligned}
d_1 &= b_1 + b_2 \frac{\Gamma\left(\nu+1/2-iz\right)}{\Gamma\left(\nu+1/2+iz\right)}, \\
d_2 &= \frac{2i}{\pi} e^{z\pi} \sinh\left(z\pi\right) \frac{\Gamma\left(\nu+1/2-iz\right)}{\Gamma\left(\nu+1/2+iz\right)} b_2.
\end{aligned} \tag{22}
$$

The new coefficients are related by

$$
\frac{2}{\pi}\sinh\left(z\pi\right)\left|d_1\right|^2 - ie^{-z\pi}\left(d_1 d_2^* - d_1^* d_2\right) = 1. \tag{23}
$$

We can also write the function $X_{\nu}^{iz}(\cosh y)$ in terms of the functions $P_{iz-1/2}^{-\nu}(\coth y)$ and $Q_{iz-1/2}^{-\nu}(\coth y)$ by using the relations

$$
\begin{aligned}
P_{\nu-1/2}^{iz}(\cosh y) &= \frac{\sqrt{2}\Gamma\left(1/2+\nu+iz\right)}{\pi^{3/2}\sinh^{1/2} y}\left\{ i\pi\sinh(\pi z) P_{iz-1/2}^{-\nu}(\coth y) \right. \\
&\quad \left. +\cos\left[\pi\left(\nu-iz\right)\right] e^{i\nu\pi} Q_{iz-1/2}^{-\nu}(\coth y)\right\}, \\
P_{\nu-1/2}^{-iz}(\cosh y) &= \frac{\sqrt{2/\pi}}{\sinh^{1/2} y}\frac{e^{i\nu\pi} Q_{iz-1/2}^{-\nu}(\coth y)}{\Gamma\left(1/2-\nu+iz\right)}.
\end{aligned} \tag{24}
$$

These relations are obtained from the formulas relating the functions $P_{\nu-1/2}^{iz}\left(x/\sqrt{x^2-1}\right)$ and $Q_{-iz-1/2}^{-\nu}(x)$ in combinations with the formulas relating the function $Q_{-iz-1/2}^{-\nu}(x)$ with $P_{iz-1/2}^{-\nu}(x)$ and $Q_{iz-1/2}^{-\nu}(x)$ (those formulas can be found in [22]).

3. Flat spacetime limit

Let us consider the mode functions for a scalar field in the limit $\alpha \to \infty$. In this limit the line element (1) takes the form

$$
ds^2 = dt^2 - t^2(dr^2 + \sinh^2 r d\Omega_{D-1}^2). \tag{25}
$$

This geometry is flat and corresponds to the Milne universe. In order to see the limiting form of the mode functions (12), we note that in the limit under consideration $\nu \approx im\alpha$ and we need the asymptotic expressions for the associated Legendre functions $P_{im\alpha-1/2}^{iz}(u)$ and $Q_{im\alpha-1/2}^{iz}(u)$ for u close to 1. One has the following relations

$$\lim_{u\to+\infty} u^{w} P_{iu-1/2}^{-w}(\cosh(\eta/u)) = J_{w}(\eta), \tag{26}$$

with $J_w(\eta)$ being the Bessel function of the first kind. Hence,

$$\lim_{\alpha\to\infty} \frac{P_{\nu-1/2}^{\pm iz}(\cosh(t/\alpha))}{\left[\alpha\sinh(t/\alpha)\right]^{(D-1)/2}} = \frac{J_{\mp iz}(mt)}{t^{(D-1)/2}} \lim_{\alpha\to\infty}(m\alpha)^{\pm iz}. \tag{27}$$

The factors $(m\alpha)^{iz}$ can be absorbed in the phase of the constants b_1 and b_2. For the mode functions we get

$$\varphi_{\sigma}^{(\mathrm{M})}(x) = \lim_{\alpha\to\infty}\varphi_{\sigma}(x) = c_1^{(\infty)} \frac{b_1 J_{-iz}(mt) + b_2 J_{iz}(mt)}{t^{(D-1)/2}\sinh^{D/2-1} r} P_{iz-1/2}^{-\mu}(\cosh r) Y(m_p;\vartheta,\phi), \tag{28}$$

where $c_1^{(\infty)} = \alpha^{(D-1)/2} c_1$ with

$$\left|c_1^{(\infty)}\right|^2 = \frac{z\sinh(\pi z)}{\pi N(m_p)} |\Gamma(\mu+1/2+iz)|^2, \tag{29}$$

and the constants $b_{1,2}$ obey the relation (19). These mode functions (28) coincide with those in the Milne universe (see [8, 25] and references therein).

The special case $b_2 = 0$ with

$$\left|b_1\right|^2 = \frac{\pi}{2\sinh(\pi z)}. \tag{30}$$

corresponds to the conformal vacuum in the Milne universe. For the adiabatic vacuum in the Milne universe one has

$$b_2 = -b_1 e^{-\pi z},\ b_1 = \frac{\sqrt{\pi}e^{\pi z/2}}{2\sinh(\pi z)}, \tag{31}$$

and

$$b_1 J_{-iz}(mt) + b_2 J_{iz}(mt) = \frac{\sqrt{\pi}}{2} e^{\pi z/2} H_{iz}^{(2)}(mt), \tag{32}$$

where $H_{iz}^{(2)}(mt)$ is the Hankel function of the second kind.

4. Conformally coupled massless field

First of all we can see that the line element (1) is conformally related to the line element of static spacetime with negative constant curvature space. In order to show that we introduce a new time coordinate η in accordance with

$$\eta = \int \frac{dt}{\sinh(t/\alpha)} = \alpha \ln\left[\tanh(t/2\alpha)\right], \tag{33}$$

or $\tanh(t/2\alpha) = e^{\eta/\alpha}$. For $0 < t < \infty$ the time coordinate η varies in the range $-\infty < \eta < 0$. By taking into account that $\sinh(t/\alpha) = -1/\sinh(\eta/\alpha)$, the line element (1) is written in a conformally static form

$$ds^2 = \frac{1}{\sinh^2(\eta/\alpha)}\left[d\eta^2 - \alpha^2(dr^2 + \sinh^2 r d\Omega_{D-1}^2)\right], \tag{34}$$

with the conformal factor

$$\Omega^2 = \frac{1}{\sinh^2(\eta/\alpha)} = \sinh^2(t/\alpha). \tag{35}$$

Note that in terms of the conformal time one has $\cosh(t/\alpha) = -\coth(\eta/\alpha)$.

For the metric tensors in the coordinate system $(\eta, r, \vartheta, \phi)$ one has the conformal relation $g_{\mu\nu} = \Omega^2 g_{\mu\nu}^{(s)}$, where the metric tensor for static spacetime is given by the expression in the square brackets of (34). The conformally coupled scalar fields in the geometries described by the metric tensors $g_{\mu\nu}$ and $g_{\mu\nu}^{(s)}$ are related by (see [8])

$$\varphi(x) = \Omega^{(1-D)/2} \varphi^{(s)}(x). \tag{36}$$

Note that for a conformally coupled field from (10) one gets $\nu = 1/2$.

The mode functions in the static spacetime have the form [28]

$$\varphi_\sigma^{(s)}(x) = c_\sigma^{(s)} \frac{P_{iz-1/2}^{-\mu}(\cosh r)}{\sinh^{D/2-1} r} Y(m_k; \vartheta, \phi) e^{-i\omega\eta}, \tag{37}$$

where ω is the corresponding energy and the normalization coefficient is given by

$$|c_{\sigma}^{(s)}|^2 = \frac{z\sinh(\pi z)}{2\pi N(m_k)\alpha^D \omega} |\Gamma(\mu + 1/2 + iz)|^2 . \tag{38}$$

Note that $|c_{\sigma}^{(s)}|^2 = |c_1|^2 / (2\alpha\omega)$. For a conformally coupled massless field the energy is expressed in terms of z by $\omega = z/\alpha$. The corresponding mode functions in dS spacetime are obtained from (12) with $\nu = 1/2$:

$$\varphi_{\sigma}(x) = c_1 \frac{X_{1/2}^{iz}(\cosh(t/\alpha))}{\sinh^{(D-1)/2}(t/\alpha)} \frac{P_{iz-1/2}^{-\mu}(\cosh r)}{\sinh^{D/2-1} r} Y(m_p; \vartheta, \phi), \tag{39}$$

where $X_{1/2}^{iz}(x) = b_1 P_0^{iz}(x) + b_2 P_0^{-iz}(x)$. For the Legendre functions in this expression one has

$$P_0^{\pm iz}(x) = \frac{1}{\Gamma(1 \mp iz)} \left(\frac{x+1}{x-1} \right)^{\pm iz/2} . \tag{40}$$

From here it follows that

$$P_0^{\pm iz}(-\coth(\eta/\alpha)) = \frac{e^{\mp iz\eta/\alpha}}{\Gamma(1 \mp iz)}, \tag{41}$$

and the mode functions are presented as

$$\varphi_{\sigma}(x) = c_1 \sinh^{(D-1)/2}(\eta/\alpha) \left[\frac{b_1 e^{-iz\eta/\alpha}}{\Gamma(1-iz)} + \frac{b_2 e^{iz\eta/\alpha}}{\Gamma(1+iz)} \right] \frac{P_{iz-1/2}^{-\mu}(\cosh r)}{\sinh^{D/2-1} r} Y(m_p; \vartheta, \phi). \tag{42}$$

Again, we see that the conformal vacuum corresponds to $b_2 = 0$ and b_1 is given by (30). By taking into account that $|b_1 / \Gamma(1-iz)|^2 = 1/(2z)$, we see that the modes (42) for the conformal vacuum are related by (36) with the modes (37) in static spacetime with a negative curvature space. The boundary-induced vacuum effects in the latter geometry have been investigated in [26-28].

5. Adiabatic vacuum

The equation for the function $X_{\nu}^{iz}(\cosh(t/\alpha))$ can be written in terms of the conformal time coordinate η as

$$\partial_{\eta}^2 h(\eta) + \omega^2(z, \eta) h(\eta) = 0, \tag{43}$$

where $h(\eta) = X_{\nu}^{iz}(-\coth(\eta/\alpha))$ and

$$\omega^2(z,\eta) = \frac{1}{\alpha^2}\left[z^2 + \frac{1/4-\nu^2}{\sinh^2(\eta/\alpha)}\right]. \tag{44}$$

From here it follows that in the region $|\eta| >> \alpha$ the time dependence of the effective frequency $\omega(z,\eta)$ is weak, $\omega(z,\eta) \approx z/\alpha$ and $|\partial_{\eta}\omega|/\omega^2 << 1$. In that region the influence of the expansion is weak and it corresponds to the adiabatic expansion regime. For a given time η the adiabatic regime is realized for large values of the quantum number z.

The solution of (43) can be presented in the WKB form (see, for example, [8])

$$h(\eta) = \frac{1}{\sqrt{2W(\eta)}} e^{-i\int^{\eta} d\eta' W(\eta')}, \tag{45}$$

where the function $W(\eta)$ obeys the equation

$$\frac{\partial_{\eta}^2 W}{W} - \frac{3}{2}\left(\frac{\partial_{\eta} W}{W}\right)^2 + 2\left[W^2 - \omega^2(z,\eta)\right] = 0. \tag{46}$$

In the adibatic regime the derivative terms in this equation are small. In the zeroth adiabatic order one has $W(\eta) = W^{(0)}(\eta) = \omega(z,\eta)$. In the same order, from (45) for the mode function we find

$$h^{(0)}(\eta) = \frac{2^{-iz/2-1/2}}{\sqrt{\omega}}\left(\frac{\alpha\omega + b\coth x}{\alpha\omega - b\coth x}\right)^{ib/2}\left(\alpha\omega\sinh x + z\cosh x\right)^{-iz}, \tag{47}$$

where $\omega = \omega(z,\eta)$, $x = \eta/\alpha$ and $b^2 = 1/4-\nu^2$. In the region $|\eta| \gg \alpha$ one finds $h^{(0)}(\eta) \propto e^{-iz\eta/\alpha}$ and the modes are reduced to the positive-energy Mnikowskian modes with the energy $\omega = z/\alpha$.

In order to specify the coefficients b_1 and b_2 in (9) for the adiabatic vacuum let us consider the mode functions in the adiabatic region $|\eta| >> \alpha$. In terms of the proper time t this corresponds to the region $t/\alpha << 1$. We need to have the asymptotic expressions for the functions $P_{\nu-1/2}^{iz}(\cosh(t/\alpha))$ when the argument is close to 1. They are directly obtained from the expressions of the associated Legendre functions in terms of the hypergeometric function [22] and are given by

$$P_{\nu-1/2}^{\pm iz}(\cosh(t/\alpha)) \approx \frac{e^{\mp iz\eta/\alpha}}{\Gamma(1 \mp iz)}, \tag{48}$$

Note that for $\nu = 1/2$ this relation is exact (see (41)). Hence, in the adiabatic region one has

$$X_\nu^{iz}(\cosh(t/\alpha)) = \frac{b_1 e^{-iz\eta/\alpha}}{\Gamma(1-iz)} + \frac{b_2 e^{iz\eta/\alpha}}{\Gamma(1+iz)}. \tag{49}$$

Now comparing with the beahvior of the zeroth adiabatic order modes $h^{(0)}(\eta) \propto e^{-iz\eta/\alpha}$, we conclude that the adiabatic vacuum corresponds to $b_2 = 0$ and b_1 from (30). Hence, the adiabatic and conformal vacua coincide and for them

$$X_\nu^{iz}(\cosh(t/\alpha)) = \sqrt{\frac{\pi}{2}} \frac{P_{\nu-1/2}^{iz}(\cosh(t/\alpha))}{\sinh^{1/2}(\pi z)}. \tag{50}$$

An alternative expression for the adiabatic vacuum is obtained by using the relation (24) for the function $P_{\nu-1/2}^{iz}(\cosh(t/\alpha))$. This fact has been already mentioned in [8] for a quantum scalar field in dS spacetime described in planar coordinates. It is of interest to note that the adibatic and conformal vacua in the Milne universe differ.

6. Wightman function

The Wightman function is defined as the vacuum expectation value $W(x,x') = \langle 0 | \varphi(x)\varphi(x') | 0 \rangle$, where $|0\rangle$ is the vacuum state. Expanding the field operator in terms of the complete set of solutions $\{\varphi_\sigma(x), \varphi_\sigma^*(x)\}$ and by using the commutation relations for annihilation and creation operators, the following mode sum formula is obtained

$$W(x,x') = \sum_\sigma \varphi_\sigma(x) \varphi_\sigma^*(x'). \tag{51}$$

Substituting the mode functions (12), with the normalization coefficient (16), for the summation over the spherical harmonics we use the addition theorem [21]

$$\sum_{m_p} \frac{Y(m_p;\vartheta,\phi)}{N(m_p)} Y^*(m_p;\vartheta',\phi') = \frac{2l+n}{nS_D} C_l^{n/2}(\cos\theta), \tag{52}$$

where the sum is taken over the integer values m_p, $p = 1, 2, \ldots, n$, in the limits defined after formula (4) and $C_l^{n/2}(\cos\theta)$ are Gegenbauer polynomials. This gives

$$W(x,x') = \frac{\alpha^{1-D}\left[\sinh(r)\sinh(r')\right]^{1-D/2}}{\pi\left[\sinh(t/\alpha)\sinh(t'/\alpha)\right]^{(D-1)/2}} \sum_{l=0}^{\infty} \frac{2l+n}{nS_D} C_l^{n/2}(\cos\theta) \times \int_0^\infty dz\, z \sinh(\pi z) P_{iz-1/2}^{-\mu}(\cosh r) P_{iz-1/2}^{-\mu}(\cosh r') \times |\Gamma(\mu+1/2+iz)|^2\, X_\nu^{iz}(\cosh(t/\alpha))[X_\nu^{iz}(\cosh(t'/\alpha))]^*, \quad (53)$$

where the function $X_\nu^{iz}(u)$ is given by (9) and the coefficients are related by (19). For the conformal and adiabatic vacua the function $X_\nu^{iz}(\cosh(t/\alpha))$ is given by (50). In a way similar to that used in [13] it can be seen that the function (53) depends on the spacetime coordinates in the form of the geodesic distance. This property is a consequence of the maximal symmetry of the dS spacetime.

7. Conclusion

In the present work we have considered a quantum scalar field in background of $(D+1)$ -dimensional dS spacetime. The geometry is described in the coordinates where the spatial sections correspond to a negative constant curvature space. The important step in the quantization is the determination of complete set of mode functions in terms of which the field operator is expanded. In the problem under consideration both the temporal and radial parts of the mode functions are expressed in terms of the associated Legendre functions and the complete set is given by (12). In addition to the normalization constant, two constants b_1 and b_2 are present, connected by the relation (19). The remaining degree of freedom is fixed by the choice of the vacuum state. In the limit of the large curvature radius we have shown that the mode functions are obtained in the Milne universe. Another special case corresponds to a conformally coupled massless scalar field. We have shown that in this case the scalar field modes are conformally related to the corresponding modes in static spacetime with constant negative curvature space. The conformal vacuum corresponds to the constant $b_2 = 0$. Next we have considered another important vacuum state corresponding to the adiabatic vacuum. It has been shown that the latter coincides with the conformal vacuum. It is of interest to note that in the Milne universe the conformal and adiabatic vacua differ. Based on the mode functions discussed we have provided an integral representation of the Wightman function for the general case of the vacuum state.

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Conflict of Interest

The authors declare no conflict of interest.

Author Contributions

The text of the paper is formulated by A.A. Saharian. The authors equally contributed to the study.

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