

New refinements of the Jensen-Mercer inequality associated to positive n -tuples

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Abstract. In this manuscript, we propose new refinements for the Jensen-Mercer as well as variant of the Jensen-Mercer inequalities associated to certain positive tuples. We give some related integral version and present applications for different means. At the end, further generalizations are given which are associated to m finite sequences.²

Key Words: Convex functions, Jensen-Mercer inequality, means

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Introduction

The celebrated Jensen inequality states that if $[a, b]$ is an interval in $\mathbb{R}$, $y_j \in [a, b]$, $\zeta_j \in \mathbb{R}^+$ ($j = 1, 2, \dots, n$), and $\psi : I \rightarrow \mathbb{R}$ is a convex function, then

$$\psi \left(\frac{\sum_{\gamma=1}^n \zeta_{\gamma} y_{\gamma}}{\sum_{\gamma=1}^n \zeta_{\gamma}} \right) \leq \frac{\sum_{\gamma=1}^n \zeta_{\gamma} \psi(y_{\gamma})}{\sum_{\gamma=1}^n \zeta_{\gamma}}. \quad (1)$$

Jensen's inequality is one of the fundamental inequalities in Mathematics, and it underlies many vital statistical concepts and proofs. Some important applications involve derivation of the AM-GM mean inequality, estimations for Shannon and Zipf-Mandelbrot entropies, the convergence property of the expectation maximization algorithm, positivity of Kullback-Leibler divergence, etc. [5, 4, 12, 6, 7]. Also this inequality has been applied to solve many problems in different fields of science and technology e.g engineering, physics, financial economics, computer science, etc.

There are several classical important inequalities which may be deduced from (1), for example Hölder, Levinson's, Ky Fan, Young's inequalities, etc.

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Due to the great importance of this inequality, several researchers have focused on this inequality and derived its many improvements, refinements and extensions. The Jensen inequality also has been given for some other generalized convex functions such as s -convex, preinvex, h -convex, η -convex functions, etc. [10, 22, 15, 17, 13, 8, 9, 21, 23].

In 2003 Mercer proved the following variant of Jensen's inequality [16].

Theorem 1 *Let $\psi : [a, b] \rightarrow \mathbb{R}$ be a convex function and let $y_\gamma \in [a, b]$, $\zeta_\gamma \in \mathbb{R}^+$, $\gamma = 1, 2, \dots, n$, $\bar{\zeta} = \sum_{\gamma=1}^n \zeta_\gamma$. Then*

$$\psi\left(a + b - \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \zeta_\gamma y_\gamma\right) \leq \psi(a) + \psi(b) - \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \zeta_\gamma \psi(y_\gamma). \quad (2)$$

The following variant of Jensen-Steffensen's inequality has been given in [1].

Theorem 2 *Let $\psi : [a, b] \rightarrow \mathbb{R}$ be a convex function and let $y_\gamma \in [a, b]$, $\zeta_\gamma \in \mathbb{R}$, $\zeta_\gamma \neq 0$, $\gamma = 1, 2, \dots, n$, $\bar{\zeta} = \sum_{\gamma=1}^n \zeta_\gamma$. If $y_1 \leq y_2 \leq \dots \leq y_n$ or $y_1 \geq y_2 \geq \dots \geq y_n$ and*

$$\sum_{\gamma=1}^n \zeta_\gamma > 0, \quad 0 \leq \sum_{\gamma=1}^k \zeta_\gamma \leq \sum_{\gamma=1}^n \zeta_\gamma, \quad k = 1, 2, \dots, n, \quad (3)$$

then

$$\psi\left(a + b - \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \zeta_\gamma y_\gamma\right) \leq \psi(a) + \psi(b) - \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \zeta_\gamma \psi(y_\gamma). \quad (4)$$

For some of the results concerning to the Jensen-Mercer inequality, we recommend [2, 18, 20, 11, 19, 3, 14].

The concept of convexity has a great impact on our everyday lives, and there are various applications of this concept in business, industry, medicine, art, etc. The applications of the convexity in equilibrium of non-cooperative games and the problems of optimum allocation of resources are significant. Jensen's inequality is one of the most important results which holds for convex (concave) functions. The classical Jensen's inequality, the complete form of Jensen's inequality, and the generalized Jensen's inequality for convex functions are important results in theoretical and applied Mathematics. In 2003 Mercer proved an interesting variant of Jensen's inequality which has been studied by many mathematicians in recent years. The purpose of this paper is to study the Jensen-Mercer inequality and to present new refinements of the Jensen-Mercer inequality in discrete as well as integral case. Also, several applications have been given for different means.

1 Refinements

Theorem 3 Let $\psi : [a, b] \rightarrow \mathbb{R}$ be a convex function and let $y_\gamma \in [a, b]$, $\zeta_\gamma, \eta_\gamma, \theta_\gamma \in \mathbb{R}^+$, $\eta_\gamma + \theta_\gamma = 1$ for each $\gamma \in \{1, 2, \dots, n\}$, and let $\bar{\zeta} = \sum_{\gamma=1}^n \zeta_\gamma$. Then

$$\begin{aligned}
& \psi\left(a + b - \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \zeta_\gamma y_\gamma\right) \\
& \leq \sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma \psi\left(\frac{\sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma (a + b - y_\gamma)}{\sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma}\right) \\
& \quad + \sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma \psi\left(\frac{\sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma (a + b - y_\gamma)}{\sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma}\right) \\
& \leq \psi(a) + \psi(b) - \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \zeta_\gamma \psi(y_\gamma).
\end{aligned} \tag{5}$$

Proof. Since $\eta_\gamma + \theta_\gamma = 1$ for $\gamma \in \{1, 2, \dots, n\}$, we can write

$$\begin{aligned}
& \psi\left(a + b - \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \zeta_\gamma y_\gamma\right) = \psi\left(\frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \zeta_\gamma (a + b - y_\gamma)\right) \\
& = \psi\left(\frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma (a + b - y_\gamma) + \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma (a + b - y_\gamma)\right) \\
& = \psi\left(\frac{\sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma}{\bar{\zeta}} \frac{\sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma (a + b - y_\gamma)}{\sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma} + \frac{\sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma}{\bar{\zeta}} \frac{\sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma (a + b - y_\gamma)}{\sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma}\right) \\
& \leq \frac{\sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma}{\bar{\zeta}} \psi\left(\frac{\sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma (a + b - y_\gamma)}{\sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma}\right) \\
& \quad + \frac{\sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma}{\bar{\zeta}} \psi\left(\frac{\sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma (a + b - y_\gamma)}{\sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma}\right) \\
& \leq \frac{\sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma}{\bar{\zeta}} \left(\psi(a) + \psi(b) - \frac{\sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma \psi(y_\gamma)}{\sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma}\right) \\
& \quad + \frac{\sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma}{\bar{\zeta}} \left(\psi(a) + \psi(b) - \frac{\sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma \psi(y_\gamma)}{\sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma}\right) \\
& = \psi(a) + \psi(b) - \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \zeta_\gamma \psi(y_\gamma).
\end{aligned}$$

The first inequality holds due to the definition of convexity, and the second inequality holds due to Theorem 1. $\square$

In the following theorem we present integral version of the above theorem.

Theorem 4 *Let $\psi : [a, b] \rightarrow \mathbb{R}$ be a convex function defined on the interval $[a, b]$. Let $p, u, v, g : [\alpha, \beta] \rightarrow \mathbb{R}$ be integrable functions such that $g(\omega) \in [a, b]$, $u(\omega), v(\omega), p(\omega) \in \mathbb{R}^+$ for all $\omega \in [\alpha, \beta]$, $v(\omega) + u(\omega) = 1$, and let $P = \int_{\alpha}^{\beta} p(\omega) d\omega$. Then*

$$\begin{aligned} & \psi \left(a + b - \frac{1}{P} \int_{\alpha}^{\beta} p(\omega) g(\omega) d\omega \right) \\ & \leq \frac{1}{P} \int_{\alpha}^{\beta} u(\omega) p(\omega) d\omega \psi \left(\frac{\int_{\alpha}^{\beta} p(\omega) u(\omega) (a + b - g(\omega)) d\omega}{\int_{\alpha}^{\beta} p(\omega) u(\omega) d\omega} \right) \\ & \quad + \frac{1}{P} \int_{\alpha}^{\beta} p(\omega) v(\omega) d\omega \psi \left(\frac{\int_{\alpha}^{\beta} p(\omega) v(\omega) (a + b - g(\omega)) d\omega}{\int_{\alpha}^{\beta} p(\omega) v(\omega) d\omega} \right) \\ & \leq \psi(a) + \psi(b) - \frac{1}{P} \int_{\alpha}^{\beta} p(\omega) \psi(g(\omega)) d\omega. \end{aligned} \quad (6)$$

If the function ψ is concave, then the reverse inequalities hold in (6).

In the following theorem we present a refinement of variant of Jensen-Steffensen's inequality:

Theorem 5 *Let $\psi : [a, b] \rightarrow \mathbb{R}$ be a convex function. Let $y_{\gamma} \in [a, b]$, $\zeta_{\gamma}, \eta_{\gamma}, \theta_{\gamma} \in \mathbb{R}$, $\zeta_{\gamma} \eta_{\gamma}, \zeta_{\gamma} \theta_{\gamma} \neq 0$ and $\eta_{\gamma} + \theta_{\gamma} = 1$ for all $\gamma \in \{1, 2, \dots, n\}$, and let $\zeta = \sum_{\gamma=1}^n \zeta_{\gamma}$. If $y_1 \leq y_2 \leq \dots \leq y_n$ or $y_1 \geq y_2 \geq \dots \geq y_n$ and*

$$\sum_{\gamma=1}^n \zeta_{\gamma} \eta_{\gamma} > 0, \quad 0 \leq \sum_{\gamma=1}^k \zeta_{\gamma} \eta_{\gamma} \leq \sum_{\gamma=1}^n \zeta_{\gamma} \eta_{\gamma}, \quad k = 1, 2, \dots, n, \quad (7)$$

$$\sum_{\gamma=1}^n \zeta_{\gamma} \theta_{\gamma} > 0, \quad 0 \leq \sum_{\gamma=1}^k \zeta_{\gamma} \theta_{\gamma} \leq \sum_{\gamma=1}^n \zeta_{\gamma} \theta_{\gamma}, \quad k = 1, 2, \dots, n, \quad (8)$$

then

$$\begin{aligned} \psi \left(a + b - \frac{1}{\zeta} \sum_{\gamma=1}^n \zeta_{\gamma} y_{\gamma} \right) & \leq \frac{1}{\zeta} \sum_{\gamma=1}^n \eta_{\gamma} \zeta_{\gamma} \psi \left(a + b - \frac{\sum_{\gamma=1}^n \zeta_{\gamma} \eta_{\gamma} (a + b - y_{\gamma})}{\sum_{\gamma=1}^n \zeta_{\gamma} \eta_{\gamma}} \right) \\ & \quad + \frac{1}{\zeta} \sum_{\gamma=1}^n \zeta_{\gamma} \theta_{\gamma} \psi \left(\frac{\sum_{\gamma=1}^n \zeta_{\gamma} \theta_{\gamma} (a + b - y_{\gamma})}{\sum_{\gamma=1}^n \zeta_{\gamma} \theta_{\gamma}} \right) \\ & \leq \psi(a) + \psi(b) - \frac{1}{\zeta} \sum_{\gamma=1}^n \zeta_{\gamma} \psi(y_{\gamma}). \end{aligned} \quad (9)$$

If the function ψ is concave, then the reverse inequalities hold in (9).

Proof. Since $\eta_\gamma + \theta_\gamma = 1$ for all $\gamma \in \{1, 2, \dots, n\}$, we have

$$\begin{aligned}
\psi \left(a + b - \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \zeta_\gamma y_\gamma \right) &= \psi \left(\frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \zeta_\gamma (a + b - y_\gamma) \right) \\
&= \psi \left(\frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma (a + b - y_\gamma) + \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma (a + b - y_\gamma) \right) \\
&= \psi \left(\frac{\sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma}{\bar{\zeta}} \cdot \frac{\sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma (a + b - y_\gamma)}{\sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma} \right. \\
&\quad \left. + \frac{\sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma}{\bar{\zeta}} \cdot \frac{\sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma (a + b - y_\gamma)}{\sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma} \right). \tag{10}
\end{aligned}$$

Note, that $\sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma > 0$ and $\sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma > 0$, and therefore $\sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma + \sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma > 0$, which gives $\bar{\zeta} > 0$.

Further, by applying convexity of ψ on the right side of (10) and Theorem 2, one can obtain

$$\begin{aligned}
\psi \left(\frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \zeta_\gamma (a + b - y_\gamma) \right) &\leq \frac{\sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma}{\bar{\zeta}} \psi \left(\frac{\sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma (a + b - y_\gamma)}{\sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma} \right) \\
&\quad + \frac{\sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma}{\bar{\zeta}} \psi \left(\frac{\sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma (a + b - y_\gamma)}{\sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma} \right) \\
&\leq \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma \left(\psi(a) + \psi(b) - \frac{\sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma \psi(y_\gamma)}{\sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma} \right) \\
&\quad + \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma \left(\psi(a) + \psi(b) - \frac{\sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma \psi(y_\gamma)}{\sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma} \right) \\
&= \psi(a) + \psi(b) - \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \zeta_\gamma \psi(y_\gamma).
\end{aligned}$$

□

Remark 1 If we add (7) and (8), then the variant of Jensen-Steffensen inequality conditions (3) will be obtained.

2 Applications to means

Let $y_\gamma \in [a, b]$, $\gamma = 1, 2, \dots, n$, and let $\zeta_1, \dots, \zeta_n > 0$ with $\bar{\zeta} = \sum_{\gamma=1}^n \zeta_\gamma$. Let $A_n(\mathbf{y}; \boldsymbol{\zeta})$, $\tilde{A}_n(\mathbf{y}; \boldsymbol{\zeta})$, $G_n(\mathbf{y}; \boldsymbol{\zeta})$, $\tilde{G}_n(\mathbf{y}; \boldsymbol{\zeta})$, $H_n(\mathbf{y}; \boldsymbol{\zeta})$, $\tilde{H}_n(\mathbf{y}; \boldsymbol{\zeta})$, and $M_n^{[r]}(\mathbf{y}; \boldsymbol{\zeta})$,

$\tilde{M}_n^{[r]}(\mathbf{y}; \zeta)$ denote the weighted arithmetic, geometric, harmonic, and power means defined as:

$$\begin{aligned} A_n(\mathbf{y}; \zeta) &:= \frac{1}{\zeta} \sum_{\gamma=1}^n \zeta_\gamma y_\gamma, & \tilde{A}_n &:= a + b - \frac{1}{\zeta} \sum_{\gamma=1}^n \zeta_\gamma y_\gamma = a + b - A_n(\mathbf{y}; \zeta), \\ G_n(\mathbf{y}; \zeta) &:= \left(\prod_{\gamma=1}^n y_\gamma^{\zeta_\gamma} \right)^{\frac{1}{\zeta}}, & \tilde{G}_n(\mathbf{y}; \zeta) &:= \frac{ab}{\left(\prod_{\gamma=1}^n y_\gamma^{\zeta_\gamma} \right)^{\frac{1}{\zeta}}} = \frac{ab}{G_n(\mathbf{y}; \zeta)}, \\ H_n(\mathbf{y}; \zeta) &:= \frac{1}{\zeta} \sum_{\gamma=1}^n \zeta_\gamma y_\gamma^{-1}, & \tilde{H}_n(\mathbf{y}; \zeta) &:= (a^{-1} + b^{-1} - H_n^{-1}(\mathbf{y}; \zeta))^{-1}, \\ M_n^{[s]}(\mathbf{y}; \zeta) &:= \begin{cases} \left(\frac{1}{\zeta} \sum_{\gamma=1}^n \zeta_\gamma y_\gamma^s \right)^{\frac{1}{s}}, & s \neq 0, \\ G_n(\mathbf{y}; \zeta), & s = 0, \end{cases} \\ \tilde{M}_n^{[s]}(\mathbf{y}; \zeta) &:= \begin{cases} \left(a^s + b^s - \left(M_n^{[s]}(\mathbf{y}; \zeta) \right)^s \right)^{\frac{1}{s}}, & s \neq 0, \\ \tilde{G}_n(\mathbf{y}; \zeta), & s = 0. \end{cases} \end{aligned}$$

Also, assume that $\eta_\gamma, \theta_\gamma \in \mathbb{R}^+$ are such that $\eta_\gamma + \theta_\gamma = 1$ for each $\gamma \in \{1, 2, \dots, n\}$.

Under the above assumptions we give the following corollaries.

Corollary 1 *The following inequalities are valid*

$$\tilde{G}_n(\mathbf{y}; \zeta) \leq \sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma \tilde{G}_n(\mathbf{y}; \zeta \cdot \boldsymbol{\eta}) + \sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma \tilde{G}_n(\mathbf{y}; \zeta \cdot \boldsymbol{\theta}) \leq \tilde{A}_n(\mathbf{y}; \zeta). \quad (11)$$

Proof. Using the function $\psi(x) = \exp(x)$ and replacing a, b , and y_γ by $\ln a, \ln b$, and $\ln y_\gamma$ in (5), we get (11). $\square$

Corollary 2 *By taking $a \rightarrow \frac{1}{a}, b \rightarrow \frac{1}{b}, y_\gamma \rightarrow \frac{1}{y_\gamma}$, in (11) we have*

$$\frac{1}{\tilde{G}_n(\mathbf{y}; \zeta)} \leq \frac{\sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma}{\tilde{G}_n(\mathbf{y}; \zeta \cdot \boldsymbol{\eta})} + \frac{\sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma}{\tilde{G}_n(\mathbf{y}; \zeta \cdot \boldsymbol{\theta})} \leq \frac{1}{\tilde{H}_n(\mathbf{y}; \zeta)}. \quad (12)$$

Corollary 3 *The following inequalities hold:*

$$\tilde{G}_n(\mathbf{y}; \zeta) \leq \left(\tilde{A}_n(\mathbf{y}; \zeta \cdot \boldsymbol{\eta}) \right)^{\sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma} \left(\tilde{A}_n(\mathbf{y}; \zeta \cdot \boldsymbol{\theta}) \right)^{\sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma} \leq \tilde{A}_n(\mathbf{y}; \zeta). \quad (13)$$

Proof. Using the function $f(x) = -\ln x$ in (5) and then applying the exponential function, we get (13). $\square$

Corollary 4 For $s \neq 0$ and $s \leq 1$, we have

$$\tilde{M}_n^{[s]}(\mathbf{y}; \boldsymbol{\zeta}) \leq \sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma \tilde{M}_n^{[s]}(\mathbf{y}; \boldsymbol{\zeta} \cdot \boldsymbol{\eta}) + \sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma \tilde{M}_n^{[s]}(\mathbf{y}; \boldsymbol{\zeta} \cdot \boldsymbol{\theta}) \leq \tilde{A}_n(\mathbf{y}; \boldsymbol{\zeta}). \quad (14)$$

Proof. Use the function $\psi(x) = x^{\frac{1}{s}}$ and replace a, b , and x_i by a^s, b^s , and x_γ^s respectively in (5) to get (14). $\square$

Corollary 5 For $t, s \in \mathbb{R}$ with $0 < t \leq s$, we have

$$\begin{aligned} \left(\tilde{M}_n^{[t]}(\mathbf{y}; \boldsymbol{\zeta}) \right)^s &\leq \sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma \left(\tilde{M}_n^{[t]}(\mathbf{y}; \boldsymbol{\zeta} \cdot \boldsymbol{\eta}) \right)^s + \sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma \left(\tilde{M}_n^{[t]}(\mathbf{y}; \boldsymbol{\zeta} \cdot \boldsymbol{\theta}) \right)^s \\ &\leq \tilde{M}_n^{[s]}(\mathbf{y}; \boldsymbol{\zeta}). \end{aligned} \quad (15)$$

Proof. Use the function $\psi(x) = x^{\frac{s}{t}}$ and replace a, b , and x_γ by a^t, b^t , and x_γ^t respectively in (5) to get (15). $\square$

Now, we generalize the given applications for the *quasi-arithmetic mean*.

Let $\phi : [a, b] \rightarrow \mathbb{R}$ be a strictly monotonic and continuous function. Then for a given n -tuple $\mathbf{y} = (y_1, \dots, y_n) \in [a, b]^n$ and positive n -tuple $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_n)$ with $\bar{\zeta} = \sum_{\gamma=1}^n \zeta_\gamma$, the value

$$M_\phi^{[n]}(\mathbf{y}; \boldsymbol{\zeta}) = \phi^{-1} \left(\sum_{\gamma=1}^n \zeta_\gamma \phi(y_\gamma) \right)$$

is well defined and is called *quasi-arithmetic mean* of $\mathbf{y}$ with weight $\boldsymbol{\zeta}$. If we define

$$\tilde{M}_\phi^{[n]}(\mathbf{y}; \boldsymbol{\zeta}) = \phi^{-1} \left(\phi(a) + \phi(b) - \sum_{\gamma=1}^n \zeta_\gamma \phi(y_\gamma) \right),$$

then we have the following result.

Corollary 6 The following inequalities hold:

$$\begin{aligned} \psi \left(\tilde{M}_\phi^{[n]}(\mathbf{y}; \boldsymbol{\zeta}) \right) &\leq \sum_{\gamma=1}^n \zeta_\gamma \eta_\gamma \psi \left(\tilde{M}_\phi^{[n]}(\mathbf{y}; \boldsymbol{\zeta} \cdot \boldsymbol{\eta}) \right) + \sum_{\gamma=1}^n \zeta_\gamma \theta_\gamma \psi \left(\tilde{M}_\phi^{[n]}(\mathbf{y}; \boldsymbol{\zeta} \cdot \boldsymbol{\theta}) \right) \\ &\leq \psi \left(\tilde{M}_\psi^{[n]}(\mathbf{y}; \boldsymbol{\zeta}) \right) \end{aligned} \quad (16)$$

provided that $\psi \circ \phi^{-1}$ is convex and ψ is strictly increasing.

Proof. Replace $\psi(x)$ by $\psi \circ \phi^{-1}(x)$ and a, b, y_γ by $\phi(a), \phi(b), \phi(y_\gamma)$ respectively in (5), then apply ψ^{-1} to get (16). $\square$

3 Further generalizations

In this section, we present a further refinement of the Jensen-Mercer as well as variant of the Jensen-Mercer inequalities concerning to m sequences whose sum is equal to unity.

Theorem 6 *Let $\psi : [a, b] \rightarrow \mathbb{R}$ be a convex function defined on the interval $[a, b]$. Let $y_\gamma \in [a, b]$, $\zeta_\gamma, \theta_\gamma^l \in \mathbb{R}^+$, $\gamma = 1, 2, \dots, n$, $l = 1, 2, \dots, m$, be such that $\sum_{l=1}^m \theta_\gamma^l = 1$ for each $\gamma \in \{1, 2, \dots, n\}$, and let $\bar{\zeta} = \sum_{\gamma=1}^n \zeta_\gamma$. Assume that L_1 and L_2 are non-empty disjoint subsets of $\{1, 2, \dots, m\}$ such that $L_1 \cup L_2 = \{1, 2, \dots, m\}$. Then*

$$\begin{aligned}
 & \psi\left(a + b - \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \zeta_\gamma y_\gamma\right) \\
 & \leq \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \sum_{l \in L_1} \theta_\gamma^l \zeta_\gamma \psi\left(\frac{\sum_{\gamma=1}^n \sum_{l \in L_1} \theta_\gamma^l \zeta_\gamma (a + b - y_\gamma)}{\sum_{\gamma=1}^n \sum_{l \in L_1} \theta_\gamma^l \zeta_\gamma}\right) \\
 & \quad + \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \sum_{l \in L_2} \theta_\gamma^l \zeta_\gamma \psi\left(\frac{\sum_{\gamma=1}^n \sum_{l \in L_2} \theta_\gamma^l \zeta_\gamma (a + b - y_\gamma)}{\sum_{\gamma=1}^n \sum_{l \in L_2} \theta_\gamma^l \zeta_\gamma}\right) \\
 & \leq \psi(a) + \psi(b) - \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \zeta_\gamma \psi(y_\gamma). \tag{17}
 \end{aligned}$$

If the function ψ is concave, then the reverse inequalities hold in (17).

Proof. Since $\sum_{l=1}^m \theta_\gamma^l = 1$ for each $\gamma \in \{1, 2, \dots, n\}$, we can write

$$a + b - \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \zeta_\gamma y_\gamma = \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \sum_{l \in L_1} \theta_\gamma^l \zeta_\gamma (a + b - y_\gamma) + \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \sum_{l \in L_2} \theta_\gamma^l \zeta_\gamma (a + b - y_\gamma).$$

Therefore, we have

$$\begin{aligned}
 & \psi\left(a + b - \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \zeta_\gamma y_\gamma\right) \\
 & = \psi\left(\frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \sum_{l \in L_1} \theta_\gamma^l \zeta_\gamma (a + b - y_\gamma) + \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \sum_{l \in L_2} \theta_\gamma^l \zeta_\gamma (a + b - y_\gamma)\right) \\
 & = \psi\left(\frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \sum_{l \in L_1} \theta_\gamma^l \zeta_\gamma \frac{\sum_{\gamma=1}^n \sum_{l \in L_1} \theta_\gamma^l \zeta_\gamma (a + b - y_\gamma)}{\sum_{\gamma=1}^n \sum_{l \in L_1} \theta_\gamma^l \zeta_\gamma} \right. \\
 & \quad \left. + \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \sum_{l \in L_2} \theta_\gamma^l \zeta_\gamma \frac{\sum_{\gamma=1}^n \sum_{l \in L_2} \theta_\gamma^l \zeta_\gamma (a + b - y_\gamma)}{\sum_{\gamma=1}^n \sum_{l \in L_2} \theta_\gamma^l \zeta_\gamma} \right)
 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \sum_{l \in L_1} \theta_{\gamma}^l \zeta_{\gamma} \psi \left(\frac{\sum_{\gamma=1}^n \sum_{l \in L_1} \theta_{\gamma}^l \zeta_{\gamma} (a + b - y_{\gamma})}{\sum_{\gamma=1}^n \sum_{l \in L_1} \theta_{\gamma}^l \zeta_{\gamma}} \right) \\
&\quad + \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \sum_{l \in L_2} \theta_{\gamma}^l \zeta_{\gamma} \psi \left(\frac{\sum_{\gamma=1}^n \sum_{l \in L_2} \theta_{\gamma}^l \zeta_{\gamma} (a + b - y_{\gamma})}{\sum_{\gamma=1}^n \sum_{l \in L_2} \theta_{\gamma}^l \zeta_{\gamma}} \right) \\
&\leq \psi(a) + \psi(b) - \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \zeta_{\gamma} \psi(y_{\gamma}). \tag{18}
\end{aligned}$$

The first inequality takes place due to the definition of convexity, and the second one holds by virtue of Jensen-Mercer's inequality. $\square$

Remark 2 We can give applications of Theorem 6 for means as given in Section 2.

The following theorem is the integral analogue of Theorem 6.

Theorem 7 Let $\psi : \mathcal{G} \rightarrow \mathbb{R}$ be a convex function defined on the interval $\mathcal{G}$. Let $p, g, u_l \in L[a, b]$ be such that $g(\omega) \in \mathcal{G}$, $p(\omega), u_l(\omega) \in \mathbb{R}^+$ for all $\omega \in [a, b]$ ($l = 1, 2, \dots, n$), and let $\sum_{l=1}^n u_l(\omega) = 1$, $P = \int_a^b p(\omega) d\omega$. Assume that L_1 and L_2 are non-empty disjoint subsets of $\{1, 2, \dots, n\}$ such that $L_1 \cup L_2 = \{1, 2, \dots, n\}$. Then

$$\begin{aligned}
&\frac{1}{P} \int_a^b p(\omega) \psi(g(\omega)) d\omega \\
&\geq \frac{1}{P} \int_a^b \sum_{l \in L_1} u_l(\omega) p(\omega) d\omega \psi \left(\frac{\int_a^b \sum_{l \in L_1} u_l(\omega) p(\omega) g(\omega) d\omega}{\int_a^b \sum_{l \in L_1} u_l(\omega) p(\omega) d\omega} \right) \\
&\quad + \frac{1}{P} \int_a^b \sum_{l \in L_2} u_l(\omega) p(\omega) d\omega \psi \left(\frac{\int_a^b \sum_{l \in L_2} u_l(\omega) p(\omega) g(\omega) d\omega}{\int_a^b \sum_{l \in L_2} u_l(\omega) p(\omega) d\omega} \right) \\
&\geq \psi \left(\frac{1}{P} \int_a^b p(\omega) g(\omega) d\omega \right). \tag{19}
\end{aligned}$$

If the function ψ is concave, then the reverse inequalities hold in (19).

In the following theorem, we present a further refinement of the variant of Jensen-Steffensen's inequality associated to m certain sequences.

Theorem 8 Let $\psi : [a, b] \rightarrow \mathbb{R}$ be a convex function defined on the interval $[a, b]$. Let $y_{\gamma} \in I$, let $\zeta_{\gamma}, \theta_{\gamma}^l \in \mathbb{R}$, $\gamma = 1, 2, \dots, n$, $l = 1, 2, \dots, m$, be such that $\zeta_{\gamma} \eta_{\gamma}, \zeta_{\gamma} \theta_{\gamma}^l \neq 0$ and $\sum_{l=1}^m \theta_{\gamma}^l = 1$ for each $\gamma \in \{1, 2, \dots, n\}$, and let $\bar{\zeta} = \sum_{\gamma=1}^n \zeta_{\gamma}$. Assume that L_1 and L_2 are non-empty disjoint subsets of

$\{1, 2, \dots, m\}$ such that $L_1 \cup L_2 = \{1, 2, \dots, m\}$. If $y_1 \leq y_2 \leq \dots \leq y_n$ or $y_1 \geq y_2 \geq \dots \geq y_n$ and for each $l \in \{1, 2, \dots, m\}$

$$\sum_{\gamma=1}^n \zeta_{\gamma} \theta_{\gamma}^l > 0, \quad 0 \leq \sum_{\gamma=1}^k \zeta_{\gamma} \theta_{\gamma}^l \leq \sum_{\gamma=1}^n \zeta_{\gamma} \theta_{\gamma}^l, \quad k = 1, 2, \dots, n, \quad (20)$$

then

$$\begin{aligned} \psi \left(a + b - \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \zeta_{\gamma} y_{\gamma} \right) & \leq \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \sum_{l \in L_1} \theta_{\gamma}^l \zeta_{\gamma} \psi \left(\frac{\sum_{\gamma=1}^n \sum_{l \in L_1} \theta_{\gamma}^l \zeta_{\gamma} (a + b - y_{\gamma})}{\sum_{\gamma=1}^n \sum_{l \in L_1} \theta_{\gamma}^l \zeta_{\gamma}} \right) \\ & \quad + \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \sum_{l \in L_2} \theta_{\gamma}^l \zeta_{\gamma} \psi \left(\frac{\sum_{\gamma=1}^n \sum_{l \in L_2} \theta_{\gamma}^l \zeta_{\gamma} (a + b - y_{\gamma})}{\sum_{\gamma=1}^n \sum_{l \in L_2} \theta_{\gamma}^l \zeta_{\gamma}} \right) \\ & \leq \psi(a) + \psi(b) - \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^n \zeta_{\gamma} \psi(y_{\gamma}). \end{aligned} \quad (21)$$

If the function ψ is concave, then the reverse inequalities hold in (21).

Proof. Since $\sum_{\gamma=1}^n \zeta_{\gamma} \theta_{\gamma}^l > 0$ for each $l \in \{1, 2, \dots, m\}$, one has $\sum_{l=1}^m \sum_{\gamma=1}^n \zeta_{\gamma} \theta_{\gamma}^l > 0$.

Also, $\sum_{l=1}^m \theta_{\gamma}^l = 1$. Hence, we can conclude that $\bar{\zeta} > 0$.

Further, proceeding in the same way as in the proof of Theorem 6 but using variant of Jensen-Steffensen's inequality instead of Jensen-Steffensen's inequality, we obtain (21). $\square$

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