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Blaschke products of given quantity index for a half-plane

G. V. Mikayelyan and F. V. Hayrapetyan

Abstract. We investigate the growth of the integral logarithmic means of Blaschke products for the half-plane. We prove the existence of Blaschke products of given quantity indices.

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Introduction

Let the sequence of complex numbers $\{w_k\}_1^\infty = \{u_k + iv_k\}_1^\infty$ lying in the lower half-plane $G = \{w : Im\,(w) < 0\}$ satisfy the condition

$$\sum_{k=1}^{\infty} |v_k| < +\infty. \tag{1}$$

Then the infinite product of Blaschke

$$B\,(w) = \prod_{k=1}^{\infty} \frac{w - w_k}{w - \overline{w}_k}$$

converges in the half-plane G, defining there an analytic function with zeros $\{w_k\}_1^\infty$.

In order to investigate the asymptotic properties of the function B, we define an integral logarithmic mean of order q, $1 \le q < +\infty$ of Blaschke products for the half-plane by the formula

$$m_q\,(v, B) = \left(\int_{-\infty}^{+\infty} |\log|B\,(u + iv)||^q \, du \right)^{1/q}, \qquad -\infty < v < 0.$$

Let us denote by $n(v)$ the number of zeros of the function B in the half-plane $\{w : Im\ w \le v\}$.

In the case of a disc the integral logarithmic mean of order $q,\ 1 \leq q < +\infty$ is defined by the formula

$$m_q(r, B) = \left(\int_{-\pi}^{+\pi} \left| \log \left| B\left(re^{i\varphi}\right) \right| \right|^q d\varphi \right)^{1/q}, \qquad 0 \leq r < 1,$$

where B is the Blaschke product for the unit disc. In this case, for $q = 2$ the problem about the boundedness of the function $m_2(r, B)$ was posed by A. Zygmund. In 1969 this problem was solved by the method of Fourier series for meromorphic functions by G.R. MacLane and L.A. Rubel [1]. In [2] V.V. Eiko and A.A. Kondratyuk investigated this problem in the general case when $1 \leq q < +\infty$. L.R. Sons [3] constructed a Blaschke product for which $m_2(r, B) \to \infty$ as $r \to 1$.

In the case of a half-plane the problem of the connection of the boundedness of $m_2(v, \pi_\alpha)$ to the distribution of zeros of the products π_α of A.M. Dzhrbashyan [5] was solved by the method of Fourier transforms for meromorphic functions (see [4]). The function π_α coincides with B for $\alpha = 0$.

In [6] estimates for $m_q(v, B)$ were obtained by $n(v)$. It is known that $\lim\limits_{v \to 0} m_1(v, B) = 0$ (see, for example [6]).

Let us define the quantity index $p(B)$ of Blaschke products for the half-plane as in [7],

$$p(B) = p - 1,$$

where $p^{-1} + q^{-1} = 1$ and $q = \sup\{s \in [1, \infty) : m_s(v, B) = O(1), v \to 0\}$.

A.A. Kondratyuk and M.O. Girnik constructed Blaschke products of given quantity index for the unit disc [7]. They used asymptotic formulas of R.S. Galoyan [8]. In [9] analogous formulas for the half-plane were proven.

1 Constructing Blaschke products

Theorem 1 . *For any $p \in [1, +\infty]$, there exists a Blaschke product of given quantity index $p(B) = p - 1$ for the half-plane.*

Proof. We will prove the theorem by establishing 3 cases:
1. $p = +\infty$. In this case, we will prove that there exists a Blaschke product such that for any $s \in (1, +\infty)$, $m_s(v, B) \to +\infty$ as $v \to 0$.
2. $p \in (1, +\infty)$. In this case, using that m_s is monotonically increasing with respect to s, we will prove that there exists a Blaschke product such that for any $s \in (q, +\infty)$, $m_s(v, B) \to +\infty$ as $v \to 0$ and for any $s \in (1, q]$, $m_s(v, B) = O(1)$ as $v \to 0$.
3. $p = 1$. In this case, we will prove that there exists a Blaschke product such that for any $s \in (1, +\infty)$, $m_s(v, B) = O(1)$ as $v \to 0$.

For each case we will find B with the zeros $\{w_k\}_1^\infty = \{iv_k\}_1^\infty$ $(-\infty < v_k < 0)$.

Case 1.

Since

$$\log\left|\frac{w-w_k}{w-\overline{w}_k}\right| = \frac{1}{2}\log\frac{u^2+(v-v_k)^2}{u^2+(v+v_k)^2} = \frac{1}{2}\log\left(1-\frac{4vv_k}{u^2+(v+v_k)^2}\right)$$

$$\leq -\frac{2|v||v_k|}{u^2+(|v|+|v_k|)^2},$$

then

$$\log|B(u+iv)| \leq -2|v|\sum_{k=1}^{+\infty}\frac{|v_k|}{u^2+(|v|+|v_k|)^2}.$$

If $|u|<|v|$ we have

$$|\log|B(u+iv)|| \geq 2|v|\sum_{k=1}^{+\infty}\frac{|v_k|}{u^2+(|v|+|v_k|)^2} \geq \frac{2}{5|v|}\sum_{v_k>v}|v_k|. \quad (2)$$

From (2) and $|B(-u+iv)| = |B(u+iv)|$ it follows that

$$m_s(r,B) = 2^{\frac{1}{s}}\left(\int_0^{+\infty}|\log|B(u+iv)||^s\,du\right)^{1/s}$$

$$\geq 2^{\frac{1}{s}}\left(\int_0^{|v|}|\log|B(u+iv)||^s\,du\right)^{1/s} \geq \frac{2^{\frac{1}{s}+1}}{5}|v|^{\frac{1}{s}-1}\sum_{v_k>v}|v_k|. \quad (3)$$

Since

$$\sum_{v_k\leq v}|v_k| = \int_{-\infty}^{v}(-t)\,dn(t) = -vn(v)+\int_{-\infty}^{v}n(t)\,dt > \int_{-\infty}^{v}n(t)\,dt,$$

we obtain $\int\limits_{-\infty}^{0} n(t)\,dt < +\infty$. Therefore, there exists

$$\lim_{v\to 0} vn(v) = \int_{-\infty}^{0}n(t)\,dt - \int_{-\infty}^{0}(-t)\,dn(t).$$

Having

$$\int_{v}^{0}(-t)\,dn(t) = \lim_{t\to 0}|t|\,n(t) - |v|\,n(v) + \int_{v}^{0}n(t)\,dt \geq \lim_{t\to 0}|t|\,n(t),$$

we can imply that $\lim_{t\to 0} |t|\, n(t) = 0$. Hence

$$\sum_{v_k > v} |v_k| = \int_v^0 (-t)\, dn(t) = -|v|\, n(v) + \int_v^0 n(t)\, dt. \tag{4}$$

Let

$$n(v) \sim \frac{1}{|v| \log^2 |v|}, \quad v \to 0. \tag{5}$$

Then from (4) and (5) we obtain that there exists $v_0\,(-\infty < v_0 < 0)$ such that when $v_0 < v < 0$:

$$\sum_{v_k > v} |v_k| > -\frac{3}{2}\frac{1}{\log^2 |v|} + \frac{1}{2}\int_0^{|v|} \frac{1}{t \log^2 t} dt = -\frac{1}{2\log|v|}\left(\frac{3}{\log|v|} + 1\right).$$

If $s > 1$, we have

$$\lim_{v\to 0} \frac{|v|^{\frac{1}{s}-1}}{\log \frac{1}{|v|}} = +\infty.$$

Then from (3) it follows that for any $s \in (1, +\infty)$, $m_s(v, B) \to \infty$ as $v \to 0$.

Case 2.

Let the counter function $n(v)$ satisfy the condition

$$n(v) \sim \frac{1}{|v|^{\alpha}},\ v \to 0, \tag{6}$$

where $0 < \alpha < 1$. Then for $0 < \varepsilon < \alpha/(2-\alpha)$ we have

$$\begin{aligned} -|v|\, n(v) + \int_v^0 n(t)\, dt &> -(1+\varepsilon)\,|v|^{1-\alpha} + (1-\varepsilon)\int_0^{|v|} t^{-\alpha} dt \\ &= |v|^{1-\alpha}\left(-1-\varepsilon + \frac{1-\varepsilon}{1-\alpha}\right) = \frac{\alpha - (2-\alpha)\,\varepsilon}{1-\alpha}\,|v|^{1-\alpha}. \end{aligned} \tag{7}$$

From (3) and (7) we obtain

$$m_s(v, B) \geq \frac{2^{\frac{1}{s}+1}}{5}\,|v|^{\frac{1}{s}-1}\sum_{v_k > v}|v_k| \geq \frac{2^{\frac{1}{s}+1}\,(\alpha - (2-\alpha)\,\varepsilon)}{5}\,|v|^{\frac{1}{s}-\alpha}. \tag{8}$$

In order to find an upperbound estimate for the integral, let's first prove integral representations. First of all, we have

$$\log|B(u+iv)| = Re\int_{-\infty}^{0}\log\frac{w-it}{w+it}dn(t)$$

$$= Re\left(-2iw\int_{-\infty}^{0}\frac{n(t)}{t^2+w^2}dt\right) = 2Im\left(w\int_{-\infty}^{0}\frac{n(t)}{t^2+w^2}dt\right). \quad (9)$$

Taking $n(t)=n(-t)$ when $t\in[-1,1]$ and $n(t)=0$ when $|t|>1$, from (9) we obtain

$$\log|B(u+iv)| = Re\left(\int_{-\infty}^{0}\frac{n(t)}{t+iw}dt - \int_{-\infty}^{0}\frac{n(t)}{t-iw}dt\right)$$

$$= Re\left(\int_{-\infty}^{+\infty}\frac{n(t)}{t+iw}dt\right) = \int_{-1}^{1}\frac{(t-v)\,n(t)}{(t-v)^2+u^2}dt. \quad (10)$$

Let $n(t)=n_1(t)+\varepsilon(t)$, where $|\varepsilon(t)|<1$. Then

$$\int_{-1}^{1}\frac{(t-v)\,\varepsilon(t)}{(t-v)^2+u^2}dt \le \int_{-1}^{v}\frac{v-t}{(t-v)^2+u^2}dt + \int_{v}^{1}\frac{t-v}{(t-v)^2+u^2}dt$$

$$= \frac{1}{2}\log\frac{(1-v)^2+u^2}{(1+v)^2+u^2}.$$

If $|v|<1/2$,

$$\frac{1}{2}\log\frac{(1-v)^2+u^2}{(1+v)^2+u^2} = \frac{1}{2}\log\left(1+\frac{4|v|}{(1+v)^2+u^2}\right)$$

$$\le \frac{2|v|}{(1-|v|)^2+u^2} \le \frac{2|v|}{\frac{1}{4}+u^2}.$$

Hence,

$$\left(\int_{0}^{+\infty}\left(\int_{-1}^{1}\frac{(t-v)\,\varepsilon(t)}{(t-v)^2+u^2}dt\right)^{s}du\right)^{1/s} \le 2|v|\left(\int_{0}^{+\infty}\frac{du}{\left(\frac{1}{4}+u^2\right)^{s}}\right)^{1/s}$$

for $|v|<1/2$.

Since we can approximate any continuous function with step functions, then without loss of generality we can take $n(v)=|v|^{-\alpha}$ where $v_0<v<0$.

From the following formula in [10]

$$\int_0^{+\infty} \frac{t^{-\alpha}}{t^2+w^2}dt = \frac{\pi}{2w^{1+\alpha}\cos\frac{\pi\alpha}{2}}, \quad 0<\alpha<1$$

and from (9), we obtain

$$\log|B(u+iv)| = \frac{\pi}{\cos\frac{\pi\alpha}{2}} Im\left(w^{-\alpha}\right) = -\frac{\pi}{\cos\frac{\pi\alpha}{2}}|w|^{-\alpha}\sin(\alpha\arg w).$$

Hence, when $v\to 0$,

$$m_s(v,B) = 2^{\frac{1}{s}}\left(\int_0^{+\infty}(\log|B(u+iv)|)^s\right)^{1/s}$$

$$\leq 2^{\frac{1}{s}}\frac{\pi}{\cos\frac{\pi\alpha}{2}}\left(\int_0^{+\infty}\left(u^2+v^2\right)^{-\frac{\alpha s}{2}}\left|\sin\left(\alpha\arctan\frac{|v|}{u}\right)\right|^s du\right)^{1/s} + o(1)$$

$$= 2^{\frac{1}{\alpha}}\frac{\pi}{\cos\frac{\pi\alpha}{2}}|v|^{-\alpha+\frac{1}{s}}\left(\int_0^{+\infty}\left(x^2+1\right)^{-\frac{\alpha s}{2}}\left|\sin\left(\alpha\arctan\frac{1}{x}\right)\right|^s dx\right)^{1/s} + o(1).$$

From here and (8) it follows that $m_s(v,B)$ is unbounded when $s>\alpha^{-1}$ and that it is bounded when $s\leq\alpha^{-1}$. Therefore, $q=\alpha^{-1}$.

Case 3. $q=+\infty$.

Let us take $n(v)\sim\log|v|^{-1}$ as $v\to 0$ and $n(v)=0$ as $v<-1$. First note that

$$\left|\log\left|\frac{w-w_k}{w-\overline{w}_k}\right|\right| = \frac{1}{2}\log\frac{u^2+(v+v_k)^2}{u^2+(v-v_k)^2} = \frac{1}{2}\log\left(1+\frac{4vv_k}{u^2+(v-v_k)^2}\right)$$

$$\leq \frac{2|v||v_k|}{u^2+(v-v_k)^2} \leq \frac{2|v||v_k|}{u^2}.$$

From here it follows that when $u>1/4$,

$$|\log|B(u+iv)|| \leq \frac{2|v|}{u^2}\sum_{k=1}^{+\infty}|v_k|.$$

Hence,

$$m_s^s(v,B) = 2\int_0^{\frac{1}{4}}|\log|B(u+iv)||^s du + 2\int_{\frac{1}{4}}^{+\infty}|\log|B(u+iv)||^s du$$

$$\leq 2\int_0^{\frac{1}{4}}|\log|B(u+iv)||^s du + 2^{s+1}|v|^s\left(\sum_{k=1}^{+\infty}|v_k|\right)^s\int_{\frac{1}{4}}^{+\infty}\frac{du}{u^{2s}}. \tag{11}$$

As in the previous case, without loss of generality, we can take $n(v) = \log|v|^{-1}$, $-1 < v < 0$. From (9) it follows that

$$\log|B(u+iv)| = 2Im\left(w\int_{-1}^{0}\frac{\log|t|^{-1}}{t^2+w^2}dt\right) = -2Im\left(w\int_{0}^{1}\frac{\log t}{t^2+w^2}dt\right).$$

From the formulas above and the following one (see [10])

$$\int_{0}^{+\infty}\frac{\log t}{t^2+w^2}dt = \frac{\pi}{2w}\log w, \quad |\arg w| < \frac{\pi}{2},$$

we obtain

$$\log|B(u+iv)| = -2\left(Im\left(w\int_{0}^{+\infty}\frac{\log t}{t^2+w^2}dt\right) - Im\left(w\int_{1}^{+\infty}\frac{\log t}{t^2+w^2}dt\right)\right)$$
$$= -2\left(\frac{\pi}{2}Im\log w - Im\left(w\int_{1}^{+\infty}\frac{\log t}{t^2+w^2}dt\right)\right).$$

Then, from the last inequality it follows that

$$|\log|B(u+iv)|| \le \pi\arctan\frac{|v|}{u} + 2|w|\int_{1}^{+\infty}\frac{\log t}{|t^2+w^2|}dt, \qquad u>0. \tag{12}$$

Let now $u \in (0,1/4]$ and $|v| \in (0,1/4)$. Since $|w| < (2\sqrt{2})^{-1}$, from (12) we obtain

$$|\log|B(u+iv)|| \le \pi\arctan\frac{|v|}{u} + \frac{1}{\sqrt{2}}\int_{1}^{+\infty}\frac{\log t}{t^2-\frac{1}{2}}dt \le \frac{\pi^2}{2} + \frac{1}{\sqrt{2}}\int_{1}^{+\infty}\frac{\log t}{t^2-\frac{1}{2}}dt.$$

Hence

$$\int_{0}^{\frac{1}{4}}|\log|B(u+iv)||^s du \le \frac{1}{4}\left(\frac{\pi^2}{2} + \frac{1}{\sqrt{2}}\int_{1}^{+\infty}\frac{\log t}{t^2-\frac{1}{2}}dt\right)^s. \tag{13}$$

From (13) and (11) the proof of case 3 is complete. □

References

[1] G. R. Maclane and L. A. Rubel, *On the growth of Blaschke product*, Canadian J. of. Math., **21**(1969), no. 3, pp. 595–601.

[2] V. V. Eiko and A. A. Kondratyuk, *Integral logarithmic means of Blaschke products*, Mat. Zametki, **64**(1998), no. 2, pp. 199–206.

[3] L. R. Sons, *Zeros of functions with slow growth in the unit disc*, Math. Japonica, **24**(1979), no. 3, pp. 271–282.

[4] G. V. Mikayelyan, *Study of the growth of Blaschke-Nevanlinna type products by method of Fourier transforms method*, Izv. AN Arm SSR, Ser. Matem., **18**(1983), no. 3, pp. 216–229.

[5] A. M. Djrbashyan, *Functions of Blaschke type for a half-plane*, DAN SSSR, **246**(1979), no. 6, pp. 1295–1298.

[6] G. V. Mikayelyan and F. V. Hayrapetyan, *On integral logaritmic means of Blaschke products for a half-plane*, Proceedings of the Yerevan state university, **52**(2018), no. 3, pp. 166–171.

[7] M. O. Ghirnyk and A. A. Kondratyuk, *Blaschke products of given quantity index*, Math. Stud., **2**(1993), pp. 49–52.

[8] R. S. Galoyan, *On the asymptotic properties of a function* $\pi_p(z, z_k)$, DAN Arm SSR, **9**(1974), no. 2, p. 65-71.

[9] A. M. Djrbashyan and G. V. Mikayelyan, *Asymptotic behavior of Blaschke like products for the half-plane*, Izv. AN Arm SSR, Ser. Matem., **21**(1986), no. 2, pp. 142–162.

[10] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, *Integrals and Series: Elementary functions*, M. Nauka, 1981.

Gagik Mikayelyan
Yerevan State University
1 Alex Manoogian St, 0025 Yerevan, Armenia.
gagik.mikaelyan@ysu.am

Feliks Hayrapetyan
Yerevan State University
1 Alex Manoogian St, 0025 Yerevan, Armenia.
feliks.hayrapetyan1995@gmail.com

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