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On Hypergraph Degree Sequence Characterisation

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The question of necessary and sufficient conditions for the existence of a simple hypergraph with a given degree sequence is a long-standing open problem. We investigate ψ_m, the set of all degree sequences of simple hypergraphs on $[n] = \{1, 2, \cdots, n\}$ having m edges. In particular, we show that ψ_m can be defined by its special part, consisting of so called upper degree sequences. Further, the set of upper degree sequences composes an ideal in a ranked poset. Analogous results are found for the complement of ψ_m, integer n-tuples, which do not correspond to any degree sequence of simple hypergraphs.

Hypergraph $\mathcal{H}$ is a pair $([n], \mathcal{E})$, where $[n] = \{1, 2, \cdots, n\}$ is its vertex set and $\mathcal{E}$, the set of edges, is a collection of subsets of $[n]$. Hypergraph is simple if it has no multiedges. The degree d_i of a vertex i of $\mathcal{H}$ is the number of edges in $\mathcal{E}$ containing i, and $d(\mathcal{H}) = (\lceil_\infty, \cdots, \lceil_\backslash)$ is the degree sequence of hypergraph $\mathcal{H}$. The hypergraph degree sequence problem - existence of a simple hypergraph with the given degree sequence - is a long-standing open problem, first stated in [2]. We bring some combinatorial counterparts of the problem. In terms of (0,1)-matrices: the incidence matrix of $\mathcal{H}$ is a $(0,1)$-matrix, with columns representing the vertices and rows representing the edges. Subsets of $[n]$ are identified with $(0,1)$-sequences of length n such that the i-th component equals '1', if and only if the i-th element of $[n]$ is included in the subset. Thus the degree sequence problem is reduced to the existence of $(0,1)$-matrices with given number of 1s in its columns (column sums) and with no repeated rows. The class of $(0,1)$-matrices under similar conditions (given column and row sums) has been studied by Ryser, who obtained necessary and sufficient conditions for the existence of such matrices, see [7]. In terms of the n-dimensional unit cube: consider the power set of $[n]$ and its partial order by inclusion. The $(0,1)$-coding of subsets maps the power set into E^n, the set of vertices of the n-dimensional unit cube: $E^n = \{(x_1, \cdots, x_n)/x_i \in \{0,1\}, i = 1, \cdots, n\}$. In this way, the hypergraph degree sequence problem is equivalent to the existence of vertex sets in E^n with given sizes of their partitions according to variables x_i. In this formulation, the question arises out of the discrete isoperimetric problem for E^n [1], where some estimations have been proven and used for quantitative characteristics of partitions of arbitrary n-cube subsets.

The hypergraph degree sequence problem is investigated by several authors and some complementary results have been obtained but the main problem is still open. In particular, the polytope of degree sequences of simple uniform hypergraphs on the vertex set $[n] = \{1, 2, \cdots, n\}$ is investigated in [3] and obtained some partial information. Several necessary and one sufficient conditions are obtained in [4] for existence of a simple 3-uniform hypergraph with the given degree sequence. It is shown in [6] that any two 3-uniform hypergraphs with the same degree sequence can be transformed into each other using a sequence of trades. We intend to give characterisation of ψ_m - the set of all degree sequences of simple hypergraphs on $[n]$ having m edges . Consider ψ_m as a subset of a ranked poset Ξ^n_{m+1}: $\Xi^n_{m+1} = \{(a_1, \cdots, a_n) : 0 \leq a_i \leq m$ for all i }, where a component-wise partial order is placed on $\Xi^n_{m+1} : (a_1, \cdots, a_n) \leq (b_1, \cdots, b_n)$ if and only if $a_i \leq b_i$ for all i and the rank of an element $(a_1, \cdots, a_n)$ is given by $a_1 + \cdots + a_n$. First we show that the area of characterisation can be narrowed: ψ_m can be defined by its special part. Define $\hat{H}$, a special subposet of Ξ^n_{m+1}: $\hat{H} = \{(a_1, \cdots, a_n) \in \Xi^n_{m+1} : a_i \geq m_{mid}$ for all i }, where a $m_{mid} = (m+1)/2$ for odd m and $m_{mid} = m/2$ for even m.

Theorem 1 *ψ_m can be given by $\psi_m \cap \hat{H}$, its part in $\hat{H}$.*

Elements of $\psi_m \cap \hat{H}$ we will call upper degree sequences. Further, we give the structure of the set of all upper degree sequences:

Theorem 2 *$\psi_m \cap \hat{H}$ is an ideal in $\hat{H}$.*

Analogous results are true for the complement of ψ_m, integer n-tuples, which do not correspond to any degree sequence of simple hypergraphs.

Thus, for defining $\psi_m \cap \hat{H}$, and hence whole set ψ_m, it is sufficient to find maximal elements of the ideal $\psi_m \cap \hat{H}$. Similarly, for defining the complement of ψ_m, integer n-tuples, which do not correspond to any degree sequence of simple hypergraphs, it is sufficient to find minimal elements of the filter $\hat{H} \setminus \psi_m$. Further investigations are in process related to finding classes of maximal elements of the ideal $\psi_m \cap \hat{H}$ and minimal elements of the filter $\hat{H} \setminus \psi_m$.

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