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# Membership Functions or $\alpha$-Cuts? Algorithmic (Constructivist) Analysis Justifies an Interval Approach

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In his pioneering papers, Igor Zaslavsky started an algorithmic (constructivist) analysis of fuzzy logic. In this paper, we extend this analysis to fuzzy mathematics and fuzzy data processing. Specifically, we show that the two mathematically equivalent representations of a fuzzy number – by a membership function and by $\alpha$-cuts – are *not* algorithmically equivalent, and only the $\alpha$-cut representation enables us to efficiently process fuzzy data.

## 1 First Result: Two Representations Are Not Equivalent

**Definition 1.** *By a* c-membership function, *we mean a tuple consisting of two real numbers* $\underline{\Delta}$ *and* $\overline{\Delta}$ *and a function* $\mu : [\underline{\Delta}, \overline{\Delta}] \to [0,1]$ *for which* $\mu(\underline{\Delta}) = \mu(\overline{\Delta}) = 0$, $\max_x \mu(x) = 1$, *and* $a \le b \le c$ *implies that* $\mu(b) \ge \min(\mu(a), \mu(c))$.

**Definition 2.** *We say that a c-membership function* $\langle \underline{\Delta}, \overline{\Delta}, \mu\rangle$ *is* computable *if both real numbers* $\underline{\Delta}$ *and* $\overline{\Delta}$ *are computable and the function* $\mu$ *is computable.*

**Definition 3.** *By a* family of $\alpha$-cuts (or simply *$\alpha$-cuts*, for short) corresponding to a c-membership function $\mu$, we mean a pair of mappings $\underline{x} : [0,1] \to \mathbb{R}$ and $\overline{x} : [0,1] \to \mathbb{R}$ for which, for every $\alpha \in [0,1]$, we have $\{x : \mu(x) \ge \alpha\} = [\underline{x}(\alpha), \overline{x}(\alpha)]$.

**Definition 4.** *We say that $\alpha$-cuts are* computable *if both mapping* $\underline{x}$ *and* $\overline{x}$ *are computable.*

**Proposition 1.** *There exists a computable c-membership function for which the corresponding $\alpha$-cuts are not computable.*

**Proposition 2.** *There exist computable $\alpha$-cuts for which the corresponding c-membership function is not computable.*

## 2 Only $\alpha$-Cuts Guarantee Algorithmic Fuzzy Data Processing

Since the two representations of fuzzy are not computationally equivalent, it is desirable to analyze which of them leads to an algorithmic fuzzy data processing. Here are the results of this analysis: fuzzy data processing is computable for $\alpha$-cuts but, in general, not computable for membership functions.

efinition 5. *Let $\mu_1, \ldots, \mu_n$ be membership functions, and let $f(x_1, \ldots, x_n)$ be a function.* *, the result of applying $f$ to fuzzy sets $\mu_1, \ldots, \mu_n$, we mean a membership function defined* *the formula* $\mu(y) = \max\limits_{x_1,\ldots,x_n: y=f(x_1,\ldots,x_n)} \min(\mu_1(x_1), \ldots, \mu_n(x_n))$.

roposition 3. *There exists a computable c-membership function $\mu_1(x_1)$ and a computable* *nction $f(x_1)$ for which the result $\mu$ of applying $f$ to $\mu_1$ is not computable.*

roposition 4. *There exists an algorithm that, given $n$ computable families of $\alpha$-cuts cor-* *sponding to the membership functions $\mu_1, \ldots, \mu_n$ and a computable function $f(x_1, \ldots, x_n)$,* *turns computable $\alpha$-cuts for the result $\mu$ of applying $f$ to $\mu_1, \ldots, \mu_n$.*

## Auxiliary Result: Why min and Not Any Other And-Operation

e want all the property to satisfy the "convexity" condition, that if $a \le b \le c$, then $\mu(b) \ge$ $\min(\mu(a), \mu(c))$. Sometimes, we know that the actual value $x$ satisfies *two* properties $S'$ and characterized by membership functions $\mu'(x)$ and $\mu''(x)$; then, the degree $\mu(x)$ to which a al number $x$ is consistent with this information can be described as $\mu(x) = f_\&(\mu'(x), \mu''(x))$. is reasonable to require that this combined property should also be "convex" (in the above nse).

efinition 6. *A function $\mu : \mathbb{R} \to [0,1]$ is called* f-convex *if $a \le b \le c$ implies that* $b) \ge \min(\mu(a), \mu(c))$.

efinition 7. *By a* generalized and-operation, *we mean a function $f : [0,1] \times [0,1] \to [0,1]$* *ich satisfies the following two properties:*

- *for all $a$, $a'$, $b$, and $b'$, if $a \le a'$ and $b \le b'$, then $f(a,b) \le f(a',b')$ (monotonicity);*
- *for all $a$, we have $f(a,1) = f(1,a) = a$.*

roposition 5. *Let $f(a,b)$ be a generalized and-operation. Then, the following two condi-* *ns are equivalent to each other:*

- *for every two f-convex functions $\mu'(x)$ and $\mu''(x)$, the function $\mu(x) = f(\mu'(x), \mu''(x))$ is also f-convex;*
- $f(a,b) = \min(a,b)$.

## eferences


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