Mathematical Problems of Computer Science 38, 49–52, 2012.

Algebras with Fuzzy Operations

S. S. Davidov, J. Hatami

Yerevan State University
Department of Algebra and Geometry, Yerevan, Armenia

In the present paper, we introduce algebras with fuzzy operations. The problem of development of algebras with fuzzy operations is formulated in [1]. We consider fuzzy operations instead of ordinary functions in structure of algebras.

1. Introduction

The problem of development of algebras with fuzzy operations is formulated in ([1], p. 136). Fuzzy approaches to various universal algebraic concepts started with Rosenfeld's fuzzy groups [2]. Since then, many fuzzy algebraic structures have been studied. Also, some authors proposed a general approach to the theory of fuzzy algebras. Another fuzzy approach to universal algebras was initiated by Bělohlavek and Vychodil [1,3], who studied the so-called algebras with fuzzy equalities and developed a fuzzy equational logic. These structures have two parts: the functional part, which is an ordinary algebra and the relational part, which is the carrier set of the algebra, equipped with a fuzzy equality which is compatible with all of the fundamental operations of the corresponding algebra. In this paper, we introduce algebras with fuzzy operations and with fuzzy equalities.

In fuzzy set theory there were different approaches to the concept of a fuzzy function. In a number of papers various kinds of fuzzy functions based on fuzzy equivalence relations were studied. In particular, such approach was used in definitions of fuzzy functions and partial fuzzy functions, given by Klawonn [4], strong fuzzy functions and perfect fuzzy functions, given by Demirci [5]. In [6], authors investigated kinds of algebras which have fuzzy sets as domains and ordinary functions as operations. In [7], authors defined uniform fuzzy relational morphisms between two algebras, and then developed fuzzy homomorphisms and proved theorems concerning them. Alexander P. Šostak, in [8], studied fuzzy categories with fuzzy functions in the role of morphisms

Fuzzy functions based on fuzzy equivalence relations have shown to be very useful in many applications in approximate reasoning, fuzzy control, vague algebra and other fields.

2. Definitions and Results

We will use complete residuated lattices $\mathcal{L} = < L, \wedge, \vee, \otimes, \rightarrow, 0, 1 >$ as the structures of truth values.

Definition 1. Let $\approx^M$ be a fuzzy equality on M. An $(n+1)-$ary fuzzy relation ρ on a set M is called an $n-$ary fuzzy operation w.r.t. $\approx^M$ and $\approx^{M^n}$ if we have the following conditions

Extensionality:

$$(p \approx^{M^n} p') \otimes (y \approx^M y') \otimes \rho(p, y) \leq \rho(p', y') \quad \forall p, p' \in M^n, \quad \forall y, y' \in M,$$

Functionality:

$$\rho(p, y) \otimes \rho(p, y') \leq y \approx^M y' \quad \forall p \in M^n, \quad \forall y, y' \in M,$$

Fully-defend:

$$\bigvee_{y \in M} \rho(p, y) = 1 \quad \forall p \in M^n,$$

where $(a_1, ..., a_n) \approx^{M^n} (b_1, ..., b_n) = \bigwedge_{i=1}^n (a_i \approx^M b_i)$. We say that ρ is a fuzzy operation on M with arity n.

Definition 2 [1]. An algebra with L-equality or $\mathcal{L}-$algebra of type $<\approx, F>$ is a triplet $\mathcal{M} = < M, \approx^M, F^M >$ such that $< M, F^M >$ is an algebra of type $< F >$ and $\approx^M$ is an $L-$equality on M such that each $f^M \in F^M$ is compatible with $\approx^M$, i.e.

$$(a_1 \approx^M b_1) \otimes ... \otimes (a_n \approx^M b_n) \leq f^M(a_1, .., a_n) \approx^M f^M(b_1, ..., b_n)$$

for each $n-$ary $f \in F$ and every $a_1, b_1, ..., a_n, b_n \in M$.

Definition 3. An algebra with fuzzy operations of type $<\approx, F>$ is a triplet $\mathcal{M} =< M, \approx^M, \mathcal{F}^M >$ such that

(i) $\approx^M$ is a fuzzy equality on the set M,

(ii) $\mathcal{F}^M$ is the set of fuzzy operations on the set M.

To simply, we call F-algebras instead of the algebra with fuzzy operations.

Theorem 1. Let $\mathcal{L}$ be a Heyting algebra ($\wedge = \otimes$) and $\mathcal{M} =< M, \approx^M, F^M >$ be an $\mathcal{L}-$algebra of type $<\approx, F>$. Let $\overline{f}^M : M^n \times M \rightarrow L$ with $\overline{f}^M(p, y) = f^M(p) \approx^M y$ for every $n-$ary $f^M \in F^M$ and for all $p \in M^n$, $y \in M$. Then $\overline{\mathcal{M}} =< M, \approx^M, \overline{F}^M >$ is an $F-$algebra of type $<\approx, F>$.

Definition 4. Let $\mathcal{M} =< M, \approx^M, \mathcal{F}^M >$ be an $F-$algebra of type $<\approx, F>$. An $\mathcal{L}-$relation (binary fuzzy relation) θ on M is called a congruence on M if

(i) θ is a fuzzy equivalence on M;

(ii) θ is compatible with $\approx^M$, i.e. $(a \approx^M b) \otimes (a' \approx^M b') \otimes \theta(a, a') \leq \theta(b, b')$ for every $a, a', b, b' \in M$;

(iii) $\bigwedge_{i=1}^n \theta(a_i, b_i) \otimes \theta(y, y') \otimes f^M(a_1, ..., a_n, y) \leq f^M(b_1, ..., b_n, y')$ for every $n-$ary fuzzy opration $f^M \in \mathcal{F}^M$ and $a_1, ..., a_n, y, y' \in M$.

The ordinary set of all congruences on $F-$algebra $\mathcal{M}$ is denoted by $Con(\mathcal{M})$.

Definition 5. Let θ be a congruence on an $F-$algebra $\mathcal{M} =< M, \approx^M, \mathcal{F}^M >$ of type $<\approx, F>$. A factor algebra of $\mathcal{M}$ by θ is an $F-$algebra $\mathcal{M}/\theta =< M/\theta, \approx^{M/\theta}, \mathcal{F}^{M/\theta} >$ of type $<\approx, F>$ such that

(i) $[a]_\theta \approx^{M/\theta} [b]_\theta = \theta(a,b)$ for each $[a]_\theta, [b]_\theta \in M/\theta$;

(ii) $f^{M/\theta}([a_1]_\theta, ..., [a_n]_\theta, [y]_\theta) = f^M(a_1, ..., a_n, y)$

for every $n-$ary operation $f^{M/\theta} \in \mathcal{F}^{M/\theta}$ and for arbitrary $a_1, ..., a_n, y \in M$, where $M/\theta = \{[a]_\theta | \; a \in M\}$ and $[a]_\theta = \{a' | \; \theta(a, a') = 1\}$.

Definition 6 [1]. Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}-$algebras of type $<\approx, F>$. A mapping $h : M \to N$ called a morphism of $\mathcal{M}$ to $\mathcal{N}$ if

(i) $h(f^M(a_1, ..., a_n)) = f^N(h(a_1), ..., h(a_n))$ for every $n-$ary $f \in F$ and arbitrary $..., a_n \in M$;

(ii) $a \approx^M b \leq h(a) \approx^N h(b)$ for every $a, b \in M$.

If $a \approx^M b = h(a) \approx^M h(b)$ for every $a, b \in M$, then h is called an embedding of $\mathcal{M}$ to $\mathcal{N}$.

Definition 7. Let $\mathcal{M} =< M, \approx^M, \mathcal{F}^M >$ and $\mathcal{N} =< N, \approx^N, \mathcal{F}^N >$ be two $F-$algebras type $<\approx, F>$. A mapping $h : M \to N$ is called a morphism of $\mathcal{M}$ to $\mathcal{N}$ if

(i) $a \approx^M b \leq h(a) \approx^N h(b)$ for all $a, b \in M$;

(ii) $f^M(a_1, ..., a_n, y) = f^N(h(a_1), ..., h(a_n), h(y))$ for every $n-$ary operation $f^M \in \mathcal{F}^M$ and arbitrary $a_1, ..., a_n, y \in M$.

A monomorphism is an injective morphism, an embedding is a morphism such that, for every $a, b \in M$,

$$a \approx^M b = h(a) \approx^N h(b),$$

for all $f^M \in \mathcal{F}^M$ with arity n and for all $a_1, ..., a_n \in M$.

An epimorphism which is an embedding is called an isomorphism.

Theorem 2. Let $\mathcal{L}$ be a Heyting algebra. Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{L}-$algebras of type $<\approx, F>$ and a mapping $h : M \to N$ be an embedding of $\mathcal{M}$ to $\mathcal{N}$. Then h is an embedding $\overline{\mathcal{M}}$ to $\overline{\mathcal{N}}$

Definition 8. For every $F-$algebras $\mathcal{M}$ and $\theta \in Con(\mathcal{M})$ a mapping $h_\theta : M \to M/\theta$, where $h_\theta(a) = [a]_\theta$ for all $a \in M$, is called a natural mapping.

Lemma 1. A natural mapping h_θ from $F-$algebras $\mathcal{M}$ to a factor $F-$algebra $\mathcal{M}/\theta$ is an epimorphism.

Theorem 3. (first isomorphism theorem). Let $h : M \to N$ be an epimorphism of -algebras. Then there is an isomorphism $g : \mathcal{M}/\theta_h \to \mathcal{N}$ such that $h_{\theta_h} \circ g = h$, where $\theta_h = \{(a, b) | h(a) \approx^N h(b)\}$.

Definition 9. Let $\mathcal{M}$ be an $F-$algebra and $\phi, \theta \in Con(\mathcal{M})$, $\theta \subseteq \phi$. Then we let ϕ/θ note an $\mathcal{L}-$relation on M/θ defined by $(\phi/\theta)([a]_\theta, [b]_\theta) = \phi(a, b)$ for all $a, b \in M$.

Theorem 4. Let $\mathcal{M}$ be an $F-$algebra and $\phi, \theta \in Con(\mathcal{M})$, $\theta \subseteq \phi$. Then $\phi/\theta \in Con(\mathcal{M}/\theta)$.

Theorem 5. (second isomorphism theorem). Suppose $\mathcal{M}$ is an $F-$algebra and $\phi, \theta \in Con(\mathcal{M})$, $\theta \subseteq \phi$. Then the mapping

$$h : (M/\theta)/(\phi/\theta) \to M/\phi$$

defined by $h([[a]_\theta]_{\phi/\theta}) = [a]_\phi$ is an isomorphism.

References

[1] Bělohlavek R., Vychodil V., Fuzzy equational logic, Springer, 2005.

[2] Rosenfeld A., Fuzzy groups, J. Math. Anal. Appl. 35(3) (1971), 512-517.

[3] Bělohlavek R., Vychodil V., Algebras with fuzzy equalities, Fuzzy Sets and Systems, 157 (2006), 161-201.

[4] Klawonn F., Fuzzy points, in: V. Novak, I. Perfilieva (Eds), Discovering World with Fuzzy Logic, Physica, Heidelberg, (2000), 431-453.

[5] Demirci M., Fuzzy functions and their fundamental properties, Fuzzy Sets and Systems, 106 (1999), 236-246.

[6] Bošnjak I., Madarasz R., Vojodic G., Algebras of fuzzy sets, Fuzzy Sets and Systems, 160 (2009), 2979-2988.

[7] Ignjatovic J., Ciric M., Bogdanovic S., Fuzzy homomorphisms of algebras, Fuzzy Sets and Systems, 160 (2009), 2345-2365.

[8] Šostak A.P., Fuzzy functions and extension of the category L-Top of CHANG-GOGUEN L-topological spaces, 271-294.