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Optimal Permissible Placement by the Height of the Transitive Directed Tree with One Root

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Abstract

In paper [1] we have introduced a new concept – the transitive directed tree with one root and have formulated its minimum permissible placement problem by the height. In the papers [1] and [2] we have introduced a couple of new concepts and obtained necessary conditions for the solution of that problem. In the present paper using the results and the introduced concepts from the papers [1] and [2] we obtain an optimal polynomial algorithm for the problem solution and present the proof of its optimality.

***Keywords*: a transitive directed graph, an optimal placement.**

1. Introduction

The definitions of the following concepts: a transitive directed graph, a permissib placement of the directed graph, a height of the placement of the directed graph, an optim placement by the height of the directed graph are given in paper [1].

From now on by the placement of the directed graph we will mean its permissib placement. All the results obtained in this paper are true for all the three definitions of the heig in paper [1].

The following concepts were defined in paper [1]:

a vertex directly following (preceding) the vertex, a basis of arcs, a transitive directed tr with one root, a branching vertex of the transitive directed tree, a branch of the directed tree, structure, a partial placement, a height of the vertex (structure) in partial placement, an inn height, an (inner) incoming (outgoing) height, delta - Δ.

The following concepts were defined in paper [2]:

a (direct) descendent of branch, a (directly) preceding branch, a leaf, a subtree of t branch, a stem of the subtree, the subtree crown, the main subtree of the given subtree, a subtr of the i-th branching level, a branch of the i-th branching level, a subcrown of the subtree, partial subtree of the subtree, a structure of the given subtree (subcrown), a boundary branch the subtree, an integrity range of the subtree.

In chapter 2 of paper [1] the problem of optimal placement by the height for graphs w introduced. The problem is NP-complete [5]. Polynomial optimal algorithms are known only f solution of certain special classes [3, 9-13]. In paper [1] the concept of transitive directed tr with one root has been introduced, the optimal permissible placement by the height problem h been formulated for it, and necessary conditions have been obtained for the solution of th problem. In papers [1] and [2] certain important results have been obtained for the solution of t problem.

n the present paper the algorithm of optimal solution of the problem and the proof of its optimality are introduced.

. Formulation of the Problem

Let us recall our definitions of transitive directed tree with one root and its optimal permissible placement problem introduced in paper [1].

The transitive directed graph the arc base of which is a directed tree with one root will be called a transitive directed tree with one root.

Problem: *Optimal placement by the height of transitive directed tree with one root.*

For the given transitive directed tree with one root $G=(V,U)$ find such a permissible placement the height of which is equal to the height of the directed tree $H(P,G)=\min_P H(P,G)$ where the minimum is taken by all permissible placements of G.

. Optimal Placement Algorithm.

Let us consider any arbitrary branch of the ortree, designate it by B (Branch), the number of its branches - by BV, the number of the vertices of the preceding branches of the branch - by BP, the number of the vertices of the descendent branches of B - by BT. The inner height of the branch is designated by H_B, the inner outgoing height - by O_B, the inner incoming height - by L_B. $L_B = BP\times(BV+BT)$, $O_B = BT\times(BP+BN)$.

The method for counting H_B is given in paper [1] in the proof of lemma 2 (for all the three definitions accepted for the height).

Let's denote the inner height of any given Str structure by H_{Str}, the inner incoming height - by L_{Str}, the inner outgoing height - by O_{Str}.

Algorithm 1. Optimal placement (ordering) by the height of the transitive ortree having m branching levels.

. Let's assign 1 to i. Consider all the leaves of the ortree as modules.

. For $i<m$, let's consider all the subtrees of the ortree having i-th branching level, for each subtree under consideration apply step 3 and step 4 of the algorithm. After having observed all the subtrees of the i-th branching level set $i+1$ to i. Repeat step 2 once more. If $i=m$ let's consider the only subtree of i-th branching level which is just the whole subtree and apply step 3 and step 5 to it.

. Order all the modules (primary and residual) of the main subtrees of the subtree under consideration taken together by the order of growth of their deltas (those having equal deltas order arbitrarily) and denote that sequence by S. Let's denote the stem of the ortree by B and P, the first module of the sequence S - by M, the partial subtree formed by P and the module M - by C.

. Apply Algorithm 2 to the subtree considered, transferring parameters B,P,M,C,S to it, in the result of which we will form the modules of i-th branching level being considered, find the inner height of the newly formed module of the subtree (called a primary module in algorithm 2), inner incoming, outgoing heights and delta. The obtained parameters of the module will be needed for considering the subtrees of higher branching level.

. (Applied only when $i=m$). The optimal placement of the ortree will be the stem of the ortree and the sequence S placed after it. To find the height of the optimal placement for all the modules in sequence S beginning from module M let's apply step 3 of algorithm 2 transferring parameters B,P,M,C,S to it. In the result of applying step 3 of algorithm 2 to the last module of the sequence S the counted quantity H_C will be the height of optimal placement of the ortree. End of the algorithm.

Algorithm 2: Formation of Subtree Modules

1. Input parameters. As inputs the algorithm takes the parameters B,P,M,C,S of subtree E transferred from Algorithm 1.
2. Checking. If $O_P < L_B$, P will be called the minimum permissible subcrown of the ortree. Its inner outgoing height is equal to O_P, inner incoming height is L_B, inner height is H_P, and its delta is equal to Δ_P. P, the module M and the modules following M in sequence S having smaller deltas than P, placed after P in the growing order of their deltas, form structure named a primary module of subtree E. The modules of the sequence S, not included in the primary module of E will be called residual modules of subtree E. Go to step 5.
3. Evaluating the quantities of C. $L_C = L_B$, $O_C = O_P - L_M + O_M$, $H_C = \max(H_P; O_P + \Delta_M)$. As P let's take C, as M - the module following the module M in sequence S, as C - C and M taken together.
4. Iterative step. Go to step 2.
5. The evaluation of the quantities of the primary module. As the sequence S we will take the ordered subsequence of modules of S following module M which have smaller deltas than P and apply for each of its module step 3 of the Algorithm with parameters B,P,M,C,S. The H_C, L_C, O_C calculated in the result of step 3 of the Algorithm for the last module of sequence S will be the inner height, inner incoming, outgoing height of the primary module respectively. The delta of the primary module will be equal to the difference of its inner height and inner incoming height. End of the algorithm.

The modules of newly formulated sequence S obtained on step 5 will be called the ta modules in the proof of Theorem 3 (these, of course, are not the modules of E).

Notice that H, L, O, Δ-s of the residual modules have already been evaluated before the were just formed as primary modules. During the further calls of the algorithm this newly formed primary module will become a residual module or a constituent part of primary module of the subtree of higher branching level.

4. Optimality of the Algorithm

Theorem 1. *In the result of Algorithm 1 the optimal permissible placement by the height of the considered ortree is obtained.*

Proof: First note that the vertices of the ortree in optimal placement must be put branch by branch (theorem 1 of paper [1]).

In the optimal placement the modules are inseparable positive structures (as the leaves) - the partial subtrees, which during the later improvements of the placement in the case of replacements will remain inseparable (that is, there must not be any other structures between their vertices), and, finally, they will appear entirely in the optimal placement.

To prove the theorem it is just enough to prove the statement below.

Statement 1. *By means of induction let's prove that in the result of iteration of the algorithm with the optimal permissible placement the partial placements of all the subtrees of i-th branching level (taken separately) are obtained. That is, the following three properties take place:*

Property 1. *In the optimal placement the modules (of subtrees of $0,\dots,i-1$ branching levels) forming the crown of the subtree of the i-th branching level must appear in the order mentioned in the algorithm.*

Property 2. *The modules of the subtree in the optimal placement appear inseparably, that is, there must not be placed any branches of other subtrees between the branches of those modules.*

Property 3. *The permissibility of the placement is not violated while ordering the modules (of the subtrees of $0,\dots,i-1$-th branching level) forming the crown of the subtree of the i-th branching level (theorem 4).*

Proof: The basis of induction we will prove for the subtrees of the first branching level.

The leaf is a positive structure. The leaf is also a subtree of zero branching level, and as it is an inseparable unit in the optimal placement we shall also consider it to be a module by itself.

Consider a subtree of an arbitrary first branching level in an arbitrary placement. As the crown of the subtree of the first branching level is entirely formed from the leaves (which are positive structures), therefore, due to theorem 2 of paper [2] we will have that no negative structures of other subtrees can be placed between the modules of the subtree crown. By applying theorem 2 of paper [1] we will obtain that by placing the modules of the crown in the order of their growth the height of the placement will not increase. According to theorem 1 of paper [2] we shall also have that the positive structures of other subtrees placed between the branches of the integrity range of the first branching level subtree must be moved from their places.

The integrity range of the subtree of the first branching level will be its minimum permissible partial subtree mentioned in Algorithm 2, and all the branches of the crown not included in that range the delta of which is less than the delta of the minimum permissible partial subtree will become the tail modules mentioned in Algorithm 2.

The minimum permissible partial subtree is a positive structure according to its construction and definition 11 of paper [2]. In the base of induction the tail modules are leaves so they are positive structures. The structures of other subtrees placed between the minimum permissible partial subtree and tail modules are also positive according to theorem 2 of paper [2]. And these structures must be moved from their place, since they can be put after the minimum permissible partial subtree if and only if their delta is greater than the delta of that partial subtree (according to theorem 2 in paper [1]), in the case of which it in itself will also be greater than the tail modules.

Thus, the minimum permissible partial subtree in the optimal placement together with the tail modules placed immediately after it forms one inseparable unit (which was called the primary module of the subtree being mentioned in Algorithm 2), and in further replacements wherever those modules are moved must be moved together.

According to lemma 3 in paper [1] the primary module is a positive structure, as it is obtained from minimum permissible partial subree and tail modules (being positive).

The property 1 of statement 3 for the basis of induction is trivial.

The Induction Step.

Let's consider an arbitrary placement. Let's assume that before this step properties 1-3 o statement 1 have been proved for all the subtrees of $0,...,i-1$ branching level and on the basis of those properties the subtrees have been ordered and formed into positive inseparable structures that is - modules (as it is mentioned in Algorithm 2) and in the result of that process the height o the placement has not increased. Prove the statement for the i-th branching level subtree.

In the way similar to that applied in the basis of induction it can be proved here that withou increasing the height of the placement we can move the negative structures placed between the modules of the crown of the considered subtree, order the modules forming the crown of the subtree in the order of the growth of their deltas and transpose the positive structures of othe subtrees placed between the modules containing the branches of the integrity range of the subtre (as, it is already supposed, the module containing the boundary branch is an already forme inseparable structure). After having made the stated displacement in the new placement th partial subtree within the stem of the observed subtree and the module contained the boundar branch (including that module) will be the minimum permissible partial subtree of the subtre being considered in Algorithm 2.

As in the basis of induction it is also proved here that the minimum permissible partia subtree in the optimal placement together with the tail modules placed directly after it forms on inseparable unit (which was called the primary module of the subtree being mentioned i Algorithm 2), and the structures placed between them must be removed from their space.

To prove property 3 in statement 1 it is sufficient to prove the following theorem;

Theorem 2. *The delta of the primary module is always less than the deltas of the residua modules.*

Proof: The delta of the residual modules is greater than the delta of the minimun permissible partial subtree according to the definition of the minimum permissible partia subtree. The inner incoming height of the minimum permissible partial subtree is equal to th inner incoming height of the primary module (as the basis of those two partial subtrees is th same, their partial placements are also the same). As the primary module includes the minimun permissible partial subtree, it means that the inner height of the primary module is greater than/o equal to the inner height of the partial subtree.

Let's show that the height of the primary module is not greater than the height of th minimum permissible partial subtree in their partial placement, that is, it is equal to it.

It means that those heights also will be equal in the general placement as in the gener optimal placement the primary module is an inseparable unit, the tail modules are place immediately after the minimum permissible subtree and the branches taken away out of th partial placement are not joined to either of the two, therefore, either they pass over the two c them or not.

We will denote the minimum permissible partial subtree by M, the tail module place immediately after it - by T, the partial subtree formed together with M and T - by E. Th inner height of T will be denoted by H_T, the inner incoming height - by l_T, the inner outgoin height - by o_T, in the partial placement of E the height of T will be denoted by H_T, th incoming height - by L_T.

As the branches of T belong to the subtree of the stem of M, it means that they are th descendents of branches of M, (or only the descendent of the stem of M, if the stem of T i the direct descendent of the stem of M). Consequently, the branches of T and their descendent are also the descendents of the preceding branches of the stem of M, which according to th incoming and outgoing heights of the structure allows us to state the following:

the set of the arcs forming the inner incoming height of T is the subset of the set of arcs forming the inner outgoing height of M.

$L_T = O_M$ where O_M is the outgoing height of M in its partial placement. $H_T = L_T + \Delta_T$ according to the definition of H_T and L_T. As in any permissible placement the number of the arcs passing over the structure is becoming greater or smaller over all the vertices of the structure equally, its delta remains the same for all the placements, where that structure exists totally.

Thus, $H_T = O_M + \Delta_T$, and as $O_M < L_M$, $\Delta_T < \Delta_M$, $\Rightarrow H_T < H_M$ that is the height of E did not increase more than the height of M: $H_E = H_M$.

$O_E = O_M - (l_T - o_T)$. As T is a positive structure $\Rightarrow l_T > o_T$, consequently $O_E < O_M < L_M$, that is E as M is a positive partial subtree and applying the same statement to E as we did in case of M, in the end we will have the height of the primary module being equal to H_M.

Thus, we have obtained that the primary module is always placed before the residual modules while ordering according to their deltas, henceforth in iterations of higher branching level the order of the crown modules (primary and residual) according to the deltas already provides the permissible placement, that is the stem of the primary module will be placed before the residual modules belonging to the crown. Theorem 4 is proved.

This implies the proof of Statement 1, by which Theorem 3 is also proved.

. Complexity

The arc base of the given $G=(V,U)$ transitive ortree can be formed by means of operations of n^2 complexity where $n=|V|$.

We denote G_B the directed graph which is obtained from G in the following way: associate a unique vertex of G_B to every branch of G so that the two vertices will follow each other in G_B if and only if when their corresponding branches in G are direct descendents of each other. The following quantities ascribe to every vertex of G_B: the vertex following that vertex, the number of the vertices of the branch corresponding to it in G, the number of the vertices of the preceding branches of the branch, the number of the vertices of the descendent branches, the number of the branching level. The formation of the orgraph $G=(V_B,U_B)$ can be realized by means of operations of the order $O(n)$, and the formation of the above mentioned quantities ascribed to its vertices, by means of the operations $O(n_B)$ (where $n_B=|V|$).

Let's assess the complexities of Algorithm 1 and Algorithm 2. During the work of the algorithm the subtree of every branch is considered (ordered) only once (all togeather n_B branches), the number of the subtree modules of that branch is less or equal to n_B, for the ordering of which less or equal $n_B \log n_B$ operations will be needed, other $o(n_B)$ operations will be needed to count the delta and the inner incoming, outgoing heights of the newly formed primary module of the subtree of that branch. The number of the branches of the whole ortree is equal to n_B, hence, for the optimal ordering of the whole ortree the algorithm will perform $O(n_B^2 \log n_B)$ operations, other $O(n_B)$ operations will be needed to count the height of the whole ortree. And as $n_B \le n = |V|$, consequently the complexity of the algorithm is $o(n^2 \log n)$.

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Մեկ արմատով տրանզիտիվ օրիենտացված ծառի օպտիմալ թույլատրելի տեղադրումն ըստ բարձրության

Ա. Խաչատուրյան

Ամփոփում

[] հոդվածում մենք ներմուծել ենք նոր հասկացություն՝ մեկ արմատով տրանզիտիվ րիենտացված ծառ և ձևակերպել նրա ըստ բարձրության օպտիմալ թույլատրելի եղադրման խնդիրը: [1] և [2] հոդվածներում մենք ներմուծել ենք մի շարք նոր ասկացություններ և ստացել այդ խնդրի լուծման անհրաժեշտ պայմաններ: գտագործելով [1] և [2] հոդվածներում ստացված արդյունքները և ներմուծված ասկացությունները, սույն հոդվածում մենք ստացել ենք նշված խնդրի լուծման պտիմալ բազմանդամային ալգորիթմը և առաջարկել նրա օպտիմալության ապացույցը:

Оптимальная допустимая расстановка по высоте транзитивно ориентированного дерева с одним корнем

А. Хачатурян

Аннотация

статье [1] мы ввели новую концепцию — транзитивно ориентированное ерево с одним корнем и сформулировали задачу его оптимально допустимой асстановки по высоте. В статьях [1] и [2] мы ввели несколько новых онцепций и получили необходимые условия для решения этой задачи. спользуя результаты и введенные концепции из статьей [1] и [2], мы в астоящей статье получили оптимальный полиномиальный алгоритм для ешения задачи и представили доказательство его оптимальности.