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Some Algebraical and Logical Properties of Two-dimensional Arithmetical Sets Representable in Presburger's System[1]

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Abstract

A classification $\Sigma_H^{(0)} \subseteq \Sigma_H^{(1)} \subseteq \Sigma_H^{(2)} \subseteq \ldots$ of arithmetical sets representable in M.Presburger's system ([1]-[4]) and a classification $\Delta_H^{(0)} \subseteq \Delta_H^{(1)} \subseteq \Delta_H^{(2)} \subseteq \ldots$ of two-dimensional sets of the same kind are considered. It is proved that these classifications are strictly monotone and complete. The operations $\cup, \cap, \circ, \lozenge, ^{-1}$ on two-dimensional arithmetical sets ([5]-[7]) and the algebras Θ^0 and Θ_1 based on these operations ([5]-[7]) are considered. The relations of these operations and algebras to the mentioned classifications are investigated.

1. Introduction

Investigations described in this paper continue the studies described in ([5]-[8]). The notion c Presburger's arithmetical system and auxiliary notions connected with it (in particular, the notion c arithmetical set expressible in this system) are defined as in [1]-[4]. The class Σ_H of all arithmetical se expressible in M.Presburger's system and its subclass Δ_H of all two-dimensional sets belonging to Σ_H are considered. Some subclasses $\Delta_H^{(n)}$ and $\Sigma_H^{(n)}$ of Δ_H and Σ_H for $n = 0,1,2,\ldots$ are introduced (note th these classes actually coincide with the classes Δ_n and Σ_n considered in [6] and [8]). It will be prove that the union of all $\Delta_H^{(n)}$ coincides with Δ_H, the union of all $\Sigma_H^{(n)}$ coincides with Σ_H, and the statemen $\Delta_H^{(n)} \subseteq \Delta_H^{(n+1)}$, $\Delta_H^{(n)} \neq \Delta_H^{(n+1)}$, $\Sigma_H^{(n)} \subseteq \Sigma_H^{(n+1)}$, $\Sigma_H^{(n)} \neq \Sigma_H^{(n+1)}$ hold for any n. The algebras Θ^0 and Θ ([5]-[7]) containing the operations $\cup$ (union), $\cap$ (intersection), $\circ$ (composition), $\lozenge$ (arithmetical sum $^{-1}$ (inversion) on arithmetical sets are considered (precise definitions will be given below). It is proved i [5] that Δ_H coincides with the class of sets representable in the algebra Θ^0. Below it will be proved th the class $\Delta_H^{(0)}$ coincides with the class of sets representable in the algebra Θ_1. It will also be proved th any class $\Delta_H^{(n)}$ is closed under the operations $\cup, \cap, \circ, ^{-1}$, but not closed under the operation $\lozenge$ (cf.[6]).

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2. Mathematical Structures

Let A be a non-empty set. The n-th Cartesian degree of A, i.e. the set of all n-tuples $(x_1, x_2, ..., x_n)$, where $x_i \in A$, will be denoted, as usually, by A^n (we admit that A^1 is A). If $B \subseteq A^n$ then an n-dimensional predicate p for A (i.e., the predicate on A^n) such that $p(x_1, x_2, ..., x_n)$ is true if and only if $(x_1, x_2, ..., x_n) \in B$, will be called a ***representing predicate for B***. In this case we shall also say that B is the ***set of truth for p***.

The set of all non-negative integers $\{0,1,2,...\}$ will be denoted by N. Any n-dimensional predicate for N (i.e. a predicate on N^n) will be called a ***n-dimensional arithmetical predicate***, and the set of truth for such a predicate will be called a ***n-dimensional arithmetical set***.

The notion of ***predicate formula*** on the basis of logical operations $\&$, $\vee$, $\rightarrow$, $\neg$, $\sim$, $\forall$, $\exists$, as well as auxiliary notions connected with it, in particular, the notion of ***term***, are defined as in [2] and [9] (cf. also [4]). ***Signature*** is defined as any set of predicate symbols, functional symbols, and symbols of constants. The notion of ***mathematical structure*** (or ***structure***) in a given signature is defined as in [4]. Namely, a structure Ψ in a given signature Γ is defined as a system consisting of some non-empty set M (which is called the ***universe*** of Ψ) and some assignment which: (1) assigns for each k-dimensional predicate symbol (correspondingly, for each k-dimensional functional symbol) belonging to Γ some k-dimensional predicate for M (correspondingly, some k-dimensional function for M, i.e. a mapping of M^k into M), and (2) assigns some element of M to each symbol of constant belonging to Γ. We say that a predicate formula F (correspondingly, a term t) is a formula (correspondingly, a term) ***in*** a signature Γ if all predicate symbols, functional symbols, and symbols of constants in F (correspondingly, all functional symbols and all symbols of constants in t) belong to Γ. If Ψ is a structure (having the universe M) in a signature Γ then any predicate formula F in Γ (correspondingly, any term t in Γ) having no free variables except $x_1, x_2, ..., x_k$ defines in an obvious way some k-dimensional predicate p for M (correspondingly, some k-dimensional function f for M). Such predicate (correspondingly, function) will be called a ***predicate*** (correspondingly, ***function***) ***expressed*** (or ***represented***) ***by the formula*** F (correspondingly, ***by the term*** t) ***in*** Ψ. The set of truth for p will be called in this case the ***set, expressed (or represented) by the formula*** F ***in*** Ψ. We shall say that a set $A \subseteq M^k$ is expressible in the structure Ψ if it is expressed by some formula F in the signature of Ψ. Below instead of "formula in the signature of a structure Ψ" or "term in the signature of a structure Ψ" we shall say in short "formula in Ψ", "term in Ψ".

We consider (cf.[4]) the following structures having the universe N (where S is interpreted as the function $S(x) = x+1$, the symbols $=, <, +, 0$ are interpreted in a usual way). By N_H we denote the structure $(N, =, 0, S, <, +)$ (it will be called below "M.Presburger's structure"). By N_L we denote the structure $(N, =, 0, S, <)$. (Note that these structures are considered in [4]; the structure N_H is denoted in [4] by N_A). The classes of sets expressible in N_H and N_L will be denoted correspondingly by Σ_H and Σ_L (this definition of Σ_H is equivalent to the definition given above – see, for example, [2], [4]). The class of two-dimensional sets belonging to Σ_H (correspondingly, Σ_L) will be denoted by Δ_H (correspondingly, Δ_L). The classes Σ_H and Σ_L can be generated by some complete deductive systems of formal arithmetic described in [2], [9] (cf. also [4]). We shall denote these deductive systems correspondingly by Ded_H and Ded_L. Formulas F and G (correspondingly terms t and r) are said to be Ded_H-equivalent or Ded_L-equivalent if the formula $(F \rightarrow G) \& (G \rightarrow F)$ (correspondingly, $t = r$) is deducible in Ded_H or Ded_L. We shall usually consider formulas and terms in N_H or N_L up to their Ded_H- or Ded_L-equivalence.

Below the term $S(S(...S(t)...))$, where t is a term, and the symbol S is repeated k times, will be denoted by $S^k(t)$. The term $S^k(0)$ will be denoted by $\overline{k}$. In particular, $S^0(t)$ is t, $\overline{0}$ is 0

3. Algebras of Arithmetical Sets

Let us recall the definitions of the operations $\cup$, $\cap$, $\circ$, $\Diamond$, $^{-1}$ on two-dimensional arithmetical sets ([5]-[7]). The operations $\cup$ and $\cap$ are defined in a usual way. The operations $\circ$, $\Diamond$, $^{-1}$ are defined by the following generating rules:
(1) if $(x,y) \in A$, $(y,z) \in B$ then $(x,z) \in A \circ B$;
(2) if $(x,y) \in A$, $(x,z) \in B$ then $(x,y+z) \in A \Diamond B$;
(3) if $(x,y) \in A$, then $(y,x) \in A^{-1}$.
We consider algebras Θ^0 and Θ_1 of two-dimensional arithmetical sets defined as in [5]-[7], namely, the algebra Θ^0 is generated by the operations $\cup$, $\cap$, $\circ$, $\Diamond$, $^{-1}$, and by the basic elements $R = \{(x,y) \mid y = x+1\}$ and $Q = \{(x,y) \mid x < y\}$; the algebra Θ_1 is generated by the operations $\cup$, $\cap$, $\circ$, $^{-1}$ and by the basic elements R, Q, and $Z = \{(x,y) \mid x = 0\}$. We say that a set is *inductively represented* in the algebra Θ^0 or Θ_1 if it can be obtained from the basic elements of the algebra by its operations.

4. Classes $\Delta_H^{(n)}$ and $\Sigma_H^{(n)}$

Now let us define the classes $\Delta_H^{(n)}$ and $\Sigma_H^{(n)}$ for $n = 0,1,2,...$. The following lemmas are proved in [2], [4], [5] (in some other terms); note that the expression $(t \equiv r)(Mod w)$ in Lemma 4.1(where t and r are terms in N_H, w is a positive natural constant) denotes the formula $\exists z((t+z+z+...+z=r) \vee (r+z+z+...+z=t))$, where the variable z is not included in t and r and is repeated w times in any part of the disjunction.

Lemma 4.1. *Any term in N_H is Ded_H-equivalent to some term having the form $n_1x_1 + n_2x_2 + ... + n_kx_k + \overline{q}$, where any expression n_ix_i denotes the term $(x_i + x_i + ... + x_i)$ in which the variable x_i is repeated n_i times, and q is a non-negative integer constant. Any formula in N_H is Ded_H-equivalent to some formula which can be obtained by & and $\vee$ from subformulas having the form $(t < r)$ or $(t \equiv r)(Mod w)$, where t and r are terms, w is a positive integer constant.*

Below we shall use a special form of formulas considered in Lemma 4.1. We say that a formula $(t < r)$ or $(t \equiv r)(Mod w)$ having the form described in Lemma 4.1 has a *reduced form* (cf. [8]) if it satisfies the following conditions: (1) no variable is included simultaneously in t and r; (2) either t or r (or, possibly, both of them) does not contain a term $\overline{q}$, where $q \neq 0$. It is easy to see that for any formula $(t < r)$ or $(t \equiv r)(Mod w)$ having the form described in Lemma 4.1 there exists a formula which is Ded_H-equivalent to the mentioned formula and has a reduced form.

Lemma 4.2. *Any term in N_L has the form $S^k(x)$ or $S^l(0)$, where x is a variable. Any formula in N_L is Ded_L-equivalent to some formula which can be obtained by & and $\vee$ from subformulas having the form $(t < r)$, where t and r are terms.*

Lemma 4.3. *The class Δ_H coincides with the class of two-dimensional arithmetical sets which can be represented by formulas in N_H obtained by & and $\vee$ from formulas having the form $kx + ly < \overline{m}$, $\overline{m} < kx + ly$, $kx + \overline{m} < ly$, $kx < ly + \overline{m}$, $kx < ly$, $kx < \overline{m}$, $\overline{m} < kx$ (or a form obtained from those by replacing x by y and y by x), where k, l are positive integer constants, m is a non-negative integer* *stant, or the form $(x \equiv \bar{l})(Modw)$, $(y \equiv \bar{l})(Modw)$, where l is a non-negative integer constant, w positive integer constant.*

Note. The expressions having the form $(t = r)$ may be added to the list of expressions $(t < r)$ and $\equiv r)(Modw)$ in the formulation of Lemma 4.1; they may be added to the list of expressions $(t < r)$ in formulation of Lemma 4.2. The statements obtained by this change are equivalent to Lemma 4.1 and mma 4.2, correspondingly. Indeed, the formula $(t = r)$ is Ded_H-equivalent to $< r + \bar{1}\ \&\ r < t + \bar{1})$; this formula is Ded_L-equivalent to $(t < S(r)\ \&\ r < S(t))$.

Let us consider (cf. [6], [8]) the monotone sequence p_n consisting of all prime numbers: $p_0 = 2$, $= 3$, $p_2 = 5$, ... The class Π_n, where $n \geq 0$ is defined as the class of all positive natural numbers ch that all their prime divisors belong to the set $\{p_0, p_1, ..., p_{n-1}\}$. For example, if $n = 0$ then the set $p_0, p_1, ..., p_{n-1}\}$ is admitted to be empty, and the class Π_0 contains only the number 1. The class Π_1 ntains all the numbers having the form 2^n, where $n \geq 0$, etc. Obviously, $\Pi_n \subseteq \Pi_{n+1}$ for any n. The ss $\Delta_H^{(n)}$ of two-dimensional arithmetical sets is defined as the class of sets which can be expressed by mulas in N_H obtained by & and $\vee$ from subformulas having one of the forms (1) $kx + ly < \bar{m}$, $< kx + ly$, $kx + \bar{m} < ly$, $kx < ly + \bar{m}$, $kx < ly$, $kx < \bar{m}$, $\bar{m} < kx$ (or a form obtained from those by lacing x by y and y by x), where $k \in \Pi_n$, $l \in \Pi_n$, or (2) the form $(x \equiv \bar{l})(Modw)$, $\equiv \bar{l})(Modw)$, where $w \in \Pi_n$ (cf. [6]).

The class $\Sigma_H^{(n)}$ of arithmetical sets is defined as the class of sets which can be represented by mulas in N_H having the form described in Lemma 4.1 and satisfying the following conditions: all formulas $(t < r)$ and $(t \equiv r)(Modw)$ have a reduced form, and all the coefficients $n_1, n_2, ..., n_k$ in representations of t and r in the form $n_1x_1 + n_2x_2 + ... + n_kx_k + \bar{q}$, as well as the numbers w in the pression $(t \equiv r)(Modw)$, belong to Π_n (cf. [8]).

Main Theorems

e following theorems will be considered below. For the reader's convenience we recall some theorems ved earlier in [5] and [7].

Theorem 1. $\Delta_H^{(n)} = \Delta_H \cap \Sigma_H^{(n)}$ *for any* $n \in N$.

Theorem 2. Δ_H *coincides with the class of sets inductively representable in the algebra* Θ^0.

Theorem 3. Δ_L *coincides with the class of sets inductively representable in the algebra* Θ_1.

Theorem 4. $\Delta_H = \bigcup_{n=0}^{\infty} \Delta_H^{(n)}$; $\Sigma_H = \bigcup_{n=0}^{\infty} \Sigma_H^{(n)}$; $\Delta_H^{(n)} \subseteq \Delta_H^{(n-1)}$; $\Sigma_H^{(n)} \subseteq \Sigma_H^{(n-1)}$ *for any* $n \in N$.

Theorem 5. $\Delta_H^{(n)} \neq \Delta_H^{(n+1)}$; $\Sigma_H^{(n)} \neq \Sigma_H^{(n+1)}$ *for any* $n \in N$.

Theorem 6. *Any class* $\Delta_H^{(n)}$ *is closed under the operations* $\cup$, $\cap$, $\circ$, $^{-1}$ *but it is not closed under the eration* $\Diamond$.

Theorem 7. $\Delta_n^{(0)}$ *coincides with the class of sets inductively representable in the algebra* Θ_1.

Theorems 2 and 3 were proved earlier: Theorem 2 - in [5], Theorem 3 - in [7].
Theorems 4 – 7 were formulated (without proofs) in [6] and [8].

6. Proofs of Theorems

Proof of Theorem 1. The statement

$$\Delta_H^{(n)} \subseteq \Delta_H \cap \Sigma_H^{(n)}$$

is obtained easily from the definitions.
The reverse statement

$$\Delta_H \cap \Sigma_H^{(n)} \subseteq \Delta_H^{(n)}$$

is proved similarly to the proof of Corollary of Lemma 3.1 in [5].

The proof of Theorem 4 follows easily from the definitions.

For the proof of Theorem 5 let us introduce some auxiliary notions and establish some lemmas.

If $A \subseteq N$ is a one-dimensional arithmetical set then the set A is said to be *eventually periodic* (cf. [4]) if there exists a non-negative integer g and a positive integer h such that for any natural $x \geq g$ the statement $x \in A$ is equivalent to $x+h \in A$. In this case we say that h is a *period* of A. If A is eventually periodic with $g=0$ then we say that A is *purely periodic*. If A is eventually periodic with the period h, and no positive $h_1 < h$ is a period of A, then we say that h is a *minimal period* of A.

Lemma 6.1. *If $A \subseteq N$ is eventually periodic with the minimal period h, then all the periods of A have the form hw, where w is a positive natural number.*

Lemma 6.2. *If $A \subseteq N$ and $B \subseteq N$ are eventually periodic with the periods h_1 and h_2, correspondingly, then $A \cup B$ and $A \cap B$ are eventually periodic with the period $h_1 h_2$.*

The proofs of these lemmas are obtained easily using the corresponding definitions.

Lemma 6.3. (cf. [4], Theorem 32F) *A set $A \subseteq N$ belongs to $\Sigma_H^{(n)}$ if and only if it is eventually periodic with the period belonging to Π_n.*

Proof. If $A \in \Sigma_H^{(n)}$ then (see the definition of $\Sigma_H^{(n)}$) A can be represented by a formula in N_H having a single free variable x and such that it can be obtained by & and $\vee$ from subformulas having the form $(kx < \bar{l})$, $(\bar{l} < kx)$, $(x \equiv \bar{l})(Mod w)$, where all the numbers k and w belong to Π_n.

But the sets represented by formulas having the forms $(kx < \bar{l})$ or $(\bar{l} < kx)$ are either finite or have finite complements; so they are eventually periodic with the period 1. Every set represented by the formula having the form $(x \equiv \bar{l})(Mod w)$ is purely periodic with the period w. So we conclude (see Lemma 6.2) that A is eventually periodic with the period belonging to Π_n.

Now let A be eventually periodic with the period $h \in \Pi_n$. Let $x \in A$ is equivalent to $x+h \in A$ when $x \geq g$. Without loss of generality we may suppose that $g > 0$; let us denote the number $g-1$ by k. Let $k_1, k_2, ..., k_t$ be all natural numbers less than g and belonging to A, $l_1, l_2, ..., l_u$ be all natural numbers greater than k (i.e. $\geq g$) and less than $g+h$. Then the set A is represented by the formula which is the disjunction of subformulas $x = \bar{k}_i$ for $1 \leq i \leq t$ and of subformulas $((\bar{k} < x) \& ((x \equiv \bar{l}_j)(Mod h)))$ for $1 \leq j \leq u$. It is easy to see that this formula represents the set A. Hence $A \in \Sigma_H^{(n)}$. This completes the proof.

Note. If a one-dimensional set A has a period belonging to Π_n, then its minimal period also belongs to Π_n (see Lemma 6.1). So in the formulation of Lemma 6.3 we may say "minimal period" instead of "period".

Lemma 6.4. *If a two-dimensional arithmetical predicate $p(x, y)$ is the representing predicate for a set $A \in \Delta_H^{(n)}$, then $\exists x p(x, y)$ and $\exists y p(x, y)$ are representing predicates for some one-dimensional sets belonging to $\Sigma_H^{(n)}$.*

Proof. Let us prove the statement of Lemma for $\exists y p(x,y)$; the proof for $\exists x p(x,y)$ is obtained similarly.

The predicate $p(x,y)$ is represented by a formula having the form described in the definition of the class $\Delta_H^{(n)}$. It will be convenient to use a shorter form of this description; this form can be obtained by including of subtraction symbol in the language of terms (cf. [4]). Namely we shall consider the expressions having the form $\pm n_1x_1 \pm n_2x_2 \pm \ldots \pm n_kx_k \pm \overline{q}$, where the expressions n_ix_i are interpreted as it is defined in the formulation of Lemma 4.1. Obviously, any formula $(t<r)$, where t and r are expressions of such kind, can be transformed to the formula without the symbol of subtraction (for example, $2x-3y<-7$ can be transformed to $2x+7<3y$). Of course, some formulas of the mentioned kind are identically true or identically false (for example, $2x+3y<-5$ is identically false). In this case such formulas are Ded_H-equivalent to $0<\overline{1}$ or $0<0$.

If the symbol of subtraction is used, then the general form of subformulas $(t<r)$ of the formula representing the predicate $p(x,y)$ can be given by the expression $kx+ly<\overline{m}$, where k,l,m are integers (possibly, negative or 0), and $|k|$, $|l|$ are zeros or belong to Π_n.

Now, let us apply the algorithm for elimination of quantifiers described in [4] to the formula F representing the predicate $\exists y p(x,y)$. We shall prove that the set of truth for the formula obtained by this algorithm, belongs to $\Sigma_H^{(n)}$.

Using some elementary logical transformations described in [4], we can transform the formula F representing $\exists y p(x,y)$ to a formula $\widetilde{F}$ which can be obtained by & and $\vee$ from subformulas of the following two kinds: (1) some of them do not contain $\exists y$ and were included previously in F; (2) some of them have the form

(6.4.1) $$\exists y(F_1 \,\&\, F_2 \,\&\ldots\&\, F_r \,\&\, G),$$

in which all F_i have the form $(k_ix+l_iy<\overline{m}_i)$ where $l_i \neq 0$, $|l_i| \in \Pi_n$, $k_i=0$ or $|k_i| \in \Pi_n$; G has the form $(y \equiv \overline{l})(Mod w)$, where $w \in \Pi_n$. The subformulas of $\widetilde{F}$ which do not contain $\exists y$ satisfy the conditions noted in the definition of $\Sigma_H^{(n)}$, so for the proof of Lemma it is sufficient to consider the process of elimination of $\exists y$ from the formulas having the form (6.4.1).

The following step of algorithm described in [4] is characterized there as "a uniformization of coefficients at y". Namely we consider the product $|l_1l_2\ldots l_r|=T$ and the numbers $l_1^*, l_2^*, \ldots, l_r^*$ such that $T=|l_i|l_i^*$, $1\le i\le r$. The formula (6.4.1) can be transformed to the form

(6.4.2) $$\exists y(F_1' \,\&\, F_2' \,\&\ldots\&\, F_r' \,\&\, G'),$$

where any F_i' is $(k_il_i^*x+l_il^*y<\overline{m_il_i^*})$, G' is $(Ty\equiv\overline{Tl})(Mod Tw)$. This formula is further transformed (by introducing a new variable $z=Ty$) to the form

(6.4.3) $$\exists z(F_1'' \,\&\, F_2'' \,\&\ldots\&\, F_r'' \,\&\, G'' \,\&\, (z\equiv 0)(Mod T)),$$

where F_i'' is $(k_il_i^*x+z<\overline{m_il_i^*})$, when $l_i>0$ and is $(k_il_i^*x-z<\overline{m_il_i^*})$, when $l_i<0$; G'' is $(z\equiv\overline{Tl})(Mod Tw)$. Obviously, any coefficient k_il^* either is equal to 0, or satisfies the condition: $|k_il_i^*| \in \Pi_n$; the modules T and Tw belong to Π_n. Any formula F_i'' can be represented in the form $z<\overline{m_il_i^*}-k_il_i^*x$ when $l_i>0$ and in the form $k_il_i^*x-\overline{m_il_i^*}<z$ when $l_i<0$. If F_i'' is $z<\overline{m_il_i^*}-k_il_i^*x$ then the expression $\overline{ml_i^*}-k_il^*x$ will be denoted below by U_i (it is an *upper bound* for z); if F_i'' is $k_il_i^*x-\overline{m_il_i^*}<z$ then the expression $k_il_i^*x-\overline{m_il_i^*}$ will be denoted below by L_i (it

is a *lower bound* for z). Following to the method described in [4] we also add the number (-1) to th set of lower bounds; this additional lower bound reflects the condition: the number z should be non negative, i.e. $z > -1$.

Let us denote by M the least common multiple of modules contained in the considered formula (i our case $M = Tw$, hence $M \in \Pi_n$). As it is proved in [4], the existing number z satisfying th condition expressed by formula

(6.4.4) $$F_1^* \& F_2^* \& ... \& F_r^* \& G^* \& (z \equiv 0)(ModT)$$

is equivalent to the following statement: this condition is satisfied for one of the numbers $L_i + \overline{q}$, wher $1 \le q \le M$. So the formula (6.4.3) is equivalent to the disjunction of formulas such that each of them obtained by the substitution of some expression $L_i + \overline{q}$, for z in (6.4.4). The disjunction of formula obtained by such substitution is the result of the mentioned algorithm; clearly, every formula F_i^* afte such substitution obtains the form $L_i < L_j + \overline{q}$, or $L_i + \overline{q} < U_j$, but any such formula contains a singl variable x , hence it is equivalent to some formula having one of the forms $x < \overline{d}$, $\overline{d} < x$, $0 < \overline{1}$ $0 < 0$. So, the conditions noted in the definition of the class $\Sigma_H^{(n)}$ are satisfied for these formula Similarly the formulas obtained from G^* or $(z \equiv 0)(ModT)$ by the substitution of $L_i + \overline{q}$, for z can b easily reduced to the formulas satisfying these conditions. Hence the set of truth for the formula obtaine by the mentioned algorithm belongs to $\Sigma_H^{(n)}$. This completes the proof.

Let us consider two-dimensional arithmetical sets $D_k = \{(x, y) \mid y = kx\}$ for $k = 1,2,3,...$, and th set $E = D_1$. Obviously, $E \in \Delta_H^{(0)}$, hence $E \in \Delta_H^{(n)}$ for any n .

Lemma 6. 5. $D_k \in \Delta_H^{(n)}$ *if and only if* $k \in \Pi_n$.

Proof. Let n be a natural number such that $k \in \Pi_n$. Then D_k is represented by the formula $(y < kx + \overline{1}) \& (kx < y + \overline{1})$. Hence $D_k \in \Delta_H^{(n)}$.

Now let n be a natural number such that $D_k \in \Delta_H^{(n)}$. The representing predicate for the set D_k $y = kx$. Let us denote this predicate by $p(x, y)$. Using Lemma 6.4 we conclude that the set of truth f the predicate $\exists y p(x, y)$ belongs to $\Sigma_H^{(n)}$. But this set is $\{x \mid (x \equiv 0)(Modk)\}$; it is a purely periodic s and its minimal period is k . Using Lemma 6.3 we conclude that $k \in \Pi_n$. This completes the proof.

Proof of Theorem 5. Let n be a natural number, $n \ge 0$, let p_n be n -th prime number. Obvious $p_n \in \Pi_{n+1}, p_n \notin \Pi_n$. Let $k = p_n$. Using Lemma 6.5 we conclude that $D_k \in \Delta_H^{(n+1)}$, $D_k \notin \Delta_H^{(n)}$ $D_k \in \Sigma_H^{(n+1)}$, $D_k \notin \Sigma_H^{(n)}$. This completes the proof.

Lemma 6. 6. *Let* A *and* B *be two-dimensional arithmetical sets,* $A \in \Delta_H^{(n)}$, $B \in \Delta_H^{(n)}$. *The* $A \circ B \in \Delta_H^{(n)}$.

Proof. Let us denote the representing predicates for A and B by $p(x, y)$ and $q(x, y$ correspondingly. Then the representing predicate for $A \circ B$ will be expressed by $\exists z(p(x, z) \& q(z, y))$ We shall prove that the set of truth for this predicate belongs to $\Delta_H^{(n)}$. We shall use (as in the proof Lemma 6.4) the algorithm for elimination of quantifiers given in [4].

The predicates $p(x, y)$ and $q(x, y)$ can be represented by formulas having the form described the definition of $\Delta_H^{(n)}$. We shall apply the mentioned algorithm to the formula $\exists z(p(x, z) \& q(z, y))$. W denote this formula by F .

The symbol of subtraction is added to the language of terms similarly to [4] and to the proof Lemma 6.4.

Similarly to the proof of Lemma 6.4 we can transform the formula F to a formula $\widetilde{F}$ which can be ...tained by & and ∨ from subformulas of the following two kinds: (1) some of them do not contain $\exists z$ ... were included previously in F; (2) some of them have the form

(6.6.1) $$\exists z((F_1 \,\&\, F_2 \,\&\, ... \,\&\, F_u) \,\&\, (G_1 \,\&\, G_2 \,\&\, ... \,\&\, G_v) \,\&\, J),$$

...which: (1) every F_i has the form $k_i x + l_i z < \overline{g}_i$, where k_i, l_i, g_i are integers (possibly, negative or ... $l_i \neq 0$, $|l_i| \in \Pi_n$, $k_i = 0$ or $|k_i| \in \Pi_n$; (2) every G_j has the form $c_j z + d_j y < \overline{h}_j$ where c_j, d_j, ...are integers (possibly, negative or 0), $c_j \neq 0$, $|c_j| \in \Pi_n$, $d_j = 0$ or $|d_j| \in \Pi_n$; (3) J has the form ... $\equiv \overline{l})(Mod w)$, where $w \in \Pi_n$. The uniformization of coefficients at z is implemented similarly to [4] ...d to the proof of Lemma 6.4. Namely, we consider the number

$$T = |l_1 l_2 ... l_u c_1 c_2 ... c_v|,$$

...d the numbers $l_1^*, l_2^*, ..., l_u^*, c_1^*, c_2^*, ..., c_v^*$ such that $T = |l_i| l_i^*$, $T = |c_j| c_j^*$, $1 \le i \le u$, $1 \le j \le v$. The ...rmula (6.6.1) is transformed to the form

(6.6.2) $\exists z((F_1' \,\&\, F_2' \,\&\, ... \,\&\, F_u') \,\&\, (G_1' \,\&\, G_2' \,\&\, ... \,\&\, G_v') \,\&\, J')$,

...which:

... every F_i' has the form $k_i l_i^* x + l_i l_i^* z < \overline{g_i l_i^*}$; (2) every G_j' has the form $c_j c_j^* z + d_j c_j^* y < \overline{h_j c_j^*}$;

... J' has the form $(Tz \equiv \overline{Tl})(Mod w)$.

Obviously, $T \in \Pi_n$, every $l_i^* \in \Pi_n$, every $c_j^* \in \Pi_n$, $l_i l_i^* = T$ or $l_i l_i^* = -T$, $c_j c_j^* = T$ or ...$c_j^* = -T$, $k_i l_i^* = 0$ or $|k_i l_i^*| \in \Pi_n$, $d_j c_j^* = 0$ or $|d_j c_j^*| \in \Pi_n$, $1 \le i \le u$, $1 \le j \le v$. The formula (6.6.2) ...further transformed (by introducing a new variable $z_1 = Tz$) to the formula

...6.3) $\exists z_1((F_1'' \,\&\, F_2'' \,\&\, ... \,\&\, F_u'') \,\&\, (G_1'' \,\&\, G_2'' \,\&\, ... \,\&\, G_v'') \,\&\, J'' \,\&\, (z_1 \equiv 0)(Mod T))$,

...which:

...) every F_i'' has the form $k_i l_i^* x + z_1 < \overline{g_i l_i^*}$ when $l_i > 0$ and the form $k_i l_i^* x - z_1 < \overline{g_i l^*}$ when $l_i < 0$;

... every G_j'' has the form $z_1 + d_j c_j^* < \overline{h_j c_j^*}$ when $c_j > 0$ and the form $-z_1 + d_j c_j^* y < \overline{h_j c_j^*}$ when ...< 0;

... the formula J'' is $(z_1 \equiv \overline{Tl})(Mod w)$.

Now *lower bounds* L_r and *upper bounds* U_p for z_1 are defined similarly to the proof of Lemma 6.4. ...mely, any lower bound L_p has the form $k_i l_i^* x - \overline{g_i l_i^*}$ or $d_j c_j^* y - \overline{h_j c_j^*}$; any upper bound U_r has the ...m $\overline{g_i l_i^*} - k_i l_i^* x$ or $\overline{h_j c_j^*} - d_j c_j^* y$. Similarly to [4] and the proof of Lemma 6.4 we add the number (-1) ...he set of lower bounds; this additional lower bound reflects the condition $z_1 > -1$. Note the fact which ...ssential for further conclusions: no expression for the lower bound or the upper bound contains two ...iables together, x and y.

By M we denote the least common multiple of modules included in the considered formula (in our ...e $M = Tw$, hence $M \in \Pi_n$). Now, similarly to [4] and to the proof of Lemma 6.4, we can conclude ...t the existence of some number z_1 satisfying the condition expressed by the formula

...6.4) $((F_1'' \,\&\, F_2'' \,\&\, ... \,\&\, F_u'') \,\&\, (G_1'' \,\&\, G_2'' \,\&\, ... \,\&\, G_v'') \,\&\, J'' \,\&\, (z_1 \equiv 0)(Mod T))$

...quivalent to the following statement: this condition is satisfied for one of the numbers $L_p + \overline{q}$, where ... is one of the lower bounds for z_1, and $1 \le q \le M$. So, the formula (6.6.3) can be transformed to the ...junction of formulas such that each of them is obtained from (6.6.4) by the substitution of some ...ression $L_p + \overline{q}$ for z_1. The formula obtained by such a way is the result of the mentioned algorithm.

It is easy to see that this formula is equivalent to some formula which can be obtained by & and $\vee$ from subformulas having one of the forms $L_\rho+\overline{q}<U_\tau$, $L_\sigma<L_\rho+\overline{q}$, $(x\equiv\overline{k})(ModM)$, $(y\equiv\overline{l})(ModM)$. If L_ρ and U_τ (L_ρ and L_σ, correspondingly) contain the same variable x or y, then the formula $L_\rho+\overline{q}<U_\tau$ ($L_\sigma<L_\rho+\overline{q}$, correspondingly) is equivalent to some formula having one of the forms $x<\overline{m}$, $\overline{m}<x$, $y<\overline{m}$, $\overline{m}<y$, $0<0$, $0<\overline{1}$. If the variables contained in L_ρ and U_τ (L_ρ and L_σ correspondingly) are different, then the formula $L_\rho+\overline{q}<U_\tau$ ($L_\sigma<L_\rho+\overline{q}$, correspondingly) is equivalent to some formula having the form noted in the definition of the class $\Delta_H^{(n)}$. So the formula obtained by the algorithm for elimination of quantifiers is equivalent to some formula having the form noted in the definition of the class $\Delta_H^{(n)}$. This completes the proof.

Proof of Theorem 6. If $A\in\Delta_H^{(n)}$, $B\in\Delta_H^{(n)}$, then using Lemma 4.3 we conclude that the set $A\cup B$, $A\cap B$, A^{-1} belong to $\Delta_H^{(n)}$. Using Lemma 6.6 we conclude that $A\circ B\in\Delta_H^{(n)}$. On the othe hand, let n be a natural number, $n\geq 0$. If $k=p_n-1$ then, obviously, $k\in\Pi_n$, $k+1\notin\Pi_n$, henc $D_k\in\Delta_H^{(n)}$, $E\in\Delta_H^{(0)}\subseteq\Delta_H^{(n)}$, but the set $D_{k+1}=D_k\Diamond E$ does not belong to $\Delta_H^{(n)}$. This completes the proof.

Proof of Theorem 7. It is sufficient to prove that $\Delta_H^{(0)}$ coincides with the class Δ_L (see Theorem 3). Let A be a two-dimensional arithmetical set. If A belongs to Δ_L then it can be represented (see Lemm 4.2) by a formula F in N_L which is obtained by & and $\vee$ from formulas having the form $(t<r)$ where t and r have one of the forms $S^k(x)$, $S^l(y)$, $S^m(0)$. But the terms $S^k(x)$ and $S^l(y)$ ar Ded_H-equivalent to $x+\overline{k}$ and $y+\overline{l}$, correspondingly. Using Lemma 4.3 we obtain that $A\in\Delta_H^{(0)}$.
Now if $A\in\Delta_H^{(0)}$ then it can be represented (see the definition of $\Delta_H^{(n)}$) by a formula F which is obtainec by & and $\vee$ from formulas having one of forms $x+y<\overline{m}$, $\overline{m}<x+y$, $x+\overline{m}<y$, $x<y+\overline{m}$, $x<y$, $x<\overline{m}$, $\overline{m}<x$ (or a form obtained from those by replacing x by y and y by x).
But $x+y<\overline{m}$ is Ded_H-equivalent to the formula

$$(x=0\,\&\,y<\overline{m})\vee(x=\overline{1}\,\&\,y<\overline{m-1})\vee....\vee(x=\overline{m-1}\,\&\,y<\overline{1}),$$

similarly, $\overline{m}<x+y$ is Ded_H-equivalent to the formu

$$(x=0\,\&\,\overline{m}<y)\vee(x=\overline{1}\,\&\,\overline{m-1}<y)\vee....\vee(x=\overline{m}\,\&\,0<y)\vee(\overline{m}<x).$$

The terms $x+\overline{m}$ and $y+\overline{m}$ are Ded_H-equivalent to $S^m(x)$ and $S^m(y)$, correspondingly. Usi Lemma 4.2 we complete the proof.

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Մ. Պրեսբուրգերի համակարգում ներկայացվող երկչափ թվաբանական բազմությունների որոշ հանրահաշվական և տրամաբանական հատկություններ

Ս. Ն. Մանուկյան

Ամփոփում

Դիտարկվում են $\Sigma_H^{(0)} \subseteq \Sigma_H^{(1)} \subseteq \Sigma_H^{(2)} \subseteq \dots$ դասերը, որոնց հաջորդականությունը ընդգրկում է Մ. Պրեսբուրգերի համակարգում ներկայացվող թվաբանական բազմությունների դասը ([1]-[4]) և $\Delta_H^{(0)} \subseteq \Delta_H^{(1)} \subseteq \Delta_H^{(2)} \subseteq \dots$ դասերը, որոնց հաջորդականությունն ընդգրկում է նման տիպի երկչափ բազմությունների դասը: Ապացուցվում է, որ նշված դասակարգումները խիստ մոնոտոն են և լրիվ: Դիտարկվում են երկչափ թվաբանական բազմությունների վրա որոշված $\cup$, $\cap$, $\circ$, $\Diamond$, $^{-1}$ գործողությունները ([5]-[7]) և այդ գործողությունների վրա հիմնված Θ^0 և Θ_1 հանրահաշիվները ([5]-[7]): Հետազոտվում են այդ գործողությունների և հանրահաշիվների փոխհարաբերությունները նշված դասակարգումների հետ:

О некоторых алгебраических и логических особенностях двумерных арифметических множеств, представимых в системе Пресбургера

С. Н. Манукян

Аннотация

Рассматривается классификация $\Sigma_H^{(0)} \subseteq \Sigma_H^{(1)} \subseteq \Sigma_H^{(2)} \subseteq \ldots$ арифметических множеств, представимых в системе Пресбургера ([1]-[4]), а также классификация $\Delta_H^{(0)} \subseteq \Delta_H^{(1)} \subseteq \Delta_H^{(2)} \subseteq \ldots$ двумерных арифметических множеств аналогичного типа. Доказывается полнота и строгая монотонность этих классификаций. Рассматриваются операции $\cup$, $\cap$, $\circ$, $\Diamond$, $^{-1}$ на двумерных арифметических множествах ([5]-[7]), а также алгебры Θ^0 и Θ_1, основанные на этих операциях ([5]-[7]). Исследуются взаимоотношения рассматриваемых операций и алгебр с вышеупомянутыми классификациями.