Mathematical Problems of Computer Science 37, 35–38, 2012.

On Maximal Dead-end Recognizing Systems in the Class of Two-Element Subsets Concerning Operations of Intersection and Complement

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Abstract

The structure of dead-end n-recognizing systems having a maximal possible length in the class of two-element subsets concerning the operations of intersection and complement is investigated. The quantity of such systems is estimated.

In [5],[6] the possible lengths of dead-end n-recognizing systems in the class of two-element subsets concerning the operations of intersections and complement were investigated. Below a similar question is considered, namely: what is the structure of n-recognizing systems of mentioned kind having a maximal possible length? Such a structure is described below. Besides, the quantity of such systems is estimated.

Let us recall some definitions given in [3]–[6]. We consider a set $[n] = \{1, 2, ..., n\}$, where $n \geq 3$, the set of all subsets of $[n]$ is denoted by $R(n)$. By $|A|$ we denote the power of the set A.

We say that the element $i \in [n]$ is recognizable by a subset $S = \{A_1, A_2, ..., A_k\}$ of $R(n)$ if a one-element set $\{i\}$ can be obtained by the operations of intersection and complement (concerning $[n]$) from the sets $A_1, A_2, ..., A_k$. A subset $S = \{A_1, A_2, ..., A_k\}$ of $R(n)$ is said to be *an n-recognizing system* if every $\{i\}$, where $\{1 \leq i \leq n\}$, is recognizable by S. We consider only such n-recognizing systems $S = \{A_1, A_2, ..., A_k\}$, in which $|A_i| = 2$ for $\{1 \leq i \leq k\}$. All definitions given below refer to such n-recognizing systems. The *representing graph* of a subset $S = \{A_1, A_2, ..., A_k\}$ of $R(n)$ such that $|A_i| = 2$ for $\{1 \leq i \leq k\}$ is defined as a graph such that there exists a one-to one correspondence between the set of its vertices and the set $[n]$, and besides, two vertices, corresponding to the numbers u and v, $u \neq v$, are connected by an edge if and only if $\{u, v\} = A_i$, where $\{1 \leq i \leq k\}$.

An n-recognizing system S is said to be dead-end if any proper subset of S is not an n-recognizing system. An n-recognizing system S is said to be minimal if there is no n-recognizing system having the power less than $|S|$.

The definitions which are not given here can be found in [7].

The following statements (1-5) are proved in [5], [6].

Statement 1: If $S = \{A_1, A_2, ..., A_k\}$ is an n-recognizing system then the union of all sets A_i cannot exclude more than one element of $[n]$.

Statement 2: The representing graph of any n-recognizing dead-end system cannot contain cycles.

Corollary 1: The representing graph of any n-recognizing dead-end system is a fores

Statement 3: Any n-recognizing system $S = \{A_1, A_2, ..., A_k\}$ cannot contain a subs A_i such that all intersections $A_i \cap A_j$, where $1 \leq i \leq n$, $i \neq j$, are empty.

Statement 4: A representing graph of any n-recognizing dead-end system cannot co tain a chain having the length more than 4.

Statement 5: The maximal possible length k of an n-recognizing dead-end syste $S = \{A_1, A_2, ..., A_k\}$ is equal to $n-2$.

Let us note that, as it is easy to see, if $n \leq 6$, then the maximal possible length of n-recognizing dead-end system is equal to the length of a minimal n-recognizing system. S below we suppose that $n \geq 7$.

The following statements are easily proved using the definitions.

Statement 6: Let $S = \{A_1, A_2, ..., A_k\}$ be a subset of $R(n)$, $n \geq 3$, such that tl representing graph of S is a tree. Then S is an n-recognizing system.

Statement 7: Let $S = \{A_1, A_2, ..., A_k\}$ be a subset of $R(n)$, $n \geq 3$, such that tl representing graph of S is a forest consisting of l trees, where the quantity of vertices in i- tree is equal to m_i, where $m_i \geq 3$ for $1 \leq i \leq l$, and $m_1 + m_2 + ... + m_l = n$.

Then S is an n-recognizing system.

Statement 8: Let $S = \{A_1, A_2, ..., A_k\}$ be a subset of $R(n)$, $n \geq 3$, such that tl representing graph of S is a forest consisting of l trees with an isolated vertex, the quanti of vertices in i-th tree is equal to m_i, where $m_i \geq 3$ for $1 \leq i \leq l$, and $m_1+m_2+...+m_l = n-$

Then S is an n-recognizing dead-end system.

Corollary 2: If a subset $S = \{A_1, A_2, ..., A_k\}$ of $R(n)$, $n \geq 7$ is an n-recognizing dea end system having the maximal possible length $n-2$, then the representing graph of contains only a single tree and only a single isolated vertex.

Indeed, in accordance with Corollary 1, the representing graph of S is a forest, clear it must contain an isolated vertex, otherwise some edge incident with a hanging vertex graph can be deleted, and the power of S can be decreased, preserving the property of bei a n-recognizing system. From the other side, if the tree in the mentioned forest is not sing then in accordance with statement 8, the length of the considered system is less than t maximal possible one.

Let us consider some auxiliary notions related to trees. We say that the tree T is *splittable tree* if there exists some edge (u, v) such that after deleting this edge from T ea of two obtained trees contains at least two edges. In the opposite case T is said to be *non-splittable tree.*

Clearly, if the representing graph of some n-recognizing system contains a splittable tı then this system cannot be dead-end (indeed, the subset, corresponding to the mention edge (u, v), can be deleted from the system, and the remaining system will be n-recognizir

A tree T is said to be *a star* if there exists a vertex w in T such that all edges in are incident to w. A tree T is said to be a *generalized star* if there exists a vertex w in such that any edge (w, u) incident to w has one of the following properties: either (1) u a hanging vertex in T, or (2) there exists a single edge (u, v), where $v \neq w$ such that v is hanging vertex in T.

(So T is a generalized star if any chain beginning from a central vertex w contains eitl one or two edges.)

Theorem 1: *If $n \geq 7$ then the representing graph of any n-recognizing dead-end syst $S = \{A_1, A_2, ..., A_k\}$ having the maximal possible length $k = n-2$ is a generalized star w a single isolated vertex.*

Proof: As it is noted in Corollary 2, the representing graph of the considered n-recognizing dead-end system contains an isolated vertex and a tree T. Clearly, T is non-splittable. Let w be a vertex in T having a local degree maximal in T. Let (w, u) be any edge in T incident to w. Clearly, the local degree of w in T is ≥ 3, T is a chain, and our n-recognizing system could not be dead-end. Hence, there exist at least two edges (w, u_1) and (w, u_2) incident to w such that $u_1 \neq u_2$, $u_1 \neq u$, $u_2 \neq u$. If u is not a hanging vertex in T then there exist some vertices (u, v_1), (u, v_2), ..., (u, v_t) incident to u and different from (w, u). But if $t > 1$ then T is splittable, so, there is only single edge (u, v_1) incident to u and different from (w, u). In this edge (u, v_1) the vertex v_1 is a hanging vertex, otherwise T would be splittable. As any edge (w, u) incident to w has the mentioned properties, hence T is a generalized star.

This completes the proof.

Now let us consider the set of all generalized stars (with an isolated vertex) having the maximal possible length $n-2$. Let T be one such generalized star. We denote by x the quantity of all two-element chains contained in T. The quantity of edges in all such chains is equal to $2x$, so we have the following unequality: $2x \leq n-2$, hence $x \leq \left\lfloor \frac{n-2}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor - 1$. So the possible values of x are $0, 1, 2, ..., \left\lfloor \frac{n}{2} \right\rfloor - 1$. Clearly, the structure of T is completely defined by the value of x (up to the isomorphism of graphs), and all mentioned values of x are actually accepted; the quantity of these values is $\left\lfloor \frac{n}{2} \right\rfloor$.

So, the following theorem is true.

Theorem 2: *The quantity of n-recognizing dead-end system having a maximal possible length $(n-2)$ (up to the isomorphism of their representing graphs) is equal to $\left\lfloor \frac{n}{2} \right\rfloor$.*

Acknowledgement

The author expresses his gratitude to Prof. I. D. Zaslavsky for valuable discussions during the article exposition.

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Հատման և լրացման գործողությունների նկատմամբ երկտարր ենթաբազմությունների դասում առավելագույն երկարություն ունեցող ճանաչող փակուղային համակարգերի մասին

Ս. Վարդանյան

Ամփոփում

Առավելագույն հնարավոր երկարություն ունեցող n-ճանաչող փակուղային համակարգերի կառուցվածքը հետազոտվում է երկտարր ենթաբազմությունների դասում լրացման և հատման գործողությունների նկատմամբ: Գնահատվում է նման տիպի համակարգերի քանակը:

О распознающих тупиковых системах, имеющих наибольшую возможную длину в классе двухэлементных подмножеств относительно операций пересечения и дополнения

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Аннотация

Исследуется структура n-распознающих тупиковых систем, имеющих наибольшую возможную длину в классе двухэлементных подмножеств относительно операций пересечения и дополнения. Описываются всевозможные типы n-распознающих максимальных тупиковых систем и дается оценка количества таких систем.