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Necessary Conditions for Optimal Permissible Placement by the Height of the Transitive Directed Tree with One Root

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Abstract:

In the graph theory the problem of the minimum placement of graph by the height, which is similarly formulated in [2] (the problem of minimum cut arrangement of graph), is known. The problem is NP-complete [3]. In the present paper a partial case of this problem, i.e. the problem of optimal permissible placement by the height of the transitive directed tree with one root (which is a such transitive directed graph, the arc base of which forms a directed tree with one root), is formulated. In this paper some new concepts are introduced and necessary conditions for optimal solving of the new formulated problem are given.

Keywords: transitive directed graph, optimal placement.

1. Introduction

Let $G=(V,U)$ be a graph on the set of vertices V, where $|V|=n$. A one-to-one function $P:V\to\{1,2,...,|V|\}$ is called the placement (numeration, arrangement) of graph G. $P(v)$ is called the position or number of the vertex $v\in V$.

In other words, to numerate graph G means to put its vertices on the coordinate line, so that the vertex $v\in V$ will be placed on the line in the position $P(v)$.

If $(u,k)\in U$ and $P(u)<P(v)<P(k)$ or $P(k)<P(v)<P(u)$, we will say that the edge (u,k) passes above the vertex $v\in V$.

Let us give the following three famous definitions of the height of the vertex.

The height of the vertex $v\in V$ for the given placement P is $h_P(v)=|\{(u,k)\in U; P(u)<P(v)<P(k)\}|$.

The height of the vertex $v\in V$ for the given placement P is $h_P(v)=|\{(u,k)\in U; P(u)\le P(v)<P(k)\}|$.

The height of the vertex $v\in V$ for the given placement P is $h_P(v)=|\{(u,k)\in U; P(u)\le P(v)\le P(k)\}|$.

The height of the graph $G=(V,U)$ in the given placement P will be the $H(P,G)=\max_{v\in V} h_P(v)$.

The height of the given graph G will be the $H(G)=\min_P H(P,G)$, i.e. the minimum from the heights of all P placements of graph G.

The aforementioned definitions are applied to directed graphs as well.

The placement P of the directed graph $G=(V,U)$ is called permissible, if $P(u)<P(v)$ for the arbitrary arc $(u,v)\in U$.

It is clear, that if a directed graph contains a directed circle, a permissible placement will not exist.

Let $G=(V,U)$ be a directed graph without directed circles. The directed graph G is called transitive, when for all pairs $x,z\in V$, if there exists a $y\in V$, for which $(x,y)\in U,(y,z)\in U$, then $(x,z)\in U$.

The Section 2 shows, that in common case, the minimum height placement problem is NP–complete for graphs, the minimum length and width placement problems for graphs and directed graphs are also NP-complete [3-6]. The optimal placement problems for some partial cases of the graphs have polynomial solutions [1, 7-11].

In this paper in Section 3 a new concept is introduced: the transitive directed tree with one root. In Section 3 we have formulated the minimum height placement problem for a transitive directed tree with one root. In Section 5 we give the necessary conditions for solving this new problem and for this purpose some important new concepts are formulated in Section 4, because the problem considers a new concept, a new subclass of transitive directed graphs, which requires a new approach.

2. Related Problems and Results

Let us present the following famous problem for the graphs, taken from the book [2]:

2.1. Minimum cut linear arrangement of the graph

Instance: Graph $G=(V,U)$, positive integer K.

Question: Is there a one-to-one function $P:V\to\{1,2,...,|V|\}$ such that for all $i, 1<i<|V|$

$$|\{(u,k)\in U: P(u)<i<P(k)\}|\le K?$$

Reference [3]. The problem is NP-complete: transformation from simple max cut problem.

Let us formulate the following famous problem for the directed graphs (as we have noted above, the placement, numeration and arrangement of the graph are similar concepts and the results obtained for one of these concepts are applied for the remaining two concepts as well):

2.2. Minimum placement by the height of the directed graph

For a given directed graph $G=(V,U)$ find a P_{opt} permissible placement of G, so that its height equals to the directed graph's height: $H(P_{opt},G)=\min_{P} H(P,G)$, where the minimum is taken from the all permissible placements of directed graph G.

This problem is NP-complete. Minimum length and width placement problems of the directed graphs are also NP-complete [4-6]. The solution algorithms for some special cases of these problems are of polynomial complexity [1, 7-11].

3. Notations and Formulation of Problem

Hereinafter, by the placement of the directed graph we will mean its permissible placement.

In the given directed graph $G=(V,U)$ vertex x directly follows vertex y (and vertex y directly precedes x) if $(x,y)\in U$.

The directed tree with one root is the tree the arbitrary vertex of which is directly preceded only by one vertex (the number of the incoming arcs is equal to one) except of the vertex which doesn't have a preceding vertex (that vertex will be called the root of the directed tree).

The arc base of the transitive directed graph $G=(V,U)$ will be called its subgraph $G'=(V,U')$ which is obtained from G by means of the following operation: for the arbitrary vertices x, z of the directed graph G, if there exists a directed path from x to z and $(x,z)\in U$, then the arc (x,z) will be removed from G': $(x,z)\notin U'$.

Definition of transitive directed tree with one root.

A transitive directed tree with one root will be the transitive directed graph the arc base of which forms a directed tree with one root.

In this paper we will discuss the necessary conditions for the following special case of minimum height placement problem of graph: the minimum height placement problem for transitive directed tree with one root.

Problem. *Minimum Permissible Placement by the Height of the Transitive Directed Tree with One Root.*

For the given $G=(V,U)$ transitive directed tree with one root find a P_{opt} permissible placement for G, so that its height equals to the height of the directed tree: $H(P_{opt},G)=\min_P H(P,G)$, where the minimum is taken from the all permissible placements of G.

4. Necessary New Definitions for Solving of the Problem

Definition 1.

The branching vertex of the given directed tree with one root $G=(V,U)$ will be called the vertex which is directly followed by more than one vertex.

Example 1: the vertices a and c are branching vertices of the directed tree in picture 1, but the vertex b is not a branching vertex of the directed tree.

Definition 2.

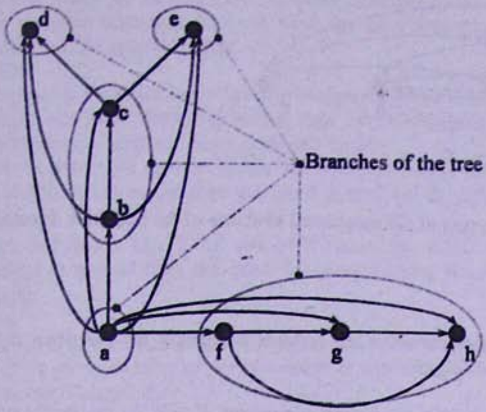
The branching vertex of the given transitive directed tree with one root $G=(V,U)$ will be called the vertex which is also the branching vertex in the arc base of that directed tree.

Example 2: the vertices a and c are branching vertices of the transitive directed tree in picture 1, the vertex b is not a branching vertex of the directed tree.

Hereinafter, saying a directed tree we mean a transitive directed tree with one root.

Definition 3.

The branch of transitive directed tree G will be called its subgraph when its arc base is connected, it doesn't possess branching vertices except of the last vertex, the last vertex is either a branching vertex or doesn't have the following vertices, and the vertex directly preceding the first vertex of the branch is a branching vertex of another branch or the first vertex of the branch is the root of the directed tree.



Picture 1. A transitive directed tree with one root and its branches

Definition 4.

In the given placement the arbitrary one or several adjacent placed branches of the given directed tree will be called a structure.

Let us define the following properties of an arbitrary structure of the transitive directed tree.

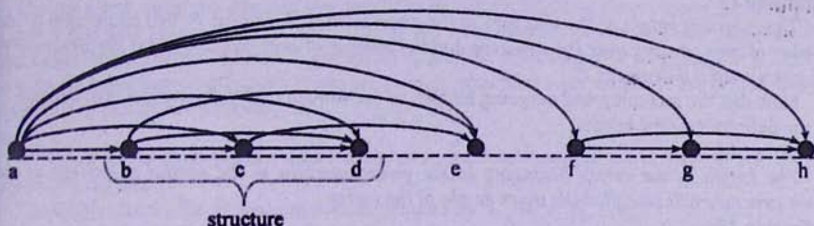
Definition 5.

Two structures are said to be connected if the vertices of those structures are joined by arcs.

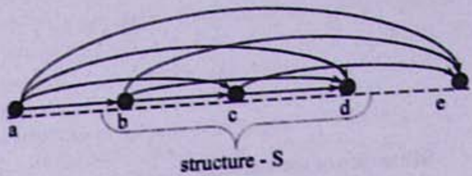
Definition 6.

The structure is said to be negative if the number of incoming arcs is less than the number of outgoing arcs. Otherwise, it will be said to be positive.

Example 3: The branches $\{b,c\}$ and $\{d\}$ from picture 1 obtain a structure mentioned in the placement of picture 2, but the branches $\{b,c\}$ and $\{e\}$ do not obtain a structure.



Picture 2. Illustration of a placement of the transitive directed tree and its structure



Picture 3. The partial placement of the mentioned structure of the transitive directed tree

Definition 9.

The height of the vertex (structure) in the general placement for the given definition of height will be called the general height.

Definition 10.

The height of the given vertex in the partial placement P will be the height of the given vertex from any of above mentioned definitions of height, when only the structures of the placement P are placed.

Example 5: The height of the vertex c in the partial placement from picture 3 is equal to 4 for the first definition of the height.

Definition 11.

The height of the given structure in partial placement P will be the maximum height of its vertices in partial placement P.

Example 6: The height of the mentioned structure in the partial placement from picture 3 is equal to 4 for the first definition of the height.

Definition 12.

The incoming height of the structure in the given partial placement P will be the sum of the number of arcs passing over the structure and the number of arcs incoming the structure in the given partial placement P.

Definition 13.

The outgoing height of the structure in the given partial placement P will be the sum of the number of arcs passing over the structure and the number of arcs outgoing from the structure in the given partial placement P.

Note that the incoming and outgoing heights of the structure (of the vertex) do not depend on the definition of the height.

Definition 14.

The height of the vertex belonging to the given structure in the partial placement of the given structure will be called the inner height of the vertex.

Definition 15.

The height of the structure in its partial placement will be called the inner height of the structure.

Definition 16.

An incoming (outgoing) height of the structure in its partial placement will be called the inner incoming (outgoing) height of the structure.

As only permissible placements are considered, the inner height of the structure and inner incoming, outgoing heights are therefore uniquely defined in the given general placement.

Example 7: When we assume first definition of height by the height of the vertex, the inner height of the structure from picture 2 will be 4, the inner incoming height – will be 4, the inner outgoing height – 3, the general height - 7.

Definition 17.

Let us define as the delta (let us denote it with Δ*) of the positive structure the difference of its inner height and inner incoming height, and for the negative structure we shall take the difference of its inner height and inner outgoing height.*

Example 8: When we assume by the height of the vertex the first definition of height, the delta of the mentioned structure from picture 2 is null and the delta of the structure $\{f, g, h\}$ is minus 1.

From the definitions 12, 13, 16 and 17 follows, that if the structure is positive, its inner incoming height is greater than the inner outgoing height, otherwise it is less than the inner outgoing height.

Definition 18.

Two structures are said to be permissibly replaceable in the given placement (according to each other) if the permissibility of the placement is not violated by putting the second structure directly before the first structure.

5. Necessary Conditions for Optimal Permissible Placement by the Height of the Transitive Directed Tree with One Root

Lemma 1.

If the positive structure is placed directly after the negative one and isn't connected to it, the height of the placement will become less by the replacing the positive structure directly before the negative one.

Proof:

The height of the positive structure will become less, as replacing it before the negative structure, makes incoming arcs (which are less in number than the outgoing arcs of negative one) of negative structure pass over it. The height of the negative structure will become less in the same way.

Lemma 2.

For the given transitive directed tree the inner incoming height of the arbitrary branch of the tree is less than the inner incoming heights of the vertex having maximum inner height in the branch and the vertices of the branch preceding it, and the inner outgoing height of the branch is less than the inner outgoing heights of the vertex with maximum inner height in the branch and the vertices of the branch following it.

Proof:

We'll call the directed graph a whole chain when it doesn't possess any branching vertex and is placed on the line.

The whole chain with m vertices will be denoted by $C_m = \{v_1, v_2, ..., v_m\}$.

The outgoing height of i-th ($1 \le i < m$) vertex of C_m (which is denoted by o_{v_i}) is equal to $o_{v_i} = i(m-i)$. The incoming height of i-th ($1 < i \le m$) vertex of C_m (denoted by I_{v_i}) is equal to the outgoing height of $i-1$-st vertex: $I_{v_i} = o_{v_{i-1}}$ (due to the definitions 12 and 13).

Thus, the plots of incoming, outgoing height of the vertex of the whole chain have the form of a turned parabola and reaches its maximum in the centre of the chain. In the case of $m = 2k$ the k-th vertex has a maximum outgoing height, and the $k+1$-st vertex has maximum incoming height. The k-th and $k+1$-st vertices have maximum outgoing height when $m=2k+1$, and, $k+1$-st and $k+2$-nd have maximum incoming height.

In case of the first definition of height, the height of the i-th $(1 \le i \le m)$ vertex in C_m chain is equal to $h_1(v_i)=(i-1)(m-i)$, therefore when $m=2k$, v_k and v_{k+1} vertices have the maximum height: $h_1(v_k)=h_1(v_{k+1})=k^2-k$, and v_{k+1} $(h_1(v_{k+1})=k^2)$ when $m=2k+1$.

In case of the second definition of height, the height of the i-th $(1 \le i \le m)$ vertex in C_m chain is equal to $h_2(v_i)=i(m-i)$, therefore when $m=2k$, v_k vertex has the maximum height: $h_2(v_k)=k^2$, and v_k, v_{k+1} vertices $(h_2(v_k)=h_2(v_{k+1})=k^2+k)$ when $m=2k+1$.

In case of the third definition of height, the height of the i-th $(1 \le i \le m)$ vertex in C_m chain is equal to $h_3(v_i)=i(m-i+1)-1$, therefore when $m=2k$, v_k and v_{k+1} vertices have the maximum height: $h_3(v_k)=h_3(v_{k+1})=k^2+k-1$, and v_{k+1} $(h_3(v_{k+1})=k^2+2k)$ when $m=2k+1$.

Putting all these together, we can state that the following property takes place: the incoming heights of the vertices preceding the vertex having maximum height in the whole chain are monotonously growing towards the vertex with maximum height and the outgoing heights of the vertices following the vertex with maximum height are monotonously decreasing.

Let us consider any branch of the given directed tree(denote it by B). Let us place the preceding chain of B, the branch B and the crown of the subtree of that branch B having an arbitrary placement on the line and denote that placement by P. Let the number of the vertices of that placement be equal to m. It is easy to notice that we can obtain placement P by taking away some arcs from C_m. The vertices of the considered branch form a subset of adjacent vertices at C_m (let us denote it by S), its height (that is the height of the vertex having maximum height) will equal to the inner height of B branch in placement P. The incoming (outgoing) height of S will be equal to inner incoming (outgoing) height of branch B in placement P and those heights will not depend on mutual order of the branches of the crown of the subtree of branch B in placement P. The vertex of C_m with maximum height we will denote by c_{max}.

The outgoing height of S is equal to the outgoing height of its last vertex (which we have already counted), and the incoming height is equal to the incoming height of the first vertex (due to the definition 12 and definition 13). If the last vertex of S precedes c_{max}, the height of S is equal to the height of its last vertex for all the definitions of height. If the first vertex of S follows c_{max}, the height of S is equal to the height of its first vertex for all the definitions of height. Otherwise, S contains c_{max}, and the height of S will be equal to the height of c_{max} (the heights of the first, last vertices of S and c_{max} we have already counted).

Taking into account all the considerations brought above and that the incoming heights of the vertices preceding c_{max} are monotonously growing towards c_{max} and that outgoing heights of the vertices following c_{max} are monotonously decreasing , we will obtain the proof of the lemma.

Theorem 1.

The vertices of the transitive directed tree in the optimal placement of the tree by the height must be placed branch by branch.

Proof:

Let us consider the arbitrary branch of the directed tree in the arbitrary placement (denote it by B). Let us denote the vertex of the branch having maximum inner height by v_{max} . group the vertices of the branch round v_{max} and show that the height of the general placement will not grow.

As the inner height of the vertices of the branch grows towards v_{max} and decreases after v_{max}, the new height of those vertices after the replacement won't be greater than the height of v_{max}. According to the lemma 2, if the vertex v of branch B precedes v_{max}, the inner incoming height of v is greater than the inner incoming height of branch B and is equal to the inner outgoing height of the vertex directly preceding v in branch B. The number of arcs of B passing above a vertex u of another branch, placed between the vertices of B is equal to the inner incoming height of the first vertex of B placed after the vertex u. Therefore, after replacing the vertices of B towards v_{max} the number of arcs of branch B passing above the vertex u will become less and the general height of u will decrease.

The case of the vertices of other branches placed between the vertices of B and following the v_{max} is proved similarly.

Lemma 3.

From the merging of adjacent placed positive (negative) structures the joint structure obtained by their unification is also positive (negative).

Proof:

Consider arbitrary two adjacent positive structures, designate the first one by S, the second one by T, and the joint structure obtained by their unification - by M. Let us find the relationship of incoming and outgoing arcs of the unified structure. Let us denote the number of the incoming arcs of the first structure by L_S, and the number of the outgoing arcs by O_S. The number of the incoming and outgoing arcs of the second structure will be denoted by L_T and O_T, and the number of the incoming and outgoing arcs of the unified structure by L_M and L_M. The number of the arcs incoming T from S will be denoted by C_{ST}. Thus $L_M = L_S + L_T - C_{ST}$, $O_M = O_S + O_T - C_{ST}$, and as S and T are positive $L_S > O_S$, $L_T > O_T$, consequently $L_M > O_M$ and therefore M is also a positive structure. The case of two positive structures is proved similarly.

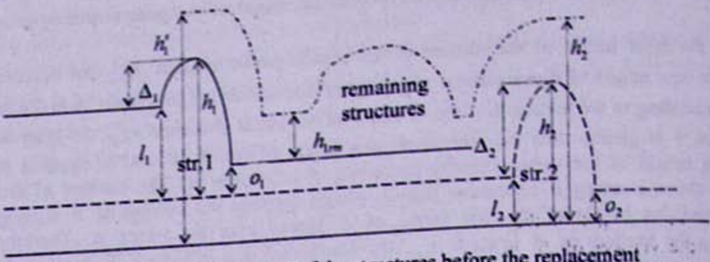
Ordering of two structures with the same sign assumes the replacement of the second structure within two positive or two negative structures directly before the first structure in the given placement.

Theorem 2.

Two permissibly replaceable structures with the same sign are ordered in the following way: two positive structures are ordered in the increasing order of their deltas – first being the smaller, and the two negative ones ordered according to the decreasing order of their deltas: first, the greater one.

Proof:

Consider the case of two positive structures.



Picture 4. Illustration of the structures before the replacement

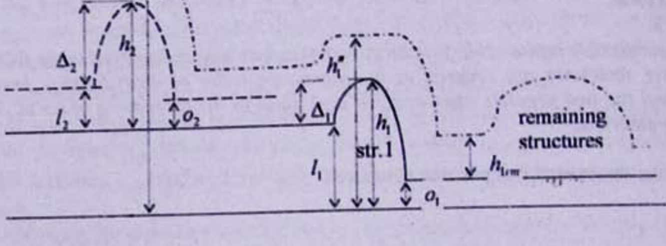
As the permissibility of the order has not been violated, the incoming and outgoing arcs have remained the same in the arbitrary structure.

In the given placement for the first given structure Let us denote its inner height by h_1, delta by Δ_1, inner incoming height by l_1, inner outgoing height by o_1. Before the replacement in the general placement the general height of the first structure is denoted by h_1', after replacing the new height will be denoted by h_1''.

The inner height of the second structure is denoted by h_2, inner incoming height by l_2, inner outgoing height by o_2, delta by Δ_2, in the general placement before replacing the height is denoted by h_2', and the new height after the replacement by h_2''.

Let the delta of the first structure is greater than the delta of the second structure: $\Delta_1 > \Delta_2$.

Except the second structure, the arcs of the remaining structures of the placement which are not connected with the first structure and pass over the first structure will be denoted by h_{1rm}, (*remainig*), $h_1' = h_1 + l_2 + h_{1rm} = l_1 + \Delta_1 + l_2 + h_{1rm}$.



Picture 5. Illustration of the structures after the replacement

Since the second structure is connected neither with the first structure, nor with the structures lying between itself and the first structure, $h_2'' = h_2 + l_1 + h_{1rm} = l_2 + \Delta_2 + l_1 + h_{1rm} < h_1'$, as $\Delta_1 > \Delta_2$, i.e. the new general height of the second structure is less than the general height of the first structure before the replacement.

As the second structure is positive, $\Rightarrow l_2 > o_2$, therefore, the new height of the first structure is less than its previous height: $h_1'' = h_1 + o_2 + h_{1rm} < h_1'$.

Due to $l_2 > o_2$ the new height of the structures lying between the first and the second structures will also decrease.

Hence, the height of the new placement didn't grow.

The case of the two negative structures is proved in the same way.

6. Conclusion

In one of the theorems the "minimum inseparable units" (branches) were obtained in the placement of transitive oriented tree, that is, it was shown that vertices of other branches must not be placed between the vertices of those branches. By grouping the vertices within the branches in an arbitrary given placement and ordering them based on the results obtained in the paper (the lemma on the structures with different signs and the theorem on the structures with the same sign), the placement becomes theoretically possible to order by considering together not only separate branches but also arbitrary adjacent placed branch sets. The obtained result is of both theoretical and practical significance. And the most important of all is that it plays a fundamental role in the optimal solution of the given problem.

References

1. A. H. Khachaturyan, "The optimal permissible placement by the height of the transitive oriented tree containing one vertex of branching", *Mathematical Problems of Computer Science*, vol. 33, pp183-186, 2010.
2. M.R. Garey, D.S. Johnson, *Computers and intractability: A guide to the theory of NP-completeness.* San Francisco, CA: W.H. Freeman, 1979.
3. F. Gavril, "Some NP-complete problems on graphs," *Proc.11th Conf. on Information Sciences and Systems, Johns Hopkins University, Baltimore, MD*, pp. 91-95, 1977.
4. M. R. Garey, R. L. Graham, D. S. Johnson and D. E. Knuth, "Complexity results for bandwidth minimization", *SIAM J. Appl. Math.*, vol. 34, pp. 477–495. 1978.
5. M. R. Garey, D. S. Johnson and L. Stockmeyer, "Some simplified NP-complete graph problems", *Theor. Comput. Sci.*, vol. 1, pp. 237–267. 1976.
6. Ch. H. Papadimitriou, "The NP-completeness of the bandwidth minimization problem", *Computing*, v. 16, pp. 263–270. 1976.
7. A.V. Petrosyan, S. E. Markosyan, Yu. H. Shukuryan, *Mathematical Problems of Automation and Projection of Calculating-Machine.* Yer., (in Russian). 1977.
8. G.G. Geoletsyan, "Flat placement of the vertices of tree with minimization of width", *DAN Arm. SSR*, issue 56, no. 4, pp. 202–207 (in Russian). 1973.
9. L. M. Goldberg and I. A. Klipker, "Minimum placement of trees on a line," *Technical Report, Physico-Technical Institute of Low Temperatures, Academy of Sciences of Ukraine SSR*, 1976.
10. Y. Shiloach, "A minimum linear arrangement algorithm for undirected trees" *Report, Dept. Of Applied Mathematics, Weizmann Institute, Rehovot, Israel.* 1976.

11. D. Adolphson and T.C. Hu, "Optimal linear ordering", *SIAM J. Appl. Math.*, vol. 25, no. 3, pp. 403–423. 1973.
12. C. Berge, *The Theory of Graphs and Its Applications*. New York: Wiley, 1962.

Մեկ արմատով փոխանցական կողմնորոշ ծառի ըստ բարձրության օպտիմալ թույլատրելի տեղադրության անհրաժեշտ պայմաններ

Արմեն Խաչատուրյան

Ամփոփում

Գրաֆների տեսության մեջ հայտնի է գրաֆի ըստ բարձրության օպտիմալ տեղադրման խնդիրը, որը համարժեք կերպով ձևակերպված է [2]-ում (գրաֆի մինիմալ կտրվածքով կարգավորման խնդիրը): Խնդիրը NP-դժվար է: Սույն աշխատանքում ձևակերպված է այս խնդրի մասնակի դեպքը՝ մեկ արմատով փոխանցական կողմնորոշ ծառի (դա այն փոխանցական կողմնորոշ գրաֆն է, որի աղեղների բազան կազմում է մեկ արմատով կողմնորոշ ծառ) ըստ բարձրության օպտիմալ թույլատրելի տեղադրման խնդիրը: Այս աշխատանքում ներկայացվել են որոշակի նոր հասկացություններ և տրվել ձևակերպված խնդրի օպտիմալ լուծման անհրաժեշտ պայմաններ:

Необходимые условия оптимальной допустимой расстановки по высоте транзитивно ориентированного дерева с одним корнем

А. Хачатурян

Аннотация

В теории графов известна проблема минимальной расстановки графа по высоте, которая аналогично сформулирована в [2] (проблема упорядочивания графа с минимальным разрезом). Проблема NP-полна 3. В настоящей статье формулирован частный случай этой проблемы: проблема оптимально допустимой расстановки по высоте транзитивно ориентированного дерева с одним корнем (это такой транзитивно ориентированный граф, база дуг которого составляет ориентированное дерево с одним корнем). В работе введены некоторые новые понятия и даны необходимые условия для оптимального решения сформулированной задачи.