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A Note on Maximum Weight Independent Set in Outer-rectangle Graph

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Abstract

An outer-rectangle graph is the intersection graph of rectangles lying inside a rectangular box and having exactly one edge on the boundary of the box. We present a polynomial-time algorithm for the problem of computing a maximum weight independent set in 2-side outer-rectangle graphs where any two rectangles lying on same edge of box do not intersect.

Keywords Weighted independent set, rectangle graphs, dynamic programming.

1. Introduction

In this paper we study the problem of computing maximum weight independent set (MWIS) in the intersection graph of rectangles. Given a family of axis-parallel rectangles in the plane, find a maximum weight subset of pairwise disjoint rectangles. This problem has been extensively studied before, and has several applications in data mining [1] and automated label placement [2, 3]. Fowler et al. [4] showed that Maximum Independent Set (weights of all the rectangles are one) in rectangle graphs is NP-hard. Asano [5] showed that Maximum Independent Set remains NP-hard even in intersection graphs of unit squares. There have been several $O(\log n)$ approximation algorithms independently suggested for this problem [2, 6]. Lewin-Eitan et al. [7] devised a $4q$-approximation algorithm for the problem, where q is the size of the maximum clique in the input graph. Chalermsook and Chuzhoy [1] were able to break the $\log n$ barrier of approximation by devising a O(log log n) randomized approximation algorithm. A simpler O(log n/ log log n)- approximation algorithm was given in [8].

In this article we will study the problem of computing MWIS for certain specific outer-rectangular graphs.

2. Problem Statement and Formulation of Result

An outer-rectangle graph is the intersection graph of $r_1, r_2, \ldots, r_n$ rectangles lying inside a rectangular box M and having exactly one edge on the boundary of the box M (Figure 1).

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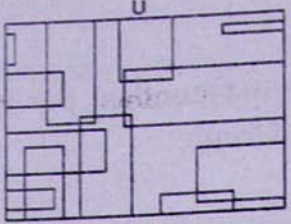
l  R

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Figure 1

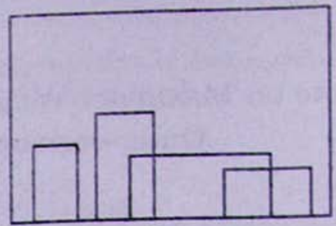

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Figure 2

The sides of the box M we will denote L (left side), R (Right), U (Upper) and D (Down). If the rectangle r_i shares an edge with L, we will say that r_i belongs to L; similarly, for other sides of M.

We will call an instance when all rectangles belong to one side of M as 1-side outer-rectangle graph. Figure 2 shows a case when all rectangles belong to D.

An instance when all rectangles belong to two sides of M we will denote as 2-side outer-rectangle graph. Figures 3 and 4 show cases when all rectangles belong to LR and LD, respectively.

For 1-side outer-rectangle graphs it is obvious that the height of rectangles does not affect the MWIS and the problem is equivelent to that of MWIS for interval graphs, which is solvable in polynomial time [11].

In the following section we will study the instance of 2-side outer-rectangle graphs when any two rectangles lying on the same edge of box M do not intersect (Figure 3, 4).

We will distinguish 2 cases: when rectangles belong to opposite and adjacent sides of M. For opposite sides we will investigate the LR case (Figure 3); for adjacent sides – LD (Figure 4). All other instances can be reduced to these cases by rotating the box M.

LR Case.

Theorem 1: *An LR instance of a 2-side outer-rectangle graph, when any two rectangles lying on the same edge of box M do not intersect, does not contain cycles.*

Proof:

It is obvious that the graph is bipartite. because rectangles that belong to L do not intersect with one another, and. similarly. rectangles that belong to R do not intersect with one another. Now let's assume that the graph contains a cycle $r_1, l_1, r_2, l_2, \ldots, r_k, l_k, r_1$. Without lose of generality we can assume that r_1 is the bottommost rectangle in the cycle. Therefore $r_2, l_2, \ldots, r_k$ are higher then r_1 and do not intersect r_1. And it is obvious that if l_k intersects with r_1. it will also intersect with rectangle l_1. which contradicts the conditions of the theorem. ∎

Chen et al. [9] present an $O(n)$ algorithm to find MWIS in cycle-free graphs. So MWIS for LR case is solvable in linear time. Note that the algorithm presented in [9] is based on dynamic programming strategy.

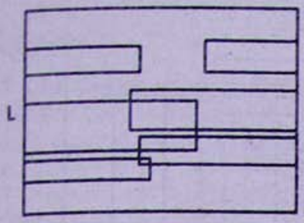

Figure 3

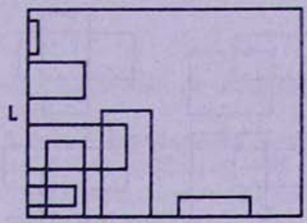

Figure 4

LD case

It is obvious that the graph is bipartite, because rectangles that belong to L do not intersect each other, and, similarly, rectangles that belong to D do not intersect each other. The problem of finding MWIS for bipartite graphs is solvable in $O(n^4)$[10]. But for this case we will present an $O(n^2)$ algorithm for finding MWIS, where n is the number of rectangles.

Lemma 1: A rectangle that belongs to L will intersect with a rectangle belonging to D if and only if the bottom edge of the rectangle from L intersects with the left edge of the rectangle from D.

Proof:

When one rectangle belongs to L, and the other to D, only 4 types of rectangle intersections can occur (shown in Figure 5). For all of these cases the statement of lemma is true. ∎

Lemma 1 shows that we can replace all rectangles from L by their bottom edge, and rectangles from D by their left edge.

Let's denote the segments of L by $l_1, l_2, \ldots, l_m$ and the segments of D by $d_1, d_2, \ldots, d_k$. The weight of any segment t is $W(t)$. Let $l_1, l_2, \ldots, l_m$ be sorted by y-coordinate in increasing order, and $d_1, d_2, \ldots, d_k$ be sorted by x-coordinate in increasing order. We will add one more vertical segment with weight 0, which is more to the right than the rightmost (vertical and horizontal) segment and is higher than all the vertical segments (Figure 6). $W(d_{k+1}) = 0$.

We will denote by $F(i)$ the weight of maximum weight independent set of the segments in the sub-problem defined by the segments lying completely inside the rectangle H bounded by d_i and M, as shown in Figure 6. Obviously $F(k+1)$ will be the weight of MWIS of the initial problem. Because all the segments lie completely inside the rectangle H, the segment d_i is the highest vertical segment and it doesn't intersect with any other segments, hence it should appear in MWIS of the sub problem.

Set of all horizontal segments from H we will denote as T_i.

Solution of sub-problem H either contains only horizontal segments (does not contain any vertical segment except d_i), or contains at least one other vertical segment. Let's assume that d_j is the highest vertical line after d_i that presents in MWIS of sub-problem H. Then H is divided into 3 parts A, B, C as shown in Figure 7.

A contains all horizontal segments higher than d_j.

B contains all vertical segments that are righter and not higher than d_j.

C contains all vertical segments that are left of j and not higher than d_j; and all horizontal segments not higher than d_j and not intersect d_j.

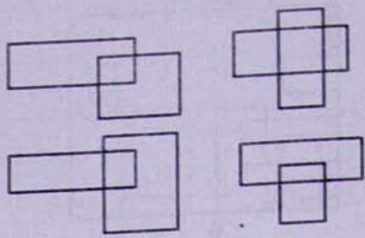

Figure 5

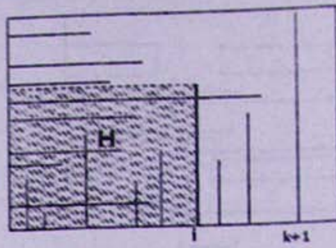

Figure 6

If solution of sub-problem H contains only horizontal segments then weight of the solution will be $W[T_i]$. Otherwise the solutions of sub-problems A, B and C are independent and the following recurrent formula holds:

$$F(i) = W(i) + \max\Big(W[T_i],\ \max_{\substack{j=1\ldots i-1 \\ h_j \le h_i}} [W(A_j) + W(B_j) + F(j)]\Big) \qquad (1),$$

where h_j is height of d_j.

We can calculate $F(i)$ using dynamic programming method, by increasing order of indices $F(1), F(2), \ldots, F(k+1)$, since $F(i)$ depends on values of F with smaller indices. The calculation of $F(i)$ requires $O(n)$ time. Consequently, the calculation of $F(1), F(2), \ldots, F(k+1)$ will require $O(n^2)$.

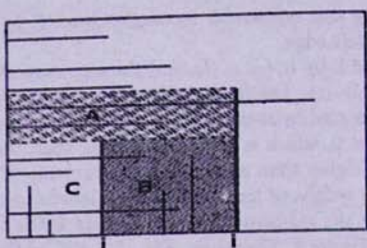

Figure 7

Theorem 2: *An LD instance of a 2-side outer-rectangle graph, when any two rectangles lying on the same edge of box M do not intersect, can be solved in $O(n^2)$ time.*

Proof: $F(k+1)$ can be calculated in $O(n^2)$ time, and it is equal to the weight of MWIS. To find out the subset of rectangles of maximum weighted independent set we should memorize $G(i)$ · the index j for which $F(i)$ reaches the maximum in formula (1). or -1 if $F(i)$ doesn't contain j.

Let's denote a sequence $S = \{s_0, s_1, \ldots, s_p\}$, where $s_0 = k+1$. $s_i = G(s_{i-1})$. $s_p \neq -1$, $G(s_p) = -1$.

The set of rectangles of MWIS will be $\{s_1, s_2, \ldots, s_p\} \cup B_{s_0} \cup B_{s_1} \ldots \cup B_{s_p}$; the union with all rectangles from L that do not intersect with those rectangles from D mentioned above. ∎

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Առավելագույն կշռով անկախ բազմություններ արտաքին ուղղանկյունների գրաֆներում

Է. Փիլիպոսյան

Ամփոփում

Արտաքին ուղղանկյունների գրաֆն այնպիսի ուղղանկյունների հատումների գրաֆ է, որոնք բոլորը ներառված են միևնույն ուղղանկյուն շրջանակի մեջ և որոնց ճիշտ մեկ կողմը հենված է այդ շրջանակի որևէ կողմի վրա: Աշխատանքում ներկայացված է արտաքին ուղղանկյունների գրաֆում առավելագույն կշռով անկախ բազմություն կառուցող բազմանդամային բարդությամբ ալգորիթմ այն դեպքերի համար, երբ ուղղանկյունները հենված են շրջանակի կողմերից միայն երկուսի վրա, և շրջանակի միևնույն կողմին հենված ուղղանկյունները չեն հատվում:

Независимые множества максимального веса в внешне прямоугольных графах

Э. Пилипосян

Аннотация

Внешнепрямоугольный граф - это граф пересечений прямоугольников, которые все расположены внутри некоторого прямоугольника - рамки, и точно одна сторона которых находится (опирается) на какой-то стороне рамки. В работе представлен полиномиальный алгоритм построения независимого множества максимального веса в внешнепрямоугольном графе в случае, когда прямоугольники опираются только на две стороны рамки, и опирающиеся на одну и ту же сторону рамки прямоугольники не пересекаются.