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A Note on Interval Edge-colorings of Graphs

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Abstract

An edge-coloring of a graph G with colors $1, \ldots, t$ is called an interval t-coloring if for all colors are used, and the colors of edges incident to each vertex of G are distinct and form an interval of integers. In this note we prove that if a connected graph G with n vertices admits an interval t-coloring, then $t \leq 2n-3$. We also show that if G is a connected r-regular graph with n vertices that has an interval t-coloring and $n \geq 2r+2$, then this upper bound can be improved to $2n-5$.

1. Introduction

All graphs considered in this paper are finite, undirected and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. The maximum degree of G is denoted by $\Delta(G)$ and the diameter of G by $d(G)$. An (a, b)-biregular bipartite graph G is a bipartite graph G with the vertices in one part all having degree a and the vertices in the other part all having degree b. A partial edge-coloring of G is a coloring of some of the edges of G such that no two adjacent edges receive the same color. If α is a partial edge-coloring of G and $v \in V(G)$, then $S(v, \alpha)$ denotes the set of colors of colored edges incident to v.

An edge-coloring of a graph G with colors $1, \ldots, t$ is an interval t-coloring if all colors are used. and the colors of edges incident to each vertex of G are distinct and form an interval of integers. A graph G is interval colorable if G has an interval t-coloring for some positive integer t. Let $\mathcal{N}$ be the set of all interval colorable graphs [1. 6]. For a graph $G \in \mathcal{N}$. the greatest value of t (the maximum span) for which G has an interval t-coloring is denoted by $W(G)$.

The concept of interval edge-coloring was introduced by Asratian and Kamalian [2]. In [2. 3] they proved the following theorem.

Theorem 1 *If G is a connected triangle-free graph and $G \in \mathcal{N}$, then*

$$W(G) \leq |V(G)| - 1.$$

In particular, from this result it follows that if G is a connected bipartite graph and $G \in \mathcal{N}$, then $W(G) \leq |V(G)| - 1$. It is worth to notice that for some families of bipartite graphs this upper bound can be improved. For example, in [1] Asratian and Casselgren proved the following

Theorem 2 *If G is a connected (a, b)-biregular bipartite graph with $|V(G)| \geq 2(a+b)$ and $G \in \mathcal{N}$, then*

$$W(G) \leq |V(G)| - 3.$$

For general graphs, Kamalian proved the following

Theorem 3 *[6]. If G is a connected graph and $G \in \mathcal{N}$, then*

$$W(G) \leq 2|V(G)| - 3.$$

Note that the upper bound in Theorem 3 is sharp for K_2, but if $G \neq K_2$, then this upper bound can be improved.

Theorem 4 *[5]. If G is a connected graph with $|V(G)| \geq 3$ and $G \in \mathcal{N}$, then*

$$W(G) \leq 2|V(G)| - 4.$$

On the other hand, in [8] Petrosyan proved the following theorem.

Theorem 5 *For any $\varepsilon > 0$, there is a graph G such that $G \in \mathcal{N}$ and*

$$W(G) \geq (2 - \varepsilon)|V(G)|.$$

For planar graphs, the upper bound of Theorem 3 was improved in [4].

Theorem 6 *If G is a connected planar graph and $G \in \mathcal{N}$, then*

$$W(G) \leq \tfrac{11}{6}|V(G)|.$$

In [3], Asratian and Kamalian investigated interval edge-colorings of connected graphs. In particular, they obtained the following two results.

Theorem 7 *If G is a connected graph and $G \in \mathcal{N}$, then*

$$W(G) \leq (d(G) + 1)(\Delta(G) - 1) + 1.$$

Theorem 8 *If G is a connected bipartite graph and $G \in \mathcal{N}$, then*

$$W(G) \leq d(G)(\Delta(G) - 1) + 1.$$

Recently, Kamalian and Petrosyan [7] showed that these upper bounds cannot be significantly improved.

In this note we give a short proof of Theorem 3 based on Theorem 1. We also derive a new upper bound for the maximum span in interval edge-colorings of regular graphs.

2. Results

Proof of Theorem 3. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and α be an interval $W(G)$-coloring of the graph G. Define an auxiliary graph H as follows:

$$V(H) = U \cup W, \text{ where}$$
$$U = \{u_1, u_2, \ldots, u_n\}, W = \{w_1, w_2, \ldots, w_n\} \text{ and}$$
$$E(H) = \{u_i w_j, u_j w_i |\ v_i v_j \in E(G), 1 \le i \le n, 1 \le j \le n\} \cup \{u_i w_i |\ 1 \le i \le n\}.$$

Clearly, H is a connected bipartite graph with $|V(H)| = 2|V(G)|$.

Define an edge-coloring β of the graph H in the following way:

(1) $\beta(u_i w_j) = \beta(u_j w_i) = \alpha(v_i v_j) + 1$ for every edge $v_i v_j \in E(G)$,

(2) $\beta(u_i w_i) = \max S(v_i, \alpha) + 2$ for $i = 1, 2, \ldots, n$.

It is easy to see that β is an edge-coloring of the graph H with colors $2, 3, \ldots, W(G) + 2$ and $\min S(u_i, \beta) = \min S(w_i, \beta)$ for $i = 1, 2, \ldots, n$. Now we present an interval $(W(G) + 2)$-coloring of the graph H. For that we take one edge $u_{i_0} w_{i_0}$ with $\min S(u_{i_0}, \beta) = \min S(w_{i_0}, \beta) = 2$, and recolor it with color 1. Clearly, such a coloring is an interval $(W(G) + 2)$-coloring of the graph H. Since H is a connected bipartite graph and $H \in \mathcal{N}$, by Theorem 1, we have

$$W(G) + 2 \le |V(H)| - 1 = 2|V(G)| - 1, \text{ thus}$$
$$W(G) \le 2|V(G)| - 3.$$

Theorem 9 *If G is a connected r-regular graph with $|V(G)| \ge 2r + 2$ and $G \in \mathcal{N}$, then*

$$W(G) \le 2|V(G)| - 5.$$

Proof:

Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and α be an interval $W(G)$-coloring of the graph G. Define an auxiliary graph H as follows:

$$V(H) = U \cup W, \text{ where}$$
$$U = \{u_1, u_2, \ldots, u_n\}, W = \{w_1, w_2, \ldots, w_n\} \text{ and}$$
$$E(H) = \{u_i w_j, u_j w_i |\ v_i v_j \in E(G), 1 \le i \le n, 1 \le j \le n\} \cup \{u_i w_i |\ 1 \le i \le n\}.$$

Clearly, H is a connected $(r + 1)$-regular bipartite graph with $|V(H)| = 2|V(G)|$.

Define an edge-coloring β of the graph H in the following way:

(1) $\beta(u_i w_j) = \beta(u_j w_i) = \alpha(v_i v_j) + 1$ for every edge $v_i v_j \in E(G)$,

(2) $\beta(u_i w_i) = \max S(v_i, \alpha) + 2$ for $i = 1, 2, \ldots, n$.

It is easy to see that β is an edge-coloring of the graph H with colors $2, 3, \ldots, W(G) + 2$ and $\min S(u_i, \beta) = \min S(w_i, \beta)$ for $i = 1, 2, \ldots, n$. Now we present an interval $(W(G) + 2)$-coloring of the graph H. For that we take one edge $u_{i_0} w_{i_0}$ with $\min S(u_{i_0}, \beta) = \min S(w_{i_0}, \beta) = 2$, and recolor it with color 1. Clearly, such a coloring is an interval $(W(G) + 2)$-coloring of the graph H. Since H is a connected $(r+1)$-regular bipartite graph with $|V(H)| \ge 2(2r+2)$ and $H \in \mathcal{N}$, by Theorem 2, we have

$$W(G) + 2 \le |V(H)| - 3 = 2|V(G)| - 3, \text{ thus}$$
$$W(G) \le 2|V(G)| - 5.$$

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Գրառում գրաֆների միջակայքային կողային ներկումների մասին

Ռ. Քամալյան և Պ. Պետրոսյան

Ամփոփում

G գրաֆի կողային նրկումը $1, \ldots, t$ գույնրով կանվանենք միջակայքային t–ներկում, եթե բոլոր գույները օգտագործվել են և յուրաքանչյուր գագաթին կից կողերը ներկված են զույգ առ զույգ տարբեր և հաջորդական գույներով: Այս աշխատանքում ապացուցվում է, որ եթե գագաթանի G կապակցված գրաֆը ունի միջակայքային t–ներկում, ապա $t \leq 2n-3$: Նաև աշխատանքում ցույց է տրվում, որ եթե n գագաթանի G կապակցված r–համասեռ գրաֆը ունի միջակայքային t-ներկում և $n \geq 2r+2$, ապա $t \leq 2n-5$:

Заметка о интервальных реберных раскрасках графов

Р. Камалян и П. Петросян

Аннотация

Интервальной t-раскраской графа G назовем правильную раскраску ребер G в цвета $1, \ldots, t$ при которой в каждый цвет i, $1 \leq i \leq t$ окрашено хотя бы одно ребро графа G, и ребра, инцидентные каждой вершине G, окрашены в последовательные цвета. В настоящей работе доказано, что если связный граф G с n вершинами имеет интервальную $t-$раскраску, то $t \leq 2n-3$. Также в данной работе показано, что если связный r-регулярный граф G с n вершинами имеет интервальную t-раскраску и $n \geq 2r+2$, то $t \leq 2n-5$